An Exploration And Analysis Of Mathematics As A Tool In The Arts

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Abstract. In this paper we explore the use of mathematics in art, focusing on linear perspective and hyperbolic tilings. First, we discuss how mathematics has been used historically as a tool in the art world. Then, we use modern day computing techniques to create our own artwork according to these methods.

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1   Introduction

Aspects of mathematics can be seen in art throughout history. Artists carefully manipulate symmetry and proportions in the process of creating their art [22]. The use of perspective (Figure 1) has a rich history in the art world, dating back to the 14th century and Brunelleschi's "false perspective" [17]. The Golden Ratio has been used time and again to create visually appealing works of art (Figure 2); some even claim it was used in the building of the great pyramids [23]. More advanced mathematical concepts, like geometric and topological forms and fractals (Figure 3), have also been used in the creation of art [26]. Tiling the plane has been deeply explored by M.C. Escher, who became fascinated by the topic after a visit to the Alhambra in Spain [13], leading to the popularity of tessellations (Figure 4).

Figure 1. Pietro Perugino, *Christ Giving the keys to St. Peter*, 1418-1482, fresco. [24]
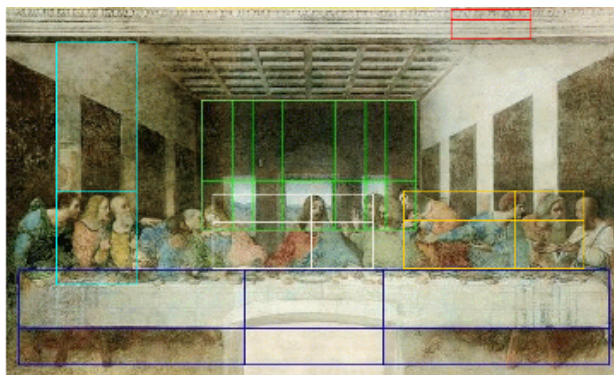
Figure 2. Leonardo da Vinci, *The Last Supper*, 1495-1498, tempera. [6]

Figure 3. Longan Drink, *i sleep only to dream of you*, 2006, digital. [9]

Figure 4. Adrien Barlow, *Tessellated Z's* 2013, digital. [2]

This paper explores the ways in which math is used intentionally in the arts, and how a knowledge of mathematics can be used in the creation of art. Specifically, we focus on two mathematical techniques used in art: linear perspective and tilings on the hyperbolic plane.

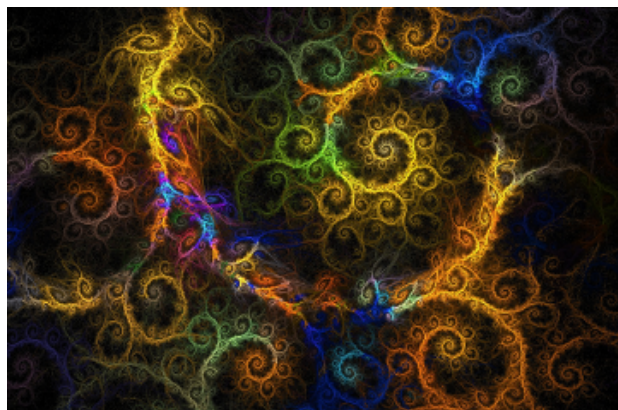We choose to focus on these techniques because we know that math is an integral aspect of their process and we believe we can streamline their application via modern computing techniques. Each section begins with a historical background on the artistic technique in question, followed by an analysis of three pieces of art. Each analysis judges the depth of the mathematics in the work and then follows up with a mathematical approach to the same artistic concept with a picture of our own.

2 Linear Perspective

2.1 A Brief History of Linear Perspective

Before we can begin our discussion of linear perspective, it feels necessary to look at Euclid's third century BC work, *Optics* [16]. This work was similar in structure to his Elements, containing pages of theorems and their proofs. Except in *Optics*, the theorems concern the geometry of vision. Sight was believed to be an active process; the eyes would send out beams and whatever they touched was what we could see. Since these beams were straight lines, it seemed only logical to apply geometry to the science of sight. One important theorem is that "objects of equal size unequally distant appear unequal and the one lying nearer to the eye always appears larger" [16]. This theorem is important because, although this text was well known throughout the middle ages, its message was largely ignored by artists in this period. The function of art at the time had less to do with expressing a scientific or realistic truth and more to do with expressing spirituality. Artists of the time would portray the importance of an object through its size, rather than how close it was to the viewer [17]. Because of this, in the majority of paintings, Christ is seen as larger than everyone else, simply because He was the most important aspect. Finally, in the 13th century, artists began to turn toward more scientifically correct compositions. It was this turning of the tide that would eventually lead to the discovery of linear perspective [17].

One prominent artist in this new movement was Giotto di Bondone (1266-1337). Through inclined lines, he was able to give his paintings an impression of depth [22]. Lines that began above eye level would incline down, lines that began below eye level would incline up, and lines that began to the left or right would tend towards the center. Although these practices seem simple, many believe that he came close to understanding linear perspective by the end of his life [17]. Another pioneer in this new age was Filippo Brunelleschi (1377-1446). Brunelleschi invented "artificial perspective" which was simply a mandate to draw proportions as the eye would see them and not as they actually were (think of Euclid's previously mentioned theorem) [17]. He is credited with the first actually correct representation of linear perspective (likely around 1413) and it is clear that he understood the need for a single vanishing point. He also understood scale and was able to compute the relationship between the size of an object in the picture and its actual size. Unfortunately for Brunelleschi, he never wrote any of this down and, even worse, his two pieces meant to showcase this rule have not survived [17].

It was Leon Battista Alberti (1404-1472) who created the first written account of con-

structing pictures in the correct perspective in his book *De Pictura (On Painting)* in 1435 [17]. This book gave mathematical descriptions of perspective and explained a method for creating the perspective of a square checkerboard pavement where one edge runs along the ground line of the picture. After his book was published, these "Albertian square-tiled pavement" paintings became very common in the art world. They were even given a name: *pavimento*, Italian for "floor" [17].

One man, Piero della Francesca (1415-1492), was a prominent artist as well as a mathematician. He wrote three volumes on perspective, focusing on mathematical principles and the geometry of vision. Some consider his work to be an extension of Alberti's, as it goes into more detail and covers more complicated examples (that is, how to draw prisms, column decorations, and even the human head in perspective) [22].

Other important artists of note in the history of linear perspective were Leonardo da Vinci, Albrecht Dürer, Frederico Commandino, Guidobaldo el Monte, and Brook Taylor. Leonardo da Vinci (1452-1519) was a successful mathematician and one of the first artists to start with a picture drawn in linear perspective and figure out where the eye must be placed to see the perspective [22]. Albrecht Dürer (1471-1528) took the topic of perspective to Germany and published a treatise about different types of mechanical objects that could be used to aid in the correct drawing of perspectives [22].

Until Frederico Commandino published *Commentarius in planisphaerium Ptolemai* in 1558, the study of perspective was focused primarily on painting; Commandino broadened the subject to stage scenery as well [22]. Guidobaldo del Monte (1545-1607) realized that any set of parallel lines which are not parallel to the plane of the picture will eventually converge to a vanishing point [22]. Brook Taylor (1685-1731) coined the phrase "linear perspective" and was the first to stress the mathematical importance of the vanishing point [22].

2.2   Analysis

We now look at three works of art which showcase perspective and analyze them in terms of correct perspective. First, we look at *The Last Supper* by Giotto di Bondone, as seen in Figure 5. From the date, we can see that this work was created before Alberti published *De Pictura*, the first written account of constructing pictures in the correct perspective. As mentioned before, Giotto was adept at depicting depth in his works, even if he didn't have the correct formulation for linear perspective. As we can see here, the lines which begin above eye level tend downwards, while the lines which begin below eye level tend upwards. Additionally, they are all moving (generally) toward the center of the piece. Although it is clear that he didn't yet understand the need for a single vanishing point, Giotto made significant progress towards displaying the depth in the room.

Figure 5. Giotto di Bondone, *The Last Supper*, 1320-1325, fresco. [8]

Figure 6. Pieter de Hooch, *A Woman Drinking with Two Men*, 1658, oil on canvas. [7]

The second work is a painting by Pieter de Hooch: *A Woman Drinking with Two Men*, as seen in Figure 6. This painting was created well after Alberti's treatise on perspective and it shows. Unlike in Giotto's work, every line meets at a single vanishing point. Another

important note is the tiled pavement floor, or *pavimento*. As mentioned before, this style became rather popular after Alberti's treatise, which detailed the process of creating a floor that is visually correct.

The third, and most recent, work is a painting by Gustave Caillebote: *Paris: A Rainy Day*, as seen in Figure 7. This painting employs a two point perspective. The idea behind a two point perspective is similar to one point; there are simply two vanishing points instead of one.

Figure 7. Gustave Caillebote, *Paris: A Rainy Day*, 1877, oil on canvas. [3]

As we can see, the two point perspective here is illustrated by the building in the center. Note how the lines on the right side of the building converge to a single vanishing point, and the lines on the left side of the building converge to a single, but different, vanishing point on the same horizon line. Now that we've looked at a few paintings, we turn our attention to the construction on which we focused for this paper.

2.3 The Mathematics

We now show how to portray a tiled floor that is visually accurate. In particular, how to correctly space the vertical and horizontal lines so that the rectangles, or tiles, appear to have equal depth. In order to do this, we walk through a basic distance point perspective construction.

1. The artist begins by picking the front edge of the floor and the desired width, measured in the number of tiles. We have labeled each point for ease of understanding later. *Note*: For this construction, the tiles must be equally spaced.
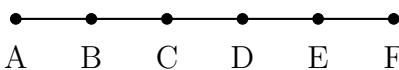

Figure 8a. Beginning the floor.

2. Then, the artist picks a vanishing point. This point is at the eye level of the viewer when the artwork is displayed.

V
•

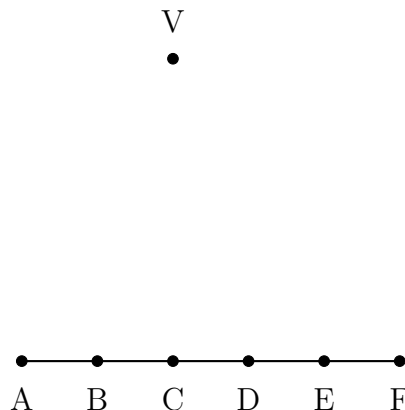•———•———•———•———•———•
A   B   C   D   E   F

Figure 8b. Placing the vanishing point.

3. Now, the artist decides upon a viewing distance. This is the desired distance from which viewers will view the artwork. For our purposes we place P anywhere on the horizon line (the line through V which is parallel to $\overline{AF}$), but the artist places the distance point based upon their desired viewing distance. *Note*: The length of $\overline{VP}$ represents the distance from the eye of the viewer to the picture plane. It is dashed here because it is not necessary for the construction of this perspective.
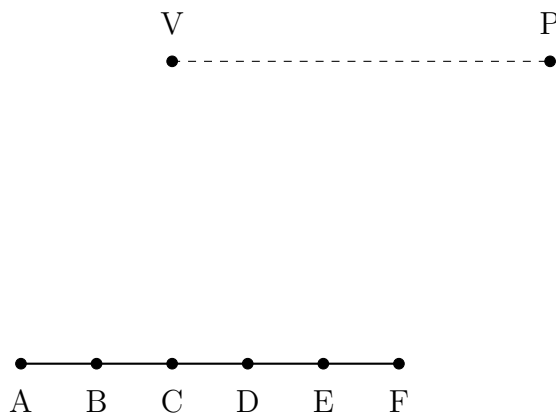
V   P
•- -•

•———•———•———•———•———•
A   B   C   D   E   F

Figure 8c. Placing the distance point.

4. The artist must draw lines from each of the initial points (A, B, C, D, E, F) to point V. These lines represent the lines on the floor which run perpendicular to the horizon line.
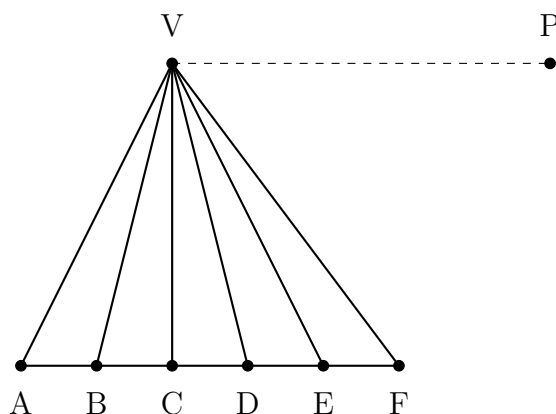
Figure 8d. Representing the perpendiculars to the horizon line.

5. Lastly, a line is drawn from point A to point P and lines parallel to $\overline{AF}$ are drawn through each point of intersection with $\overline{AP}$.
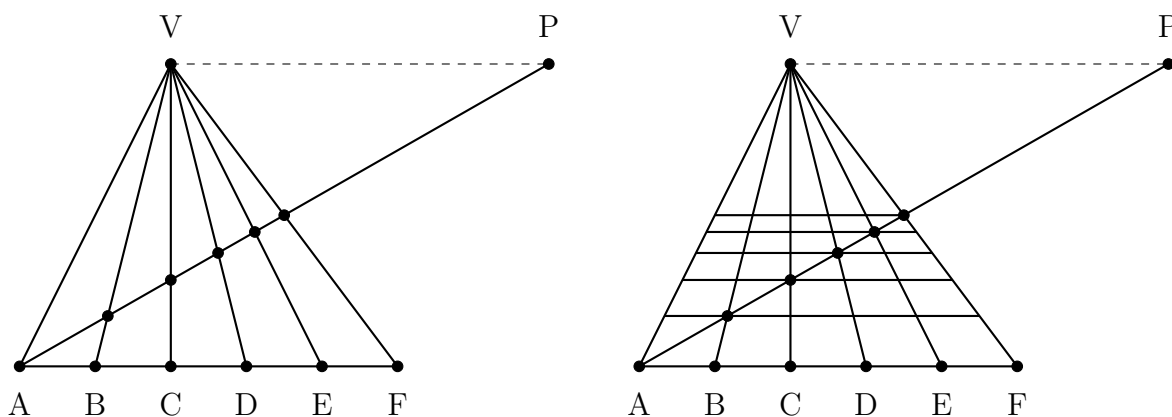

Figure 8e. Completing the floor.

Once we understand the construction, we can streamline the design process. Wouldn't it be nice if we could take a grid of our chosen dimensions and apply a perspective transformation to it? To do this, we create a program that will take as inputs the length of our desired checkerboard and the location of our vanishing and distance points (see Appendix A for code). The program successfully takes this information and creates a correct perspective of our choice.

For example, if we want to construct a 4x4 checkerboard under a perspective with a vanishing point in the center of the horizon line and a distance point two units to the right of the edge of the board, then the described process above (with the help of our program) produces the following result.
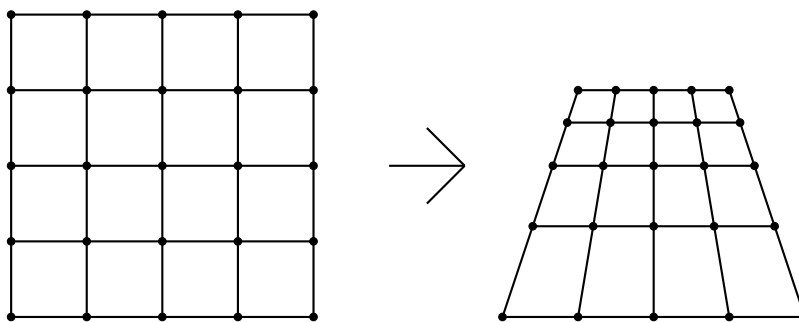
2.4 Our Approach

The last thing to do is to try our own hand at making a *pavimento*. For our piece, we decide to create a floor that is tiled with the Saint Martin's University logo. Using the program we wrote, we apply a linear perspective transformation to our chosen grid. Then, all we have to do is fill in the squares (with the help of Photoshop CS6 [1]). The final product is pictured in Figure 9.
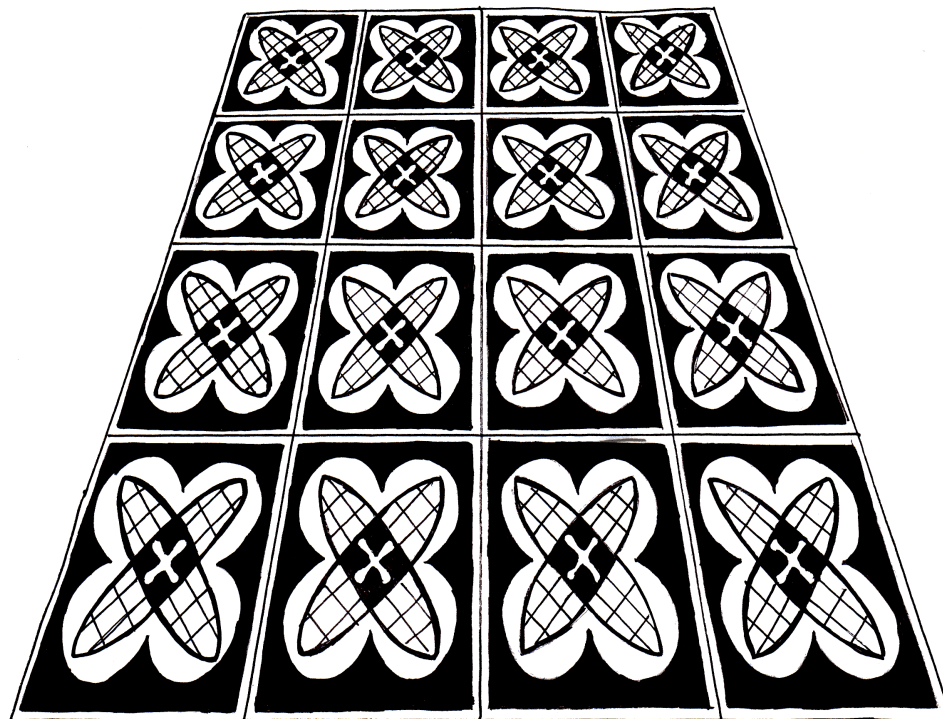

Figure 9. Perspective transformation of Saint Martin's University logo.

3   Tessellations/Tilings

3.1   A Brief History of Tessellations/Tilings

A **tessellation** is a repeating pattern without any gaps or overlaps [4]. Tessellations are commonly found in architecture throughout history. Four examples of tessellations in the construction of buildings can be seen in Figures 10-13 below.

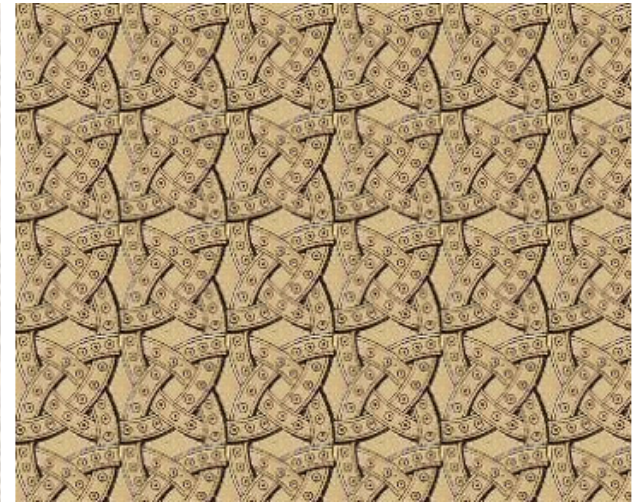**Figure 10.** Lincoln Cathedral, England, 1072 [19]

Figure 11. Lucca Cathedral, Italy, 1204 [20]

Figure 12. El-Hedim Square, Morocco, 1672 [21]

Figure 13. Alhambra, Spain, 889 [28]

We now discuss four different types of tessellations. A **regular tessellation** (Figure 14) is a tessellation of a regular polygon (a shape with equal sides and equal angles), where the same shape is used throughout [4]. A **semi-regular tessellation** (Figure 15) is a type of tessellation which uses more than one type of regular polygon [4]. A **monohedral tessellation** (Figure 16) consists of only one shape, but this shape can be rotated or flipped [4]. Lastly, an **aperiodic tessellation** (Figure 17) is a non-repeating tessellation [4].
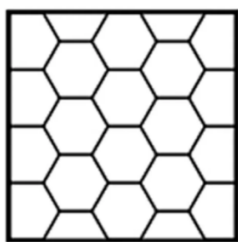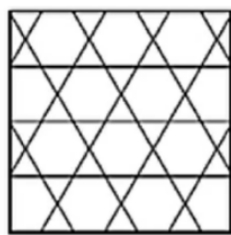

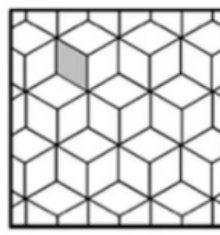**Fig 14.** A Regular Tessellation [4]   **Fig 15.** A Semi-Regular Tessellation [4]   **Fig 16.** A Monohedral Tessellation [4]   **Fig 17.** An Aperiodic Tessellation [4]
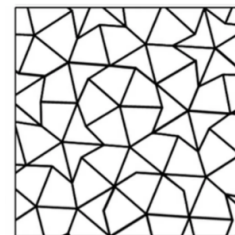
It is harder to place these developments into a historical context, but the artist most well known for tessellations is M.C. Escher. Additionally, even though Escher didn't consider himself a mathematician, we know he deliberately used mathematics in the creation of his art. He once said, "Although I am absolutely innocent of training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists" [25]. It is for this reason that we decide to focus our analysis on M.C. Escher.

3.2   Analysis

We now turn our attention to three pieces by Escher, all of which he classified as "regular divisions of the plane" [14].

This first piece (Figure 18, on the next page) is an example of a monohedral tessellation. As we can see, it is a tessellation of all one shape (pegasus) but this shape is not a regular polygon. Escher called this piece an "example of translation" [13]. A translation is when all the points of a shape are moved the same distance in the same direction without resizing, rotating, or reflecting. As we can see, the pegasus has simply been translated over and over. An important note: Escher's use of the proper mathematical terms show his understanding of the math being applied to these works.

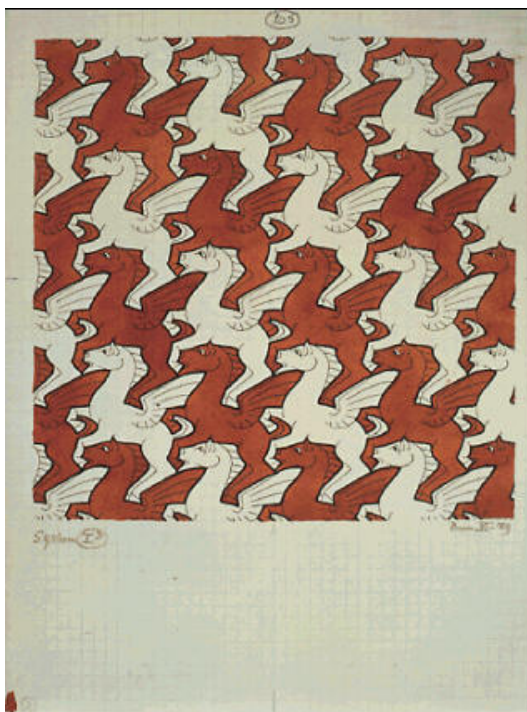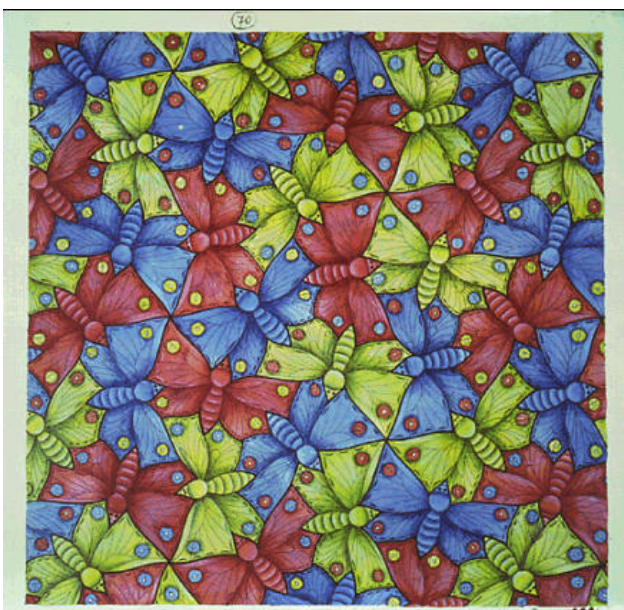**Figure 18.** M.C. Escher, *Symmetry Work no. 105*, 1959, india ink, pencil, watercolor. [15]

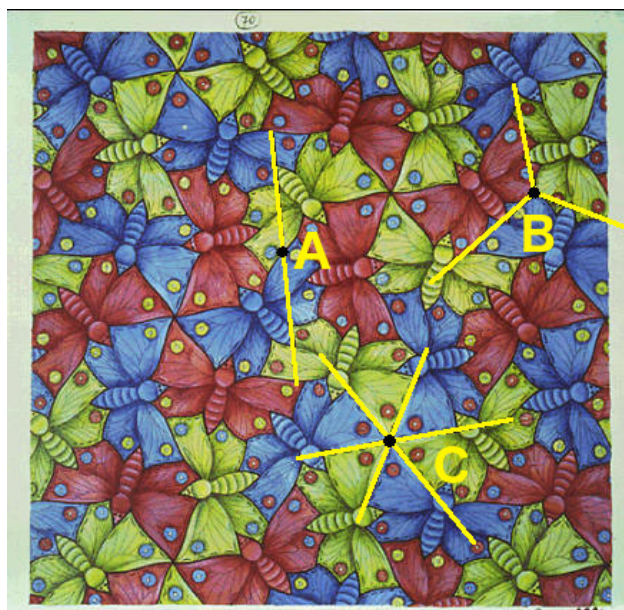**Figure 19.** M.C. Escher, *Symmetry Work no. 70*, 1948, ink, watercolor. [14]

Figure 20. *Symmetry Work no. 70*, example of axes.

The second piece (Figure 19) is another of Escher's Symmetry Works. This is, again, a monohedral tessellation, but here we have the rotation of shapes which was absent in the

last one. In fact, Escher himself referred to this piece as "an example of axes" [13]. This is because he repeated the butterflies by rotating them around binary, ternary, and senary axes (as seen in Figure 20, on the previous page).

A rotation around a binary axis means that a figure is rotated 180° once around a single point. This can be seen with either the green and blue butterflies or the two red butterflies rotated around point A in Figure 20. Rotation around ternary axes means that a figure is rotated three times, at 120° each, around a single point. We see this with the three butterflies rotated around point B in Figure 20. Lastly, rotation around senary axes means that the figure is rotated six times, at 60° each around a single point, formed from the intersection of three binary axes. This can be seen in Figure 20 with the butterflies rotated around point C.

Figure 21. M.C. Escher, *Circle Limit IV*, 1960, woodcut. [12]

We now look at the final piece: *Circle Limit IV* (Figure 21). Escher was fascinated by the ways in which he could portray a three-dimensional world in a two-dimensional piece of art [11]. It was this fascination that caused Escher to turn to hyperbolic geometry (a non-Euclidean geometry). Escher called this piece a "non-Euclidean problem" [13] and consulted a mathematician (and close friend), Professor H.S.M. Coxeter, for help in its construction. Coxeter helped Escher base *Circle Limit IV* on the Poincarè disk model of the hyperbolic plane [5]. As mathematicians, we are intrigued by the use of a non-Euclidean geometry (and recognition of doing so) by an artist. It is because of this reason that we decide to use hyperbolic geometry to analyze this piece further.

3.3   The Mathematics

For the hyperbolic geometry portion of this paper, we use the Poincaré disk model of hyperbolic geometry. Hyperbolic geometry is different from Euclidean geometry in one way: the parallel postulate. In Euclidean geometry, every line has exactly one line parallel to it through any given point. But in hyperbolic geometry, every line has infinite lines parallel to it through any given point. The Poincaré disk model is a 2D representation of hyperbolic geometry [27]. For this paper, we have to understand two things: how to construct a hyperbolic line and how to construct a hyperbolic circle.

To construct a hyperbolic line in this model, two points are needed. The hyperbolic line is constructed by constructing the Euclidean circle that goes through both points and is orthogonal to the edge of the disk. A hyperbolic circle is constructed the same way as a Euclidean circle. It is a set of points that are equidistant from a given point. However, in hyperbolic geometry, distances are skewed.
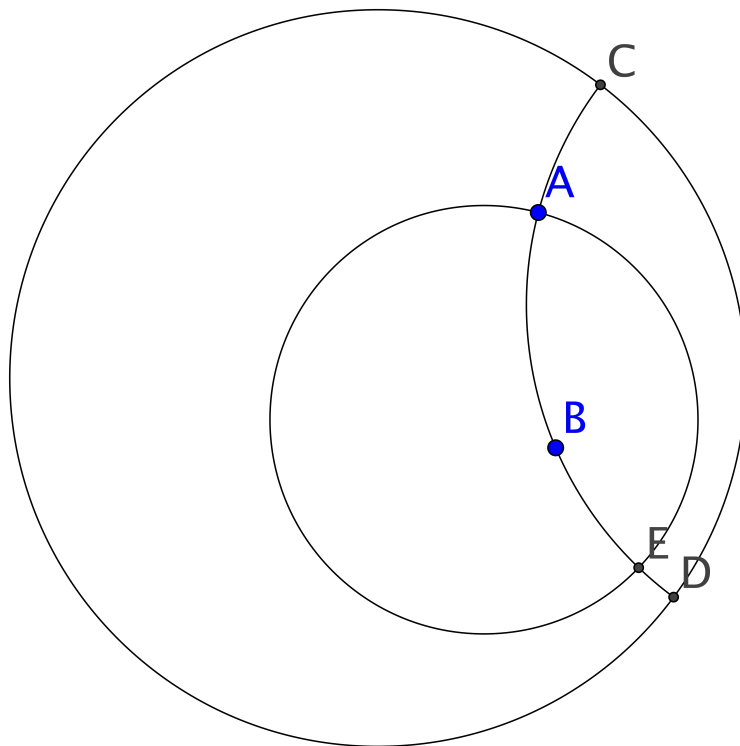
Figure 22. Example of hyperbolic lines and circles in the Poincaré's disk model.

In Figure 22 we see Poincaré's disk with a hyperbolic line through A and B and a hyperbolic circle centered at B with radius AB which is equal to radius BE. This illustrates how two distances that are equal in the Euclidean plane are not equal in the Poincarè disk model. The disk is infinite and so, as we approach the edge but never reach it, segments that look equal will become longer and longer as there is more distance packed into them.

Now, let's take another look at *Circle Limit IV*.

Figure 23. Hyperbolic tiling of 45-45-60 degree isosceles triangles.
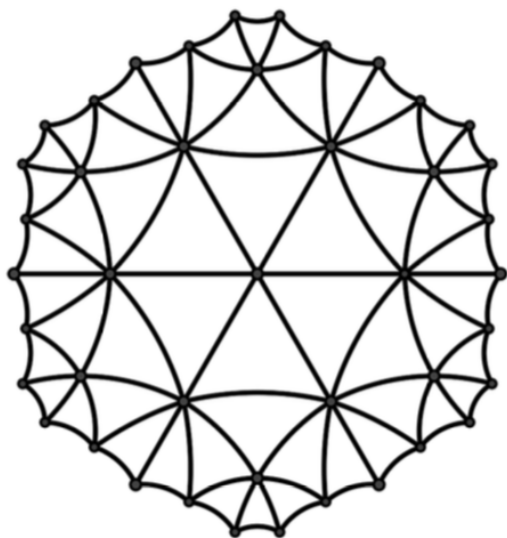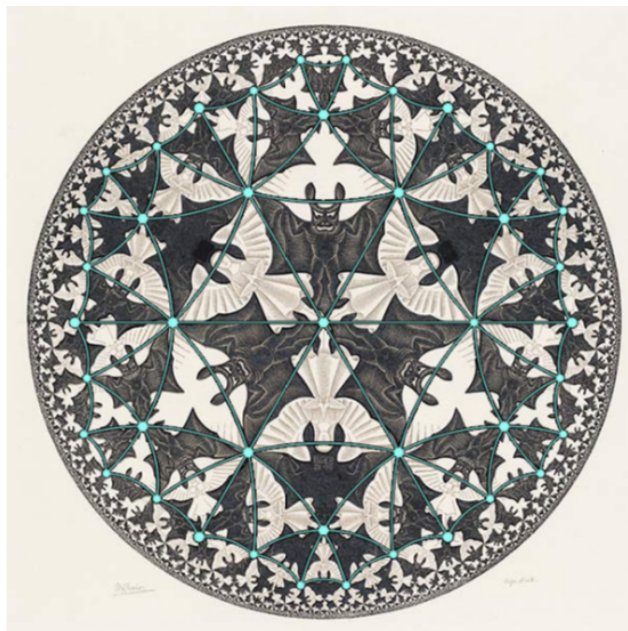
Figure 24. Hyperbolic tiling overlaid on Escher's *Circle Limit IV*.

We now create a tiling of isosceles triangles of degree 45-45-60, using Geogebra [18], pictured on the left. It is important to note that the lines in this tiling meet at equal angles at each intersection. That is, all triple intersections meet at 60° and all quadruple intersections meet at 45°. Also note that each isosceles triangle tile meets one triple intersection and two quadruple intersections.

On the right, we have this same tiling overlaid on top of Escher's woodcut. The tiling that we have created is a fairly close match to what Escher himself created. However, we can see some slight variations. These variations are likely due to error on our part and error on the part of Escher. For our part, there are some errors that occurred when tiling the plane with GeoGebra. At the points where six triangles meet, all of the angles are exactly 60°. But, where eight triangles meet, the angles are all $45° \pm 1$ when rounded to the nearest tenth. As for Escher, despite the extensive research and collaboration with H.S.M. Coxeter, it is important to remember the medium that Escher was working with. Had *Circle Limit IV* been a drawing or painting, there likely would be few errors, but this piece is a woodcut. This means that Escher had to carve the entire piece into a large chunk of wood. It is highly likely that this process, however impressive, would lead to a certain amount of error.

3.4 Our Approach

Yet again, we attempt to create this style on our own. Since our shape is square (we again use the Saint Martin's University logo) we need a square tiling. For the sake of time, we

use a {4,5} square tiling (a regular tessellation of 4-gons wherein 5 meet at each vertex at equal angles of 72° - that is, each square has 72° corners!) created by Douglas Dunham at University of Minnesota Duluth [10]. Just like our *pavimento*, all we have to do is fill in the blanks (with the help of Photoshop CS6 [1]) to create our own Escher styled print. The final product is pictured in Figure 25.

Figure 25. Hyperbolic transformation of Saint Martin's University logo.

4   Conclusion

To say that the modern, computerized approach to these classic construction methods is fast and easy would be a lie. The time required to write the computer programs that are necessary to creating the images is significant. However, now that we have these programs, portraying a floor under a desired perspective or tiling an image in the hyperbolic plane, is much faster and easier. It now takes seconds to represent a tiled floor with accurate perspective and

we can put together a hyperbolic tiling in about an hour on the computer. The best thing about streamlining classic mathematics with computers, in my opinion, is that it gives us a template for our work of art. Then, all that's left to do is fill in the blanks. However, that is not to say that filling in the blanks is easy. Patience and some semblance of artistic ability are still required for the task. Overall, though, the strictly computerized version of these mathematical constructions are very effective and an incredibly time-saving tool. In my own experience, as a mathematician and an artist, it is a very worthwhile approach to the traditional pen-and-paper method of construction these images.

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5 Appendix A: Program

1. Perspective Program (gen3)

```
def gen3(n,C,D,B):
    m1=B/D
    m0=B/C
    mn=B/(C-n)
    Gen3=[]
    for i in [0..n]:
        if i==C:
            beta = m1*C
        else:
            mi=B/(C-i)
            beta =(-i*m1*mi)/(m1-mi)
        alpha=(beta*C)/B
        delta=(beta/mn)+n
        for x in [0..n]:
            gamma=alpha+(delta-alpha)*x/n
            Gen3.append((gamma,beta))
    return Gen3
```