Rose-Hulman Undergraduate Mathematics Journal

Volume 18
Issue 2

Article 2

The Convex Body Isoperimetric Conjecture in the Plane

John Berry
Williams College

Eliot Bongiovanni
Michigan State University, eliotbonge@gmail.com

Wyatt Boyer
Williams College

Bryan Brown
Pomona College

Paul Gallagher
University of Pennsylvania

See next page for additional authors

The Convex Body Isoperimetric Conjecture in the Plane

Authors

John Berry, Eliot Bongiovanni, Wyatt Boyer, Bryan Brown, Paul Gallagher, David Hu, Alyssa Loving, Zane Martin, Maggie Miller, Byron Perpetua, and Sarah Tammen

Rose-Hulman Undergraduate Mathematics Journal

The Convex Body Isoperimetric Conjecture in $\mathbb{R}^2$

John Berry[a], Eliot Bongiovanni[b], Wyatt Boyer[c], Bryan Brown[d], Matthew Dannenberg[e], Paul Gallagher[f], David Hu[g], Jason Liang[h], Alyssa Loving[i], Zane Martin[j], Maggie Miller[k], Byron Perpetua[l], Sarah Tammen[m], Yingyi Zeng[n]

Volume 18, No. 2, Fall 2017

Sponsored by

Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
mathjournal@rose-hulman.edu
scholar.rose-hulman.edu/rhumj

[a]Williams College
[b]Michigan State University
[c]Williams College
[d]Pomona College
[e]Harvey Mudd College
[f]University of Pennsylvania
[g]Georgetown University
[h]University of Chicago
[i]University of Hawaii
[j]Williams College
[k]University of Texas at Austin
[l]Williams College
[m]University of Georgia
[n]St. Mary's College of Maryland

Rose-Hulman Undergraduate Mathematics Journal
Volume 18, No. 2, Fall 2017

The Convex Body Isoperimetric Conjecture in $\mathbb{R}^2$

John Berry, Eliot Bongiovanni, Wyatt Boyer, Bryan Brown, Matthew Dannenberg, Paul Gallagher, David Hu, Jason Liang, Alyssa Loving, Zane Martin, Maggie Miller, Byron Perpetua, Sarah Tammen, Yingyi Zeng

Abstract. The Convex Body Isoperimetric Conjecture states that the least perimeter needed to enclose a volume within a ball is greater than the least perimeter needed to enclose the same volume within any other convex body of the same volume in $\mathbb{R}^n$. We focus on the conjecture in the plane and prove a new sharp lower bound for the isoperimetric profile of the disk in this case. We prove the conjecture in the case of regular polygons and show that in a general planar convex body, the conjecture holds for small areas.

Acknowledgements: This paper is the work of the Williams College NSF "SMALL" 2013, 2014, and 2015 Geometry Groups. We thank our advisor Professor Frank Morgan for his support. We would like to thank the National Science Foundation, Williams College, and the MAA for supporting the "SMALL" REU and our travels to MathFest 2013, MathFest 2014, and MathFest 2015.

1 Introduction

The Convex Body Isoperimetric Conjecture states that the least perimeter needed to enclose a volume within a ball is greater than the least perimeter needed to enclose the same volume within any other convex body of the same volume in $\mathbb{R}^n$ [Mo1]. We focus on the case $n = 2$. Even in this simple case, little is known.

An *isoperimetric region* is a least-area enclosure of a given volume in a particular space. The spaces considered here are convex bodies in $\mathbb{R}^2$. An *isoperimetric curve* is the boundary of an isoperimetric region. Figure 1 provides an example of isoperimetric regions of a given area for a square and circle.

Outline of Paper

In Section 2, we review and prove some fundamental properties of isoperimetric minimizers in these regions. In Section 3, we prove some simple estimates of the relative perimeter in terms of the area. The main theorem of this section, Theorem 3.8, provides a sharp lower bound for the isoperimetric profile of the disk. The proof of this theorem relies heavily on a result of Kuwert [K, Theorem 1.1], which shows that the square of the isoperimetric profile is a concave function of the area. In Section 4, we examine the special case of when the convex bodies are regular polygons. Using simple geometric transformations, we prove in Theorem 4.1 the conjecture in the case that the convex body is a convex regular polygon. In Section 5, we examine the conjecture for small areas, and we show in Theorem 5.2 that any convex body satisfies the conjecture for a sequence of sufficiently small areas. We prove this theorem by comparing the curvature of the disk with that of the convex body. In Section 6, we discuss obstructions to applying symmetrization arguments to the convex body isoperimetric problem. In Section 7, we suggest further lines of research.

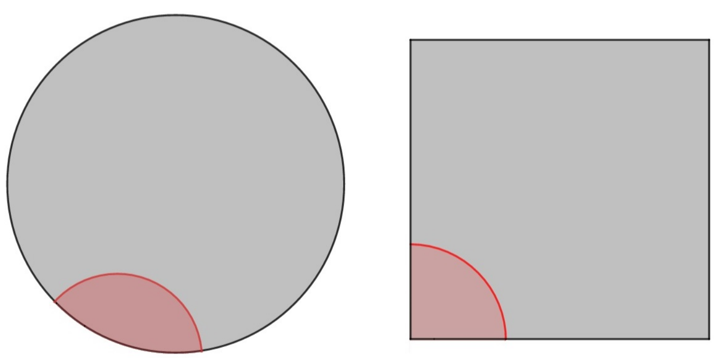

Figure 1: It is easy to see that enclosing a fraction of the unit-area circle requires more perimeter (red border) than enclosing the same fraction of the unit square. Theorem 4.1 proves that the least perimeter needed to enclose a fraction of the unit-area circle is greater than the least perimeter needed to enclose the same fraction of any unit-area regular polygon.

2 The Two-dimensional Convex Body Isoperimetric Conjecture

This section establishes notation, definitions, and fundamental properties of isoperimetric curves used throughout the paper. We conclude that for all convex bodies in the plane, isoperimetric regions are connected and enclosed by a curve that meets the boundary orthogonally.

Definition 2.1. For a convex body B, we denote by $P(A)$ or $P_B(A)$ the length of the isoperimetric curve enclosing an area A within B.

A result of Kuwert, restated here, characterizes the isoperimetric length as a function of area and will later serve to imply a sharp lower bound of the least perimeter needed to enclose given area in the disk. It also holds for general convex bodies via approximation by C^2 smooth convex bodies.

Lemma 2.2. *[K, Theorem 1.1] For any C^2 convex body, $P^2(A)$ is concave.*

The following proposition has been shown for C^3 regions in general dimension by Sternberg and Zumbrun [S, Theorem 2.1]. Interior and boundary regularity of isoperimetric surfaces within a C^3 convex body are known in general dimensions; see [Mo2, 8.5,12.3]. The following proposition treats the 2-dimensional case without any smoothness hypothesis.

Proposition 2.3. *A least-perimeter curve enclosing a given area within a convex region in the plane consists of circular arcs or straight lines meeting the boundary orthogonally.*

Proof. Isoperimetric curves in Euclidean space are regular [Mo5, Theorem 2.3] with constant curvature, so isoperimetric curves within a convex region must consist of circular arcs and line segments. If an isoperimetric curve does not intersect the boundary of the convex region at all, the curve can be translated so as to meet the boundary tangentially. Suppose an isoperimetric curve in any convex region in the plane intersects the boundary at an angle $\theta \neq \pi/2$. Without loss of generality, take $\theta < \pi/2$. Then cutting the corner with angle θ of the isoperimetric region results in a change in perimeter vs. area described by

$$\frac{dP^2}{dA} = -2\cot\theta < 0.$$

For any sufficiently small $\epsilon > 0$, we can choose a distance h (see Figure 2) such that drawing a normal to the boundary of length h and connecting the endpoint to a point on the isoperimetric curve such that the resulting line segment is tangent to the original curve (see Figure 2), or to the other end of the isoperimetric curve if no such point exists, reduces perimeter while preserving area and decreasing perimeter by at least ϵ, a contradiction. Therefore, an isoperimetric curve consists of circular arcs or straight lines and each component must intersect the boundary orthogonally. □

Sternberg and Zumbrun [S, Theorem 2.6] prove that an isoperimetric surface in a C^3 convex body in $\mathbb{R}^n$ must be connected. The following propositions treat the 2-dimensional case without any smoothness hypothesis.

Proposition 2.4. *Isoperimetric regions in a convex body are connected.*

Proof. For some convex region B with total area A_{tot}, one has $P_B^2(0) = 0$. Lemma 2.2 states that P_B^2 is concave. If we wish to enclose some area $A_{enc} \in (0, A_{tot}]$, then for any smaller area $A < A_{enc}$ it follows that

$$P_B^2(A) \geq \frac{A}{A_{enc}} P_B^2(A_{enc}).$$

Suppose that the region with total area A_{enc} enclosed by an isoperimetric curve consisted of $n > 1$ connected components within B each with nonzero area. Let L_i denote the length of the curve enclosing the i-th connected component and A_i the area of the i-th component. As all A_i sum to A_{enc}, one obtains

$$\left(\sum_{i=1}^{n} L_i\right)^2 > \sum_{i=1}^{n} L_i^2 \geq \sum_{i=1}^{n} \frac{A_i}{A_{enc}} P_B^2(A_{enc}) = P_B^2(A_{enc}).$$

Hence, any curve enclosing a region with multiple connected components must require more length than a true isoperimetric curve in B. □

Proposition 2.5. *Isoperimetric curves inside a convex region have exactly one component.*

Proof. Suppose that an isoperimetric curve enclosing an area A_{enc} within a convex body B of total area A_{tot} consisted of $n > 1$ components. As B is convex and the isoperimetric region must be connected by Proposition 2.4, the complement of the isoperimetric region within B is not connected. Yet, the complement of the isoperimetric region must itself be an isoperimetric region with area $A_{tot} - A_{enc}$ within B. Hence, any isoperimetric curves within a convex region consist of one connected component. □

Remark 2.6. It follows from Propositions 2.3 and 2.4 that an isoperimetric curve in the disk is a constant curvature arc normal to the boundary. Burago and Maz'ya [B1] prove this result in $\mathbb{R}^n$, showing that an isoperimetric hypersurface inside a round ball is a spherical cap normal to the boundary (see also [B2, 18.1.3]).

The next proposition shows that we may freely choose the area of the convex bodies to work with. We typically work with convex bodies of area π, but on occasion it is convenient to work with unit-area convex bodies.

Lemma 2.7. *To prove the Convex Body Isoperimetric Conjecture, it suffices to prove maximality of a disk of any area A_0 for prescribed area less than $A_0/2$.*

Proof. By scaling, we may assume area A_0. We note further that a curve enclosing area $A < A_0$ can also be considered to be enclosing area $A_0 - A$. Since least perimeter is continuous and symmetric about $A_0/2$, it suffices to consider areas less than $A_0/2$. □

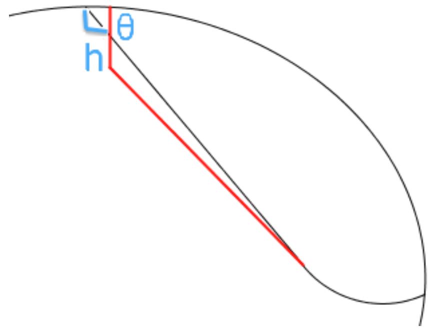

Figure 2: An isoperimetric curve inside a convex region in the plane must intersect the boundary orthogonally.

Proposition 2.8. *It suffices to prove that the unit disk requires more perimeter than any convex equilateral polygon.*

Proof. Approximate a convex region B by a polygon in the following way: let $\epsilon > 0$ be such that any ball of radius ϵ centered on ∂B has exactly two intersection points with ∂B. Pick a point p on ∂B, and center a disk there. Draw a line from p to the point p' clockwise from p where the boundary of the disk intersects ∂B. Now use p' as the center of the disk and continue in this fashion until the original disk is hit again. Connecting all these gives a polygon with all side lengths equal to ϵ. Now dilate this polygon so that it has the same area as the convex region. Moving every point ϵ in a direction out of the convex region makes the polygon have area strictly greater than B, so the dilation must move all points less than ϵ. Thus every point is less than ϵ away from the boundary, and we have a good approximation of our convex region. □

3 Estimates and n-gons for Small n

Here we prove some simple estimates of the relative perimeter in terms of the area. The main theorem of this section, Theorem 3.8, provides a sharp lower bound for the isoperimetric profile of the disk.

Recall Definition 2.1 that for a convex body B we denote by $P(A)$ or $P_B(A)$ the length of the isoperimetric curve enclosing an area A within B. We denote the disk by $\mathbb{D}$.

Lemma 3.1. *For $A \in [0, \pi/2]$,*

$$P_{\mathbb{D}}^2(A) \geq \frac{8}{\pi}A.$$

Proof. By Lemma 2.2, $P_{\mathbb{D}}^2(A)$ is a concave function of A. Since $P_{\mathbb{D}}^2(0) = 0$ and $P_{\mathbb{D}}^2(\pi/2) = 4$, $P_{\mathbb{D}}^2(A)$ must lie above the line between these two points, which is $P^2 = (8/\pi)A$. □

The constant $8/\pi$ is the best possible, as $P_C^2(\pi/2) = (8/\pi)(\pi/2)$. We also have means of approximating lengths of isoperimetric curves enclosing an area on polygonal bodies in terms of area for small areas.

Proposition 3.2. *Let B be a convex polygon with smallest interior angle θ formed by incident sides. Then $P_B^2(A) \leq 2\theta A$ for all A.*

Proof. A circular arc within B centered at the vertex of the angle θ enclosing an area A has length given by $P^2(A) = 2\theta A$ if A is sufficiently small so that the arc intersects both of the two sides adjacent to the angle θ. The isoperimetric curve must then satisfy $P_B^2(A) \leq 2\theta A$ for these sufficiently small A. Since by Lemma 2.2 P_B^2 is a concave function, $P_B^2(A) \leq 2\theta A$ for all A. □

While Propositions 3.1 and 3.2 are very simple estimates, with Lemma 2.2, they prove the conjecture for a large class of polygons.

Proposition 3.3. *If B is a polygon of area π in which the smallest interior angle θ satisfies $\theta \leq 4/\pi$, then $P_B^2(A) \geq P_{\mathbb{D}}^2(A)$ for all A.*

Proof. Let B be such a polygon. By Proposition 3.2, $P_B^2(A) \leq 2\theta A$ for all A. But by Proposition 3.1 we have that

$$P_B^2(A) \leq 2\theta A \leq \frac{8A}{\pi} \leq P_{\mathbb{D}}^2(A).$$

So $P_B \leq P_{\mathbb{D}}$, as desired. □

As a corollary, we immediately have that all triangles satisfy the conjecture.

Corollary 3.4. *For any planar triangle B with area π, $P_B^2(A) > P_{\mathbb{D}}^2(A)$ for all A.*

To prove that any parallelogram has greater perimeter than the disk, the lower-bound for $P_{\mathbb{D}}^2$ from Lemma 3.1 is not tight enough. However, by choosing more secant lines we can find better lower-bounds for $P_{\mathbb{D}}^2$ that are sufficient for this purpose.

Proposition 3.5. *If B is a parallelogram of area π, then $P_B^2(A) \geq P_{\mathbb{D}}^2(A)$ for all A.*

Proof. Let L be such a parallelogram. Remark 2.6 indicates that the only possible minimizers in the parallelogram are circular arcs about the four vertices of the parallelogram or the two altitudes of the parallelogram. If $\theta \leq \pi/2$ is the smallest interior angle of the parallelogram, then by Lemma 3.2 we obtain

$$P_L^2(A) \leq 2\theta A \leq \pi A.$$

For areas close to $\pi/2$, the shorter altitude of height h of the parallelogram is the minimizer. As P_L^2 is concave, it must be bounded above by its tangent line at any point. Taking an area sufficiently close to $\pi/2$, this tangent line is $P^2 = h^2$. As L has area π, we also have that $h \leq \sqrt{\pi}$, so $P_L^2 \leq \pi$. Combining the two estimates on P_L^2 we have the following:

$$P_L^2(A) \leq \begin{cases} \pi A & \text{if } 0 \leq A \leq 1 \\ \pi & \text{if } 1 < A \leq \frac{\pi}{2}. \end{cases}$$

As $P^2_{\mathbb{D}}$ is concave it is bounded below on the interval $[0,1]$ by a secant line meeting $P^2_{\mathbb{D}}$ when $A = 0$ and $A = 1$. Similarly, it is bounded below on the interval $[1,\pi]$ by a secant line meeting $P^2_{\mathbb{D}}$ when $A = 1$ and $A = \pi$. As $P^2_{\mathbb{D}}(0) = 0 = P^2_L(0)$ and $P^2_{\mathbb{D}}(\pi/2) = 4 \geq P^2_L(\pi/2)$, it suffices to show that $P^2_L(1) \leq \pi \leq P^2_{\mathbb{D}}(1)$. Recalling Proposition 2.3, a direct computation reveals that $P^2_{\mathbb{D}}(1) = 3.5081$ up to four decimal places, so the claim follows. □

E. Milman [[M], Cor. 1.4] includes the next result; F. Morgan [Mo4] gives an alternate proof.

Proposition 3.6. *For any $A \in [0,\pi]$, we have*

$$P^2_{\mathbb{D}}(A) \geq A(\pi - A).$$

While the above bound is quadratic in A, on most of the interval it is dominated by the linear bound from Lemma 3.1. The following lemma allows us to find a much tighter lower bound for $P^2_{\mathbb{D}}$.

Lemma 3.7. *If $f : [0,1] \to \mathbb{R}$ is a function such that*

(1) $f(0) = f(1) = 0$,

(2) $f \in C^2((0,1))$,

(3) $\lim_{t\to 1^-} f'(t) = 0$,

(4) $\lim_{t\to 1^-} f''(t) > 0$, *and*

(5) f'' *is increasing on* $(0,1)$,

then $f \geq 0$ *on* $[0,1]$.

Proof. Since f'' is increasing on $(0,1)$, if $\lim_{t\to 0^+} f''(t) \geq 0$, f must be convex on $[0,1]$. Therefore, for any $y \in [0,1]$, we have $f(x) \geq f(y) + f'(y)(x-y)$ for $x \in [0,1]$. Taking $y \to 1$, we have that $f(x) \geq 0$, as desired. Therefore, we may assume that $\lim_{t\to 0^+} f''(t) < 0$.

As f'' is increasing, and $\lim_{t\to 0^+} f''(t) < 0$ and $f''(1) > 0$, there must exist a unique $c \in (0,1)$ such that $f''(c) = 0$ and $f'' \leq 0$ on $(0,c]$ and $f'' \geq 0$ on $[c,1)$. It follows that f is concave on $[0,c]$ and convex on $[c,1]$. Using the same convexity argument as above, we find that $f \geq 0$ on $[c,1]$. Now, as f is concave on $[0,c]$, we have that

$$f(x) \geq \frac{f(c) - f(0)}{c - 0} x = \frac{f(c)}{c} x,$$

for $x \in [0,c]$. But as shown above, $f(c) \geq 0$, so $f \geq 0$ on $[0,c]$ as desired. □

In the following theorem, the quadratic bound on $P^2_{\mathbb{D}}$ from Proposition 3.6 is made sharp. The constant $16/\pi^2$ is the best possible, as $P^2_{\mathbb{D}}(\pi/2) = 4 = (16/\pi^2)(\pi/2)(\pi - \pi/2)$.

Theorem 3.8. *For any A we have*

$$P_{\mathbb{D}}^2(A) \geq \frac{16}{\pi^2} A(\pi - A).$$

Proof. It suffices to show that $d^2 P_{\mathbb{D}}^2 / dA^2$ is increasing on $(0, \pi/2)$, as then Lemma 3.7 can be applied to the function

$$P_{\mathbb{D}}^2\left(\frac{\pi A}{2}\right) - \frac{16}{\pi^2} \cdot \left(\frac{\pi A}{2}\right)\left(\pi - \frac{\pi A}{2}\right)$$

to show that it must be non-negative on $[0, 1]$.

Let $\kappa(A)$ denote the curvature of an isoperimetric minimizer that encloses area A in the disk. Since $dP_{\mathbb{D}}/dA = \kappa$, the Leibniz and chain rules for derivatives imply that

$$\frac{d^2 P_{\mathbb{D}}^2}{dA^2} = 2\kappa^2 + 2P_{\mathbb{D}} \frac{d\kappa}{dA}. \tag{1}$$

For each area A, consider an isoperimetric curve within the unit disk enclosing that area. Let θ be the angle of the circular arc on the boundary of the unit disk between the two intersection points with that isoperimetric curve. By Propositions 2.3 and 2.4 as well as the symmetry of a disk, $A(\theta)$ is an injection, so parametrize A by θ. Elementary geometry shows that

$$P_{\mathbb{D}}(A) = (\pi - \theta) \tan \frac{\theta}{2}, \tag{2}$$

$$\kappa(A) = \cot \frac{\theta}{2}, \tag{3}$$

and

$$\frac{d\kappa}{dA}(A) = \frac{\cot^3 \frac{\theta}{2}}{\sin\theta + \theta - \pi}. \tag{4}$$

Substituting (2), (3), and (4) into (1), we find that

$$\frac{d^2 P_{\mathbb{D}}^2}{dA^2} = \cot^2 \frac{\theta}{2} + (\pi - \theta) \frac{\cot^2 \frac{\theta}{2}}{\sin\theta + \theta - \pi}. \tag{5}$$

As $A(\theta)$ is strictly increasing, it suffices to show that the function of θ, denoted $f(\theta)$ in (5) is an increasing function of θ on $[0, \pi]$. The derivative of f is

$$\frac{df}{d\theta} = \frac{(\pi - \theta)(2 - \cos\theta) - 3\sin\theta}{(\sin\theta + \theta - \pi)^2} \cot^2 \frac{\theta}{2}.$$

The only non-square term in the above expansion is the numerator of the fraction. Simple numerics then show that this expression is non-negative on $[0, \pi]$, completing the proof. □

The following proposition considers the hemisphere with density $d = 1/(2r^2)$.

Proposition 3.9. *Let $P_{\mathbb{H}}(A)$ denote the least weighted perimeter to enclose a weighted area A on a hemisphere of radius r and density $d = 1/(2r^2)$. Then*

$$P_{\mathbb{H}}^2(A) = 2dA(\pi - A).$$

Proof. It is well known that an isoperimetric region on the sphere is bounded by a circle. We claim that an isoperimetric region on a hemisphere is bounded by a circular arc meeting the boundary orthogonally. Otherwise reflecting a better region on the northern hemisphere onto the southern hemisphere would yield a better region on the sphere. The rest of the claim then follows from a standard surface area calculation.

Recall that the surface area and perimeter of regions bounded by a circle on the surface of a unit sphere with constant density 1 are related by $P^2 = A(4\pi - A)$. Scaling the sphere causes P to change linearly and Q to change quadratically with the factor of dilation, while changing the density affects both P and Q linearly. Hence

$$P_{\mathbb{H}}^2(A) = 2dA\left(\pi - \frac{1}{2r^2 d}A\right) = 2dA\left(\pi - A\right)$$

as desired.

□

By Proposition 3.9, to prove Theorem 3.8 it would suffice to find a weighted area-preserving diffeomorphism from the disk to $\mathbb{H}_{\pi/4}$ that is weighted-perimeter nonincreasing. In practice finding such a map is difficult. However, we can find such a map from the disk to weighted hemispheres of a larger radii. While this is a much more appealing geometric argument, the best bound we can obtain by this argument is the same as the quadratic bound in Proposition 3.6.

Alternate proof of Proposition 3.6. Consider the map $f : \mathbb{D} \to \mathbb{H}_r$ given by

$$f(x, y) = r\left(\sqrt{1 - \frac{x^2 + y^2}{2}}x, \sqrt{1 - \frac{x^2 + y^2}{2}}y, -1 + x^2 + y^2\right).$$

The map

$$g(x, y) = \left(\sqrt{1 - \frac{x^2 + y^2}{4}}x, \sqrt{1 - \frac{x^2 + y^2}{4}}y, -1 + \frac{x^2 + y^2}{2}\right)$$

is the Lambert azimuthal equal-area projection of the disk of radius $\sqrt{2}$ onto the unit lower hemisphere. As $f(x, y) = rg(\sqrt{2}x, \sqrt{2}y)$, a region $R \subseteq \mathbb{D}$ is first scaled by $\sqrt{2}$ to the dilated disk $\sqrt{2}\mathbb{D}$ increasing its area by a factor of 2. Then $\sqrt{2}\mathbb{D}$ is mapped to the unit lower hemisphere with no change in area. Finally, the unit lower hemisphere is stretched to the lower hemisphere of radius r, scaling the area by a factor of r^2. Therefore, the region R is mapped to a region of area $2r^2|R|$. But then the weighted area of $f(R)$ is $|R|$, because $2r^2 =$

$1/d$, where d is the density of $\mathbb{H}_r$. It follows that f is weighted area-preserving diffeomorphism from the disk to $\mathbb{H}_r$.

Next, we show that if $r > 2\sqrt{2}$, then f is weighted-perimeter nonincreasing. Let $\gamma : [0, L] \to \mathbb{D}$ be a curve parameterized by arc length. We show that

$$\int_0^L d|(f \circ \gamma)'(t)|\, dt \leq \int_0^L |\gamma'(t)|\, dt.$$

It suffices to show that for all t that

$$d|(f \circ \gamma)'(t)| \leq |\gamma'(t)|,$$

as if $F(\ell) = \int_0^\ell d|(f \circ \gamma)'(t)|\, dt$ and $G(\ell) = \int_0^\ell |\gamma'(t)|\, dt$, then this is just the statement that $F' \leq G'$. As $F(0) = G(0)$, this would imply that $F(\ell) < G(\ell)$ for any ℓ. As γ is parameterized by arc length, $|\gamma'(t)| = 1$ for any t. So we show that

$$d^2|(f \circ \gamma)'(t)|^2 \leq 1.$$

Furthermore, we may assume that if $\gamma(t) = (x(t), y(t))$ then $x'(t) = \sqrt{1 - y'(t)^2}$, as f is symmetric in x and y, so we need only show the result for t such that $(x(t), y(t))$ is in the first or fourth quadrant. Calculation reveals that

$$\begin{aligned} |(f \circ \gamma)'|^2 = &-\frac{1}{2(-2 + x^2 + y^2)} \big(-4x^4 + x^2(8 - 3y^2) \\ &+ (-2 + y^2)^2 + (-12 + 5x^2 + 5y^2)y'((x - y)(x + y)y' - 2xy\sqrt{1 - y'^2})\big) \end{aligned}$$

Using the standard gradient methods from calculus, it can be shown that $|(f \circ \gamma)'|$ achieves a maximum of 4 over the set $0 \leq x, x^2 + y^2 \leq 1$, and $0 \leq y' \leq 1$. Therefore, as long as $4d^2 \leq 1$, f will be weighted-perimeter nonincreasing. But this clearly happens only when $d \leq 1/2$, or equivalently $1 \leq r^2$. Therefore, if $r = 1$, f does not increase weighted perimeter. Thus,

$$P_{\mathbb{D}}^2(A) \geq A(\pi - A),$$

as desired. □

4 Regular Polygons

In this section we examine the special case of when the convex bodies are regular polygons. Theorem 4.1 proves the Convex Body Isoperimetric Conjecture in the case that the convex body is a convex regular polygon.

Theorem 4.1. *The least perimeter needed to enclose an area within the unit-area disk is greater than the least perimeter needed to enclose the same area within any planar unit-area regular n-gon.*

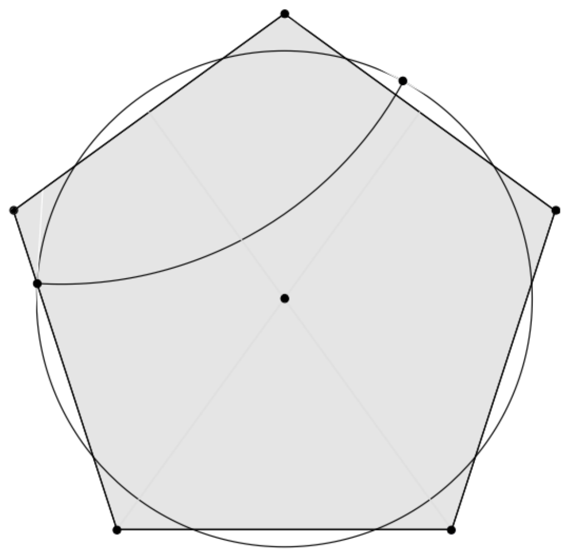

Figure 3: A “good” region in the proof of Proposition 4.1: the isoperimetric arc on the disk encloses more area in the polygon with less perimeter.

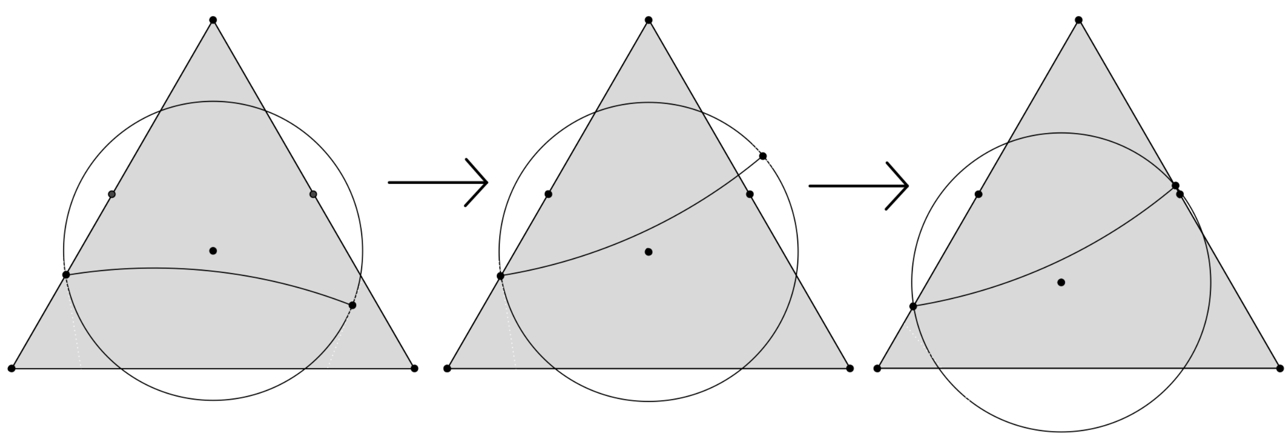

Figure 4: Reflection and translation of the disk used to fix a “bad” region in the proof of Proposition 4.1 (second case, where the ending point of the arc is outside the triangle after reflecting the disk).

Proof. Consider a unit-area regular n-gon and a unit-area disk in the plane sharing the same center. There are n regions of the polygon that protrude from the disk and n regions of the disk that protrude from the polygon. By symmetry, since the disk and the polygon have the same area, all of the disk protrusions and polygon protrusions have the same area.

Choose an area $A \in [0, 1/2)$. Let an isoperimetric arc enclose that area on the disk such that the starting point of the arc is at a disk–polygon intersection point, and such that at that starting point the arc bends towards a polygon protrusion. If the ending point of the arc is outside the polygon, then the portion of the arc inside the polygon encloses more area in the polygon using less perimeter than the entire arc encloses inside the disk (Figure 3). Thus the perimeter required to enclose the area A inside the polygon is less than the perimeter required to enclose A inside the disk.

If the other ending point of the arc is inside the polygon, then that point borders either the polygon protrusion adjacent to the starting point or some other polygon protrusion. In the first case, translate the disk and its isoperimetric arc along the axis of the side of the polygon on which lies the starting point of the arc toward the polygon vertex nearest the starting point, so that the ending point lies on the polygon. Now the arc encloses more area on the polygon than A while using the same perimeter. Translating the disk slightly further in the same direction results in the arc's enclosing a region on the polygon that still has more area than A while using less perimeter. Since minimal perimeter on the polygon as a function of area is nondecreasing, enclosing the area A on the polygon requires less perimeter than on the disk.

In the second case, reflect the disk about the axis passing through its radius and the arc's starting point. If the ending point now lies inside the polygon, then rotate the disk so that the ending point is at an intersection point of the disk and polygon; the arc then encloses more area in the polygon using less perimeter, as desired. If the ending point is outside the polygon, then translate the disk and its arc along the axis of the side of the polygon on which lies the starting point of the arc toward the polygon vertex nearest the starting point, so that the ending point lies on the polygon. After this translation, the disk protrusion adjacent to the starting point has not changed in area, the disk protrusions once fully or partially enclosed by the arc have each shrunk in area, and the polygon protrusions enclosed by the arc have each increased in area. Therefore, the arc now encloses more area in the polygon than in the disk using the same perimeter. By the same argument used in the first case, we can enclose the area A on the polygon using less perimeter than on the disk.

We need only show that after this translation, the starting point of the arc remains on the polygon. The distance the starting point can be moved toward the nearest polygon vertex while remaining on the polygon is $d_1 = h_1 \sec(\theta/2)$, where h_1 is the height of the triangle whose base is formed by connecting the points of intersection nearest the vertex and θ is the internal angle of the polygon. The maximum translation distance is $d_2 = h_2 \sec(\theta/2)$, where h_2 is the height of each disk protrusion. Therefore, to show $d_1 > d_2$, it is enough to show

$h_1 > h_2$. Express h_1 as the radius of the disk subtracted from circumradius of the polygon:

$$h_1 = \csc\left(\frac{\pi}{n}\right)\sqrt{\frac{\tan(\pi/n)}{n}} - \frac{1}{\sqrt{\pi}}.$$

Express h_2 as the inradius of the polygon subtracted from the radius of the disk:

$$h_2 = \frac{1}{\sqrt{\pi}} - \frac{1}{\sqrt{n\tan(\pi/n)}}.$$

Since $1/k + k > 2$ for all positive k, we have

$$\frac{1}{\sqrt{\cos(\pi/n)}} + \sqrt{\cos\left(\frac{\pi}{n}\right)} > 2,$$

for all $n \geq 3$. Furthermore, note that $1/\sqrt{n\sin(\pi/n)}$ is greater than $1/\sqrt{\pi}$ for all $n \geq 3$. Thus

$$\begin{aligned}
\frac{1}{\sqrt{n\sin(\pi/n)}}\left(\frac{1}{\sqrt{\cos(\pi/n)}} + \sqrt{\cos\left(\frac{\pi}{n}\right)}\right) &> \frac{2}{\sqrt{\pi}}\\
\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{\sin(\pi/n)\cos(\pi/n)}} + \frac{\sqrt{\cos(\pi/n)}}{\sqrt{\sin(\pi/n)}}\right) &> \frac{2}{\sqrt{\pi}}\\
\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{\sin(\pi/n)\cos(\pi/n)}}\right) - \frac{1}{\sqrt{\pi}} &> \frac{1}{\sqrt{\pi}} - \frac{1}{\sqrt{n}}\left(\frac{\sqrt{\cos(\pi/n)}}{\sqrt{\sin(\pi/n)}}\right)\\
\csc\left(\frac{\pi}{n}\right)\sqrt{\frac{\tan(\pi/n)}{n}} - \frac{1}{\sqrt{\pi}} &> \frac{1}{\sqrt{\pi}} - \frac{1}{\sqrt{n\tan(\pi/n)}}\\
h_1 &> h_2,
\end{aligned}$$

as desired. □

We suspect that the regular n-gon is the extremal case of all equilateral n-gons.

Conjecture 4.2. *For any given n, the least perimeter needed to enclose an area within the regular unit-area n-gon is greater than the least perimeter needed to enclose the same area within any other convex equilateral unit-area n-gon.*

By Proposition 2.8 and Theorem 4.1, this conjecture is sufficient to prove the full Convex Body Isoperimetric Conjecture. For $n = 3$, the conjecture is trivial. We show it for $n = 4$.

Proposition 4.3. *The least perimeter needed to enclose an area within a unit-area square is greater than the least perimeter needed to enclose the same area within any other unit-area rhombus.*

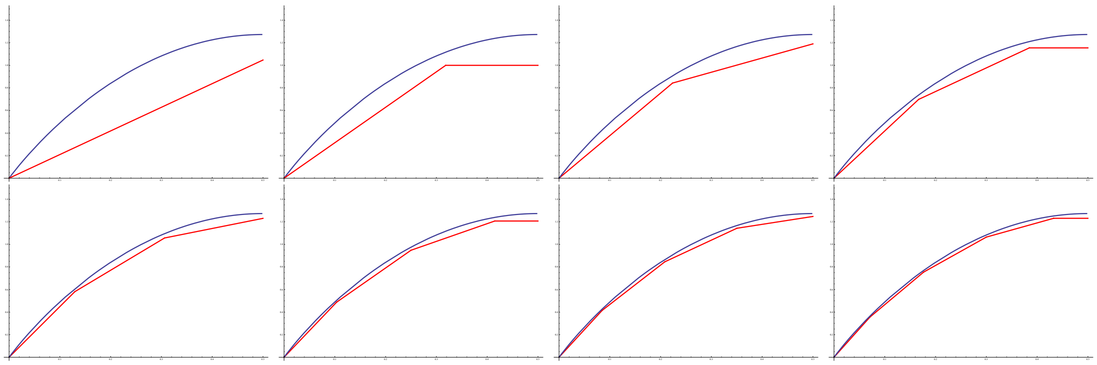

Figure 5: Left to right, top to bottom: P^2 vs. A for the regular 3-, 4-, 5-, 6-, 7-, 8-, 9-, and 10-gon in red (lower piecewise linear curve) against P^2 vs. A for the disk in blue (upper curve).

Proof. Let R be a unit-area rhombus; let S be a unit-area square. By Proposition 2.3, an isoperimetric curve in R is either a circular arc orthogonal to two adjacent sides or a line orthogonal to two opposite sides. Let s denote the side-length of R and $\theta < \pi/2$ denote its smallest interior angle. By using circular arcs of radius r about this angle, we obtain $P_R^2(A) \leq 2\theta A$ for A between 0 and $(\theta/2)s^2$. For $A \geq s^2 \sin(2\theta)/4$, using a straight line orthogonal to the base yields the bound $P_R^2(A) \leq \sin\theta$. The only other possible minimizers for R are lines orthogonal to the other sides, and circular arcs about the angles of $\pi - \theta$. Thus

$$P_R(A)^2 = \begin{cases} 2\theta A & \text{if } A \leq \sin\theta/(2\theta) \\ \sin\theta & \text{if } \sin\theta/(2\theta) < A \leq \frac{1}{2} \end{cases}$$

as the first are beaten by the shorter heights and the latter do no better than circular arcs about the angle θ. For S, this computation yields

$$P_S(A)^2 = \begin{cases} \pi A & \text{if } A \leq 1/\pi \\ 1 & \text{if } 1/\pi < A \leq \frac{1}{2}. \end{cases}$$

Comparison of the two formulae yields the desired result. □

Here is a partial result towards proving Conjecture 4.2 for the $n = 5$ case. This proof demonstrates the difficulty of working with equilateral polygons as the number of sides increases.

Proposition 4.4. *Let T be the unit-area convex polygon made up of an equilateral triangle and a square that share a side. Let s denote the common side length. Then*

$$P_T(A)^2 = \begin{cases} \frac{2\pi}{3} A & \text{if } 0 \leq A \leq \frac{\sqrt{3}}{4} s^2 \\ \frac{\pi}{3}\left(A + \frac{\sqrt{3}}{4} s^2\right) & \text{if } \frac{\sqrt{3}}{4} s^2 < A \leq \frac{12-\sqrt{3}\pi}{4\pi} s^2 \\ s^2 & \text{if } \frac{12-\sqrt{3}\pi}{4\pi} s^2 < A \leq \frac{1}{2}. \end{cases}$$

Figure 6: In a rhombus, minimizers enclosing a single angle have less perimeter than those in a square. Furthermore, minimizers enclosing two angles both have less perimeter and are available to use for smaller areas in the rhombus.

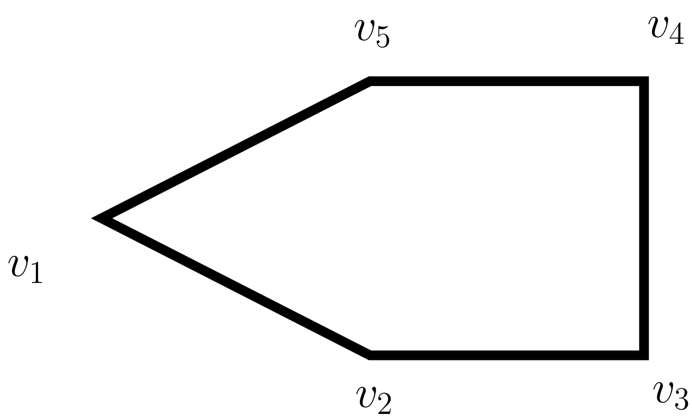

Figure 7: The polygon in Proposition 4.4

Furthermore, when $A < \sqrt{3}s^2/4$*, the unique minimizer is a circular arc about the angle of this polygon with measure* $\pi/3$*. When* $A > (12 - \sqrt{3}\pi)s^2/4\pi$*, the unique minimizer is an altitude of the square. If* A *is between those values, there are two minimizers of equal length, each a circular arc between a side of the triangle and a side of the square with a side of the triangle between them.*

Proof. There are six possible isoperimetric curves on T. We list them according to type (see Figure 7).

1. Circular arcs about v_1.
   Such arcs imply that $P_T^2(A) \leq (2\pi/3)A$, with equality when these are the minimizer.

2. Circular arcs about v_2 or v_5.
   Such arcs imply that $P_T^2(A) \leq (5\pi/3)A$, with equality when these are the minimizer.

3. Circular arcs about v_3 or v_4.
   Such arcs imply that $P_T^2(A) \leq \pi A$, with equality when these are the minimizer.

4. Circular arcs from $\overline{v_1v_2}$ to $\overline{v_4v_5}$ or from $\overline{v_1v_5}$ to $\overline{v_2v_3}$.
   Such arcs imply that $P_T^2(A) \leq \pi/3(A + \sqrt{3}s^2/4)$, with equality when these are the minimizer.

5. Circular arcs from $\overline{v_1v_2}$ to $\overline{v_3v_4}$ or from $\overline{v_1v_5}$ to $\overline{v_3v_4}$.
   Such arcs imply that $P_T^2(A) \leq (2\pi/3)(A + s^2/2\sqrt{3})$, with equality when these are the minimizer.

6. Perpendicular Lines from $\overline{v_2v_3}$ to $\overline{v_4v_5}$.
   Such arcs imply that $P_T^2(A) \leq s^2$, with equality when these are the minimizer.

All of these curves intersect T at right angles (Prop. 2.3). As these are the only possible isoperimetric curves, $P_T^2(A)$ is given by the minimum of them for each A. By Proposition 2.7, we need only consider $0 < A < 1/2$. In this region, clearly the bounds given by (2), (3), or (5) are never sharp, as the bound from (1) lies strictly below them. Thus curves of type two, three, or five cannot be minimizers. Taking the minimum over all the remaining possibilities yields

$$P_T(A)^2 = \begin{cases} \frac{2\pi}{3}A & \text{if } 0 \leq A \leq \frac{\sqrt{3}}{4}s^2 \\ \frac{\pi}{3}\left(A + \frac{\sqrt{3}}{4}s^2\right) & \text{if } \frac{\sqrt{3}}{4}s^2 < A \leq \frac{12-\sqrt{3}\pi}{4\pi}s^2 \\ s^2 & \text{if } \frac{12-\sqrt{3}\pi}{4\pi}s^2 < A \leq \frac{1}{2}. \end{cases}$$

Finally, note that for any A, the curve of type (1) or type (6) is unique, but there are two curves of type (4). This gives the desired result. □

Finding the length of isoperimetric curves within equilateral polygons becomes a far more difficult task as the number of sides increases. One potential solution is to consider equiangular polygons instead, as the length of isoperimetric curves within a polygon largely depend on the angles of that polygon.

Proposition 4.5. *The least perimeter needed to enclose an area within a unit-area equiangular pentagon is at most that needed to enclose the same area within the unit-area regular pentagon.*

Proof. Let R_5 denote the unit-area regular pentagon and E_5 denote another equiangular pentagon. By Proposition 2.3, isoperimetric curves within an equiangular pentagon are either circular arcs intersecting adjacent sides or circular arcs intersecting sides with one side between them. Hence

$$P^2_{R_5}(A) = \begin{cases} \frac{6\pi}{5}A & \text{if } 0 \le A \le A_0 \\ \frac{2\pi}{5}\left(A + \frac{1}{4}s^2\tan\left(\frac{2\pi}{5}\right)\right) & \text{if } A_0 < A \le 1/2. \end{cases}$$

where s denotes the side-length of R_5 and A_0 is the point where the two formulae coincide.

By Proposition 3.2, $P^2_{E_5}(A) \le (6\pi/5)A$ for all A. Let ℓ denote the length of the smallest side of E_5. Then a circular arc orthogonally intersecting the sides adjacent to the shortest side will satisfy

$$P^2(A) = \frac{2\pi}{5}\left(A + \frac{1}{4}\ell^2\tan\left(\frac{2\pi}{5}\right)\right).$$

But ℓ must be less than s, so this line must meet the line $(6\pi/5)A$ before A_0. Thus $P^2_{E_5}(A) \le P^2_{R_5}(A)$ for all A. □

While finding the isoperimetric profile of equiangular polygons may be easier than for equilateral polygons, equiangular polygons do exhibit some counter-intuitive phenomena. The next proposition demonstrates that a regular polygon does not always require more perimeter to enclose a given area than all equiangular polygons with the same number of sides.

Proposition 4.6. *The least perimeter needed to bisect some equiangular hexagon is greater than the least perimeter needed to bisect the regular unit-area hexagon.*

Proof. Isoperimetric curves in an equiangular hexagon can intersect adjacent sides, sides separated by a single side, or opposite sides. In the case of a regular hexagon R_6 having a side length a, these three cases yield an isoperimetric profile:

$$P^2_{R_6} = \begin{cases} \frac{4\pi}{3}A & \text{if } 0 \le A \le A_0 \\ \frac{2\pi}{3}\left(A + \frac{\sqrt{3}}{4}a^2\right) & \text{if } A_0 < A < A_1 \\ 3a^2 & \text{if } A_1 \le A \le \frac{1}{2} \end{cases}$$

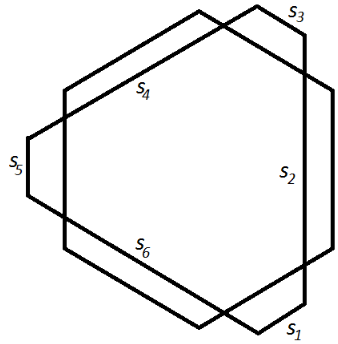

Figure 8: Some equiangular hexagons require more perimeter to divide in half than the reqular hexagon. This is an equiangular hexagon E_6 superimposed over a regular hexagon R_6. All trapezoids must have the same area, so comparison of opposite trapezoids suggests that the distance between opposite sides in E_6 is greater than in R_6.

where A_0 and A_1 are the points where the first two formulas and the last two formulas coincide, and $0 < A_0 < A_1 < 1/2$. For a unit-area regular hexagon, $a = \sqrt{2/\sqrt{3^3}}$. Consider an equiangular hexagon E_6 with sides of alternating lengths $s_1 = s_3 = s_5 = \epsilon$ and $s_2 = s_4 = s_6 = r$ with $\epsilon < r$ as in Figure 8. The isoperimetric profile of E_6 is:

$$P_{E_6}^2 = \begin{cases} \frac{4\pi}{3}A & \text{if } 0 \leq A \leq A_2 \\ \frac{2\pi}{3}\left(A + \frac{\sqrt{3}}{4}\epsilon^2\right) & \text{if } A_2 < A < A_3 \\ \left[\frac{\sqrt{3}}{2}\left(\sqrt{\frac{4}{\sqrt{3}} + 3\epsilon^2} - \epsilon\right)\right]^2 & \text{if } A_3 \leq A \leq \frac{1}{2} \end{cases}$$

where A_2 and A_3 are the points where the first two formulas and the last two formulas coincide.

When $\epsilon = 3a/4$, numerical calculations can demonstrate that all three of these possible formulas are greater than the least perimeter needed to bisect R_6.

As each possible candidate isoperimetric curve enclosing area $1/2$ in E_6 has length greater than the corresponding isoperimetric curve in R_6, the least perimeter necessary to bisect E_6 is larger than that needed to bisect R_6. □

5 The Conjecture for Small Areas

In this section we examine the Convex Body Isoperimetric Conjecture for small areas. We show in Theorem 5.1 that any convex body satisfies the conjecture for a sequence of sufficiently small areas.

Theorem 5.1. *For a sufficiently small area A, an isoperimetric curve enclosing area A in a convex unit-area region whose boundary is not differentiable is shorter than an isoperimetric curve enclosing the same area in the unit-area disk.*

Proof. If a unit-area convex region in the plane has a corner with angle $\theta < \pi$, then any sufficiently small area A may be enclosed within the region with perimeter $P \leq \sqrt{2\theta A}$. For some $\epsilon > 0$, any sufficiently small area A must be enclosed in the unit disk with minimum perimeter $P > \sqrt{2(\theta + \epsilon)A} > \sqrt{2\theta A}$. □

Proposition 5.2. *Given a convex unit-area region R in the plane which is not the disk D, there exists a sequence of areas $A_n \to 0$ such that the isoperimetric curve in the region that encloses area A_n is shorter than the isoperimetric curve enclosing area A_n within the unit-area disk.*

Proof. By Proposition 5.1, if the convex region's boundary ∂R is not differentiable, then the proof is complete. Assume ∂R is differentiable. Since R is convex, the curvature of the boundary must exist almost everywhere and any singular contributions must be positive, so that the integral of the curvature is at most 2π. If the curvature of ∂R is greater than or equal to that of ∂D almost everywhere, then the perimeter of R must be at most that of D. By the isoperimetric theorem, as they both enclose the same area, R must in fact be D, a contradiction. Therefore, we may assume that the curvature of ∂R is less than that of ∂D at some point x.

Place D on R such that their oriented boundaries ∂D and ∂R are tangent at x. Locally ∂D lies inside R. Since D and R have the same area, elsewhere ∂R must lie inside D, as in Figure 9. Translate R slightly in the direction of the inward normal at x, so that we now have at least two maximal intervals where ∂D lies inside R. Choose one subtending a central angle less than π. By moving R farther in that direction, we may assume that the interval is arbitrarily small and that the circular arc inside D remains inside R, and encloses more area than R than in D. To enclose the same area inside D thus requires more perimeter. □

Remark 5.3. Proposition 5.2 is difficult to extend to general dimension. In the plane, any two intersection points of the boundary of the unit-area disk and an overlaid unit-area convex region may be the endpoints of an isoperimetric arc in the disk. An isoperimetric surface in the unit-volume n-disk with boundary in the intersection of the disk with an n-dimensional unit volume convex body only exists if the intersection includes an $(n-2)$-sphere of any radius, which is not guaranteed. As an example in three dimensions, consider the intersection of the boundaries of a solid ball with a solid cylinder whose axis does not pass through the center of the sphere, as seen in Figure 10.

6 Flows on Polygons

In this section, we seek to apply symmetrization arguments towards proof of the Convex Body Isoperimetric Conjecture. Symmetrization has proved a useful tool for approaching isoperimetric problems in the past. Steiner, spherical, and Schwarz symmetrizations are the most well studied and have simplified the study of isoperimetric regions in many cases. All of

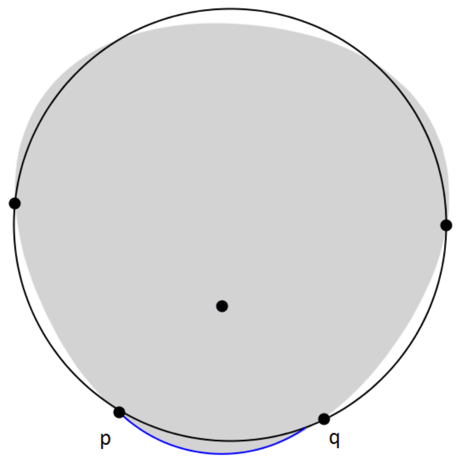

Figure 9: A unit-area region with differentiable boundary (gray) overlaid on the unit-area disk (black boundary). If the disk is placed with boundary tangent to the region's boundary at x outside the disk, the curve between p and q lies inside the disk. After translating the region slightly, the isoperimetric curve in the disk between intersection points p and q encloses more area in the convex region than in the disk.

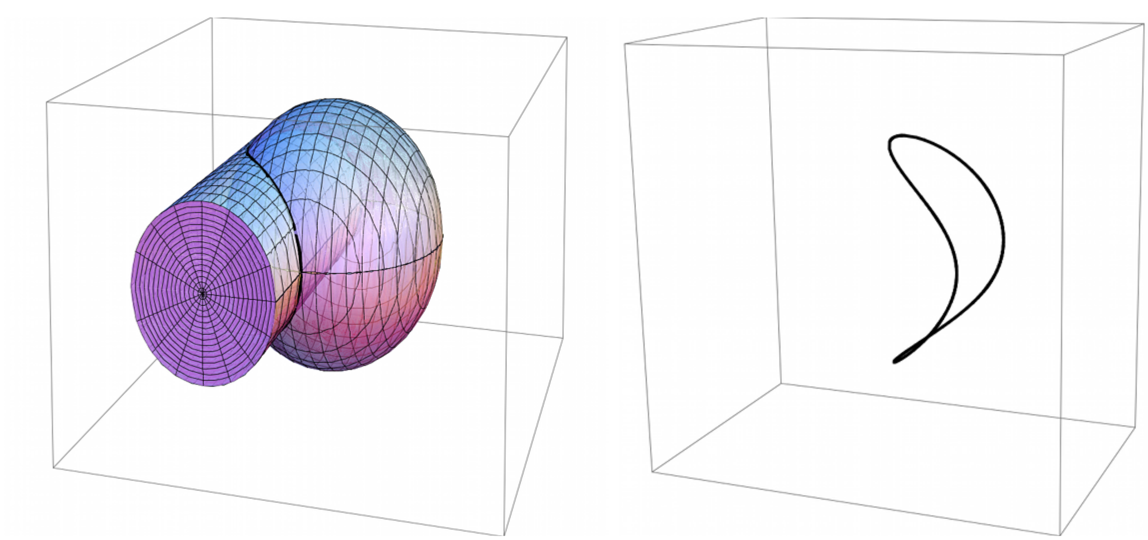

Figure 10: In dimension $n > 2$, the intersection of the unit-volume n-disk with a unit-volume convex body of dimension n need not contain a $n-2$ sphere. Left: a solid ball and cylinder in 3-space. Right: the intersection of the boundaries of the solid ball and solid cylinder.

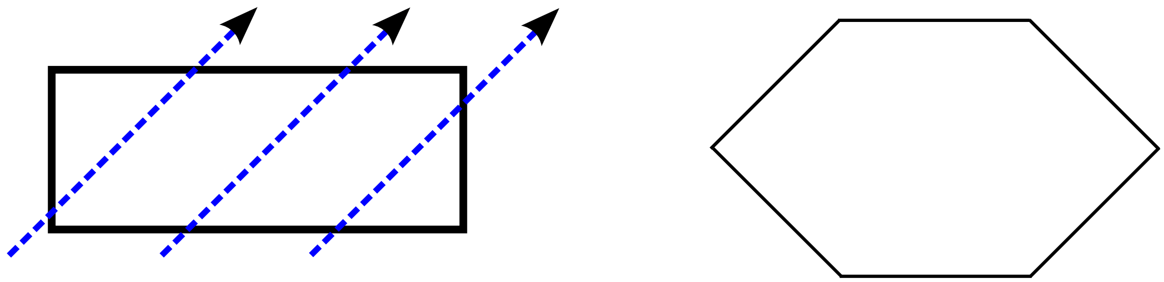

Figure 11: The Steiner symmetrization with respect to the vector v (dashed blue) is the hexagon pictured here.

these symmetrizations have the property of maintaining a body's volume, while decreasing its perimeter.

Definition 6.1. The **Steiner symmetrization** *of a set* $S \subseteq \mathbb{R}^n$ *with respect to the vector* v *is the set* T *that satisfies the following condition: for any line* L *orthogonal to* v, $L \cap T$ *is a closed line segment centered on* V *such* $L \cap T$ *and* $L \cap S$ *have the same Hausdorff measure, and* $L \cap T$ *is empty if and only if* $L \cap S$ *is. We denote the Steiner symmetrization of a set* S *in direction* v *as* $\mathrm{Sym}_v(S)$.

Example Consider the rectangle in figure 11. Steiner symmetrizing it about the vector v in the direction of the dashed blue arrow yields the hexagon in the same figure. The Steiner symmetrization has the useful property that it has the same area as its input, yet less perimeter.

The next proposition shows that Steiner symmetrization does not improve the isoperimetric profile of a convex body for small areas.

Proposition 6.2. *Let* P *be a convex polygon and* v *be a unit vector in* $\mathbb{R}^2$. *Then the smallest interior angle of* $\mathrm{Sym}_v(P)$ *is at least as big as the smallest interior angle of* P.

Proof. Consider a convex polygon, P', that is Steiner symmetric with respect to v. Without loss of generality, we may assume $v = e_1$, the first standard basis vector. If P is a convex polygon such that $\mathrm{Sym}_v(P) = P'$, then P can be obtained from P' by antisymmetrizing - that is, translating line segments in P' orthogonal to e_1 in the direction they are pointed in. So we may obtain P' by translating any line segment of P' in the direction e_2 up or down. If p is any vertex of P', then P' also has a vertex p' at the reflection of p through the e_1-axis. If p is not on this axis, then p and p' are distinct. Translating line segments close close to p will then increase the angle of one of the angles about p and p' and decrease the other in the same amount. Therefore the smallest of any of the angles around a vertex not on the e_1-axis will be at least as big as the corresponding angle in the antisymmetrization. A similar argument shows that the same holds for any vertex on the e_1 axis. $\square$

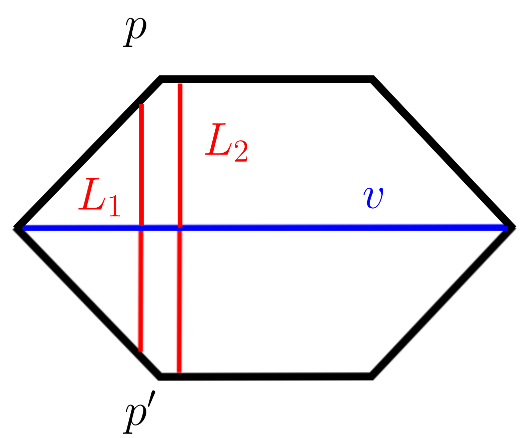

Figure 12: Translating the lines L_1 and L_2 up or down will increase one of the angles about p or p' and decrease the other by the same quantity. Because of this, Steiner symmetrization can only increase the minimum angle of a polygon.

The above proposition suggests that Steiner symetrization cannot decrease the isoperimetric profile of a convex body.

Conjecture 6.3. *For any convex body P and any unit vector $v \in \mathbb{R}^2$, $\text{Sym}_v(P)$ has an isoperimetric profile at least as large as that of P.*

The next example shows that the minimizers of a convex body P are not always mapped to minimizers of $\text{Sym}_v(P)$, for a unit vector v.

Example Consider the equilateral polygon, T, consisting of an equilateral triangle and a square that share a side. Then an altitude parallel to the shared side that cuts the polygon into two regions of equal area is not mapped to a minimizer in the Steiner symmetrization of T with respect to a vector parallel to the shared side. (See Figure 13)

Definition 6.4. A **flow** of polygons is a smooth family of equilateral polygons.

We show that there exists a canonical flow on an equilateral polygon such that the three smallest interior angles increase, and the three largest interior angles decrease.

Proposition 6.5. *Let P be a convex equilateral polygon with three largest exterior angles $\alpha_1 \geq \alpha_2 \geq \alpha_3$ and three smallest angles $\beta_1 \leq \beta_2 \leq \beta_3$. Then there is a unique flow on P such that $\alpha_i'(t) = -\beta_i'(t) \leq 0$, and the other angles remain constant.*

Proof. We denote the successive exterior angles of P by $\tau_1, \tau_2, ..., \tau_n$. We obtain the following

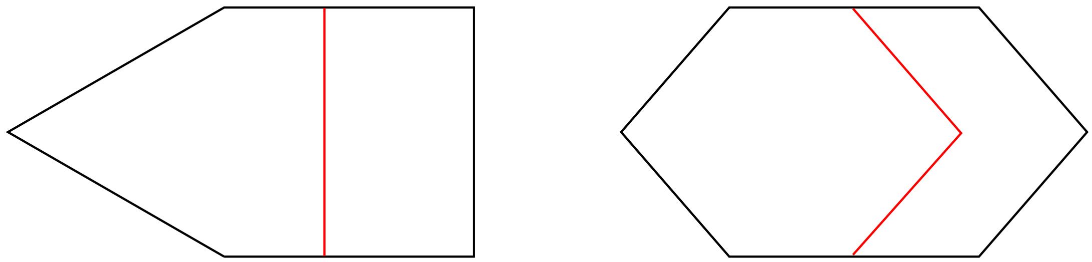

Figure 13: The red line in the convex body on the left is a minimizer. However, its image in the Steiner symmetrization of the convex body is not a minimizer. This behavior makes it difficult to relate the isoperimetric profiles of the two regions.

two conditions by noting the sum of the sides of the polygon as vectors must be 0:

$$1 + \sum_{i=1}^{n-1} \cos\left(\sum_{j=0}^{i-1} \tau n - j(t)\right) = 0$$

$$\sum_{i=1}^{n-1} \sin\left(\sum_{j=0}^{i-1} \tau n - j(t)\right) = 0.$$

Differentiating each with respect to t, and using the conditions of the flow, we obtain two homogeneous, linear equations in α_1', α_2', and α_3'. These conditions are independent as the first only requires that the sum of the sides of the polygon as vectors has abscissa 0, while the second only requires that the same sum has ordinate 0. It follows that for any $y > 0$, we may pick a set of vectors that have abscissa 0, but ordinate y. Clearly, we may also do this in a way such that the vectors are a smooth function of y and so that as $y \to 0$, they approach the sides of P. The resulting $\tau_i(t)$ then satisfy the first condition, but not the second for any $t > 0$. Therefore, these conditions must be independent, as for any P we may construct functions which satisfy one but not both. An example of the first condition being satisfied but not the second is presented in figure 14. Since the conditions are independent, there must exist a one-parameter family of solutions for the α_{j_k}'. As a result, for some fixed t,

$$\begin{pmatrix} \alpha_{j_1}'(t) \\ \alpha_{j_2}'(t) \\ \alpha_{j_3}'(t) \end{pmatrix} = m(t)v(t).$$

for some unique choice of real-valued continuous function m and vector-valued function v. Integration with respect to t then gives us that the unique solution is of the form $\alpha_{j_k}(t) = \int_0^t m(x)v_k(x)\,dx + c_k$ for some constants c_k determined by the initial values of the α_{j_k}s. $\square$

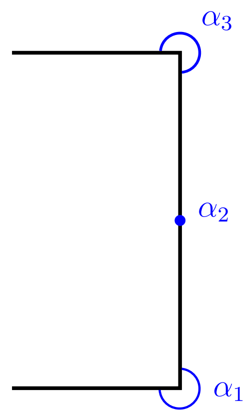

Figure 14: An instance of where the condition $1 + \sum_{i=1}^{n-1} \cos\left(\sum_{j=0}^{i-1} \alpha_{n-j}(t)\right) = 0$ is satisfied but the condition $\sum_{i=1}^{n-1} \sin\left(\sum_{j=0}^{i-1} \alpha_{n-j}(t)\right) = 0$ is not. While this is not a convex equilateral polygon, from the perspective that the α_{j_k} are simply solutions to some differential equation, this does not matter.

For a given convex equilateral polygon P we refer to the convex equilateral polygon at time $t = 1$ of the above flow as the *evolved polygon* of P.

Conjecture 6.6. *The isoperimetric profile of a convex equilateral polygon is smaller than that of its evolved polygon.*

The above conjecture is clear for when minimizers enclose only a single angle, but is difficult to prove for minimizers enclosing multiple angles.

7 Directions for Future Work

Along with the conjectures in this paper, another vein of future work is exploring similar problems with different norms on $\mathbb{R}^2$. For example, is there a convex body such that the least perimeter needed to enclose an area inside that convex body is greater than the least perimeter needed to enclose the same area within any other convex body of the same total area when length and area are calculated with respect to the metric induced by a p-norm? It is unclear if this variant of the problem is more or less tractable than the standard conjecture. We would like to thank Professor Michael Gage of the University of Rochester for proposing this variant of the conjecture.

References

[B1] Y.D. Burago and V. Maz'ya, Potential theory and function theory for irregular regions (translated from Russian), Seminars in Mathematics, V. A. Steklov Mathematical

Institute, Leningrad, Vol. 3, Consultants Bureau, New York, (1969).

[B2] Y.D. Burago, V.A. Zalgaller, Geometric Inequalities, Grundlehren der Mathematischen Wissenschaften 285, Springer-Verlag, Berlin, (1988).

[E] L. Esposito, V. Ferone, B. Kawohl, C. Nitsch, and C. Trombetti, The longest shortest fence and sharp Poincaré-Sobolev inequalities, Arch. Ration. Mech. Anal. 206 (3) (2012), 821-851.

[Go] E. Gonzalez, U. Massari, and I. Tamanini, On the regularity of boundaries of sets minimizing perimeter with a volume constraint, Indiana Univ. Math. J. 32 (1983), 25-37.

[Gr] M. Grüter, Boundary regularity for solutions of a partitioning problem, Arch. Rat. Mech. Anal. 97 (1987), 261-270.

[K] E. Kuwert, Note on the isoperimetric profile of a convex body, Geometric Analysis and Nonlinear Partial Differential Equations, S. Hildebrandt and H. Karcher, eds., Springer, Berlin, 2003, 195-200.

[M] E. Milman, Sharp isoperimetric inequalities and model spaces for curvature-dimension-diameter condition, J. Eur. Math. Soc.17 (2015), 1041-1078.

[Mo1] F. Morgan, Convex body isoperimetric conjecture, `http://sites.williams.edu/Morgan/2010/07/03/convex-body-isoperimetric-conjecture/` (2010).

[Mo2] F. Morgan, Geometric Measure Theory: A Beginner's Guide, Academic Press, 2016.

[Mo3] F. Morgan, Regularity of isoperimetric hypersurfaces in Riemannian manifolds, Trans. Amer. Math. Soc. 355 (2003), 5041-5052.

[Mo4] F. Morgan, Sharp isoperimetric bounds for convex bodies, `http://sites.williams.edu/Morgan/2013/10/30/sharp-isoperimetric-bounds-for-convex-bodies/` (2013).

[Mo5] F. Morgan, Soap bubbles in $\mathbb{R}^2$ and in surfaces, Pacific J. Math. 165 (2) (1994), 347-361.

[S] P. Sternberg and K. Zumbrun, On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint, Comm. Anal. Geom. 7 (1) (1999), 199-220.