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Abstract. A *triquadratic number field* is a number field of degree 8 that is created by adjoining the square roots of three rational, squarefree integers to $\mathbb{Q}$. We often denote a number field by K and its *ring of integers* by $\mathcal{O}_K$. The ring if integers is defined to be the set of all elements of K that are zeros of a monic polynomial with coefficients in $\mathbb{Z}$. It is already well-known that the ring of integers for a quadratic field $\mathbb{Q}(\sqrt{a})$ is given by all integer linear combinations of either $\{1, \sqrt{a}\}$ or $\{1, (1+\sqrt{a})/2\}$ depending on the value of a. These sets are called an *integral basis* for the ring of integers of the number field $\mathbb{Q}(\sqrt{a})$. The integral bases for the rings of integers for biquadratic fields are also already known. In this paper, we determine the structure of the ring of integers for triquadratic fields. There are five different cases, depending on the value of the radicands modulo 4. We treat each of these five cases separately and give an explicit form for the integral basis in each case.

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1 Introduction

We study the integral bases of a number field because we want to understand the structure of the ring of integers better, and to have a concrete way to express these integers. The ring of integers of a field can help describe other properties of the field as well, such as how elements factor into irreducibles, what ideals of the ring of integers look like, and how the Euclidean algorithm is defined for the ring. In this paper, we look for the integral bases of a specific classification of number field: the multiquadratic number field.

A number field is a field that is a finite degree extension of the field of $\mathbb{Q}$. Thus, to form a number field, you begin with $\mathbb{Q}$ and "add in" some other real or complex numbers (which must be non-transcendental!). For example, $\mathbb{Q}\left(\sqrt{2}\right)$ is a number field, where every element is of the form $r + s\sqrt{2}$ with $r, s \in \mathbb{Q}$. Here, multiplication and addition behave the same way they would on any real numbers.

One component of a number field is its ring of integers. The ring of integers for a number field is the set of elements of the field which are zeros of a monic polynomial with integer coefficients. For example, in $\mathbb{Q}\left(\sqrt{2}\right)$, the element $\sqrt{2}$ is part of the ring of integers because $\sqrt{2}$ is a zero of the monic polynomial $x^2 - 2$.

Further, a ring of integers has an integral basis, which describes the entire ring of integers as integer linear combinations of these basis elements. An integral basis for $\mathbb{Q}\left(\sqrt{2}\right)$ is simply the set $\{1, \sqrt{2}\}$, so any element of the ring of integers can be written as $n \cdot 1 + m \cdot \sqrt{2}$ for $m, n \in \mathbb{Z}$.

An *n-quadratic number field*, $n \in \mathbb{N}$, is any number field K of degree 2^n over $\mathbb{Q}$ that is created by adjoining the square roots of rational, squarefree integers to $\mathbb{Q}$. That is, an n-quadratic number field K has the form $K = \mathbb{Q}\left(\sqrt{A_1}, \ldots, \sqrt{A_m}\right)$ for $m \geq n$ and $A_1, \ldots, A_m$ squarefree rational integers. If $n > 1$, the field is also known as a *multiquadratic number field*.

Ideally, we like to define n-quadratic number fields by adjoining exactly n square roots to $\mathbb{Q}$. Notice that $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ and $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}, \sqrt{6}\right)$ both represent the same 2-quadratic (that is, *biquadratic*) number field, but the first representation is written more concisely, and the presence of $\sqrt{6}$ in this field can be easily obtained by multiplying $\sqrt{2}$ and $\sqrt{3}$. Thus, in this paper, we will always express n-quadratic number fields by adjoining exactly n square roots to $\mathbb{Q}$.

This means that whenever we create an n-quadratic field $K = \mathbb{Q}\left(\sqrt{A_1}, \ldots, \sqrt{A_n}\right)$, we are assuming $A_1, \ldots, A_n \in \mathbb{Z}$ are squarefree, with the additional property that for any $I \subset \{1, \ldots, n\}$, we have that $sf\left(\prod_{i \in I} A_i\right)$ is not in the set $\{A_1, \ldots, A_n\}$ whenever I contains at least two elements. Here, the notation sf refers to the squarefree part of an integer. for example, $sf(20) = 5$.

The generalization of the integral bases of rings of integers of quadratic and biquadratic fields are well known, which are summarized here. If $A \equiv 1 \pmod 4$, then the integral basis of $\mathcal{O}_{\mathbb{Q}(\sqrt{A})}$ is $\{1, \frac{1+\sqrt{A}}{2}\}$. If $A \equiv 2, 3 \pmod 4$, then the integral basis for $\mathcal{O}_{\mathbb{Q}(\sqrt{A})}$ is $\{1, \sqrt{A}\}$. The integral bases for rings of integers for the different classifications of biquadratic fields are given by Kenneth S. Williams [2]. For a biquadratic field $\mathbb{Q}(\sqrt{A}, \sqrt{B})$, where $\gcd(A, B) = G$,

and $A_1 = A/G$, $B_1 = B/G$, Table 1 describes the integral bases for $\mathcal{O}_{\mathbb{Q}(\sqrt{A},\sqrt{B})}$.

Basis	**Condition**
$\left\{1, \frac{1}{2}(1+\sqrt{A}), \frac{1}{2}(1+\sqrt{B}), \frac{1}{2}(1+\sqrt{A}+\sqrt{B}+\sqrt{A_1A_1})\right\}$	$A, B, A_1, B_1 \equiv 1 \pmod 4$
$\left\{1, \frac{1}{2}(1+\sqrt{A}), \frac{1}{2}(1+\sqrt{B}), \frac{1}{4}(1-\sqrt{A}+\sqrt{B}+\sqrt{A_1B_1})\right\}$	$A, B \equiv 1 \pmod 4, A_1, B_1 \equiv 3 \pmod 4$
$\left\{1, \frac{1}{2}(1+\sqrt{A}), \sqrt{B}, \frac{1}{2}(\sqrt{B}+\sqrt{A_1B_1})\right\}$	$A \equiv 1 \pmod 4, B \equiv 2 \pmod 4$
$\left\{1, \sqrt{A}, \sqrt{B}, \frac{1}{2}(\sqrt{A}+\sqrt{A_1B_1})\right\}$	$A \equiv 2 \pmod 4,\ B \equiv 3 \pmod 4$
$\left\{1, \sqrt{A}, \frac{1}{2}(\sqrt{A}+\sqrt{B}), \frac{1}{2}(1+\sqrt{A_1B_1})\right\}$	$A, B \equiv 3 \pmod 4$

Table 1: The generalization of the integral bases of all biquadradic fields

The results in this paper are summarized and extended from the work of D. Chatelain [1]. This paper aims to make explicit and simplify these results in the case of triquadratic fields, to obtain bases that are easier to understand and apply. The integral bases of rings of integers for triquadratic fields fall into 3 general classifications as laid out by D. Chatelain [1].

We summarize our results here:

Let $A, B, C \in \mathbb{Z}$ be squarefree integers not equal to 0 or 1 such that $\mathbb{Q}\left(\sqrt{A}, \sqrt{B}, \sqrt{C}\right)$ is a triquadratic field. Define

$$\gamma := \frac{1}{8}\left(1+\sqrt{A}+\sqrt{B}+\sqrt{C}+\sqrt{sf(AB)}+\sqrt{sf(AC)}+\sqrt{sf(BC)}+\sqrt{sf(ABC)}\right).$$

Then, the integral bases of a triquadradic field of the form $\mathbb{Q}\left(\sqrt{A}, \sqrt{B}, \sqrt{C}\right)$ can be given as in Table 2.

Basis	**Condition**
A normal integral basis given by all 8 conjugates of γ	$A, B, C \equiv 1 \pmod 4$
$\left\{1, \sqrt{B}, \sqrt{C}, \frac{1+\sqrt{A}}{2}, \frac{1+\sqrt{B}+\sqrt{sf(AB)}}{2}, \frac{1+\sqrt{B}+\sqrt{sf(BC)}}{2}, \frac{\sqrt{C}+\sqrt{sf(AC)}+\sqrt{sf(BC)}}{2}, \frac{\sqrt{B}+\sqrt{sf(AB)}+\sqrt{sf(BC)}+\sqrt{sf(ABC)}}{2}\right\}$	$A \equiv 1, B \equiv 2, C \equiv 3 \pmod 4$
$\left\{1, \frac{1+\sqrt{A}}{4}, \frac{1+\sqrt{B}}{4}, \frac{1+\sqrt{sf(AB)}}{4}, \frac{1+\sqrt{C}+\sqrt{sf(AC)}}{2}, \frac{1+\sqrt{C}+\sqrt{sf(BC)}}{2}, \frac{\sqrt{sf(AB)+sf(AC)+sf(BC)+sf(ABC)}}{4}, \frac{1+\sqrt{A}+\sqrt{B}+\sqrt{sf(AB)}}{8}\right\}$	$A \equiv B \equiv 1 \pmod 4), C \equiv 2, 3 \pmod 4$

Table 2: The generalization of the integral bases of all triquadratic fields

Section 2 details our results. Each subsection outlines the generalization for a particular classification of triquadratic fields, based on the properties of its generating elements, $\sqrt{A}$, $\sqrt{B}$, and $\sqrt{C}$.

2 Integral Bases for Triquadratic Fields

Throughout this section we will take $A_i = \alpha_i^2$, $A_i \in \mathbb{Z}$ squarefree, $1 \le i \le 7$. Further, we will always assume $A_1, A_2, A_3 \equiv 1 \pmod 4$. When the context is clear, we will use the shorthand (A_i, A_j) in place of $\gcd(A_i, A_j)$, $1 \le i \le 7$.

We define α_i, $4 \le i \le 7$ as follows:

$$\alpha_4 := \frac{\alpha_1\alpha_2}{(A_1, A_2)}, \quad \alpha_5 := \frac{\alpha_1\alpha_3}{(A_1, A_3)},$$
$$\alpha_6 := \frac{\alpha_2\alpha_3}{(A_2, A_3)}, \quad \alpha_7 := \frac{\alpha_1\alpha_2\alpha_3}{(A_1, A_2)\cdot(A_3, A_4)}.$$

Since we are finding integral bases for all triquadratic fields, we need to consider different cases, depending on the forms of the radicands. The three cases presented in the remainder of this paper are sufficient to describe the integral bases for all triquadratic fields, because even if a field is not in one of the prescribed forms, it can be re-written to be in one of the forms described below.

2.1 An Integral Basis for $K = \mathbb{Q}\left(\alpha_1, \sqrt{2}\alpha_2, \sqrt{-1}\alpha_3\right)$

In this section, we give a simplified version of an integral basis for triquadratic fields that can be written in the form $K = \mathbb{Q}\left(\alpha_1, \sqrt{2}\alpha_2, \sqrt{-1}\alpha_3\right)$. In the original paper by Chatelain,

he addresses number fields of the form $L = \mathbb{Q}\left(\alpha_1, \sqrt{-2}\alpha_2, \sqrt{-1}\alpha_3\right)$ separately. However, L can be re-written to be in the same form as K, by replacing α_2 with α_6. We prove this in the following lemma:

Lemma 1. *Let L be a triquadratic field that is written $L = \mathbb{Q}\left(\nu_1, \sqrt{-2}\nu_2, \sqrt{-1}\nu_3\right)$ with $\nu_i^2 \equiv 1 \pmod 4$, $1 \le i \le 3$. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that $L = \mathbb{Q}\left(\alpha_1, \sqrt{2}\alpha_2, \sqrt{-1}\alpha_3\right)$ and $\alpha_i^2 \equiv 1 \pmod 4$ are integers, $1 \le i \le 3$.*

Proof. Define $N_1 := \nu_1^2$, $N_2 := \nu_2^2$ and $N_3 = \nu_3^2$. Define $\alpha_1 := \nu_1$, $\alpha_3 := \nu_3$ and $\alpha_2 := \dfrac{-\nu_2\nu_3}{(N_1, N_2)}$. Then, since $\sqrt{-2}\nu_2 \cdot \sqrt{-1}\nu_3 = \sqrt{2}(-\nu_2\nu_3)$, we clearly have that $L = \mathbb{Q}\left(\alpha_1, \sqrt{2}\alpha_2, \sqrt{-1}\alpha_3\right)$, and we just need to prove that $\alpha_2^2 \equiv 1 \pmod 4$ to prove the lemma.

We have that

$$(\alpha_2)^2 = \left(\frac{-\nu_2\nu_3}{(N_1, N_2)}\right)^2 = \frac{\nu_2^2\nu_3^2}{(N_2, N_3)^2} = \frac{N_2 N_3}{(N_2, N_3)^2}.$$

Clearly (N_2, N_3) divides both N_2 and N_3, so $N_2/(N_2, N_3)$ and $N_3/(N_2, N_3)$ are both odd integers. Further, they are congruent to each other modulo 4. Thus their product will be congruent to 1 modulo 4, so $\alpha_2^2 \equiv 1 \pmod 4$ as well. □

Proposition 1. *Take α_i, $1 \le i \le 7$ to be as defined above. Let K be any triquadratic number field that can be written in the form $\mathbb{Q}\left(\alpha_1, \sqrt{2}\alpha_2, \sqrt{-1}\alpha_3\right)$. Then an integral basis for $\mathcal{O}_K$ is*

$$\left\{1, \sqrt{2}\alpha_2, \sqrt{-1}\alpha_3, \frac{1}{2}(1+\alpha_1), \frac{\sqrt{2}}{2}(\alpha_2+\alpha_4), \frac{\sqrt{2}}{2}\left(\alpha_2+\sqrt{-1}\alpha_6\right),\right.$$
$$\left.\frac{\sqrt{-1}}{2}\left(\alpha_3+\alpha_5+\sqrt{2}\alpha_6\right), \frac{\sqrt{2}}{4}\left(\alpha_2+\alpha_4+\sqrt{-1}\alpha_6+\sqrt{-1}\alpha_7\right)\right\}.$$

Proof. Chatelain ([1], Theorem 10) shows us how to find the integral basis for fields of this form. We must first define 8 quantities, β_i for $0 \le i \le 7$:

$$\begin{array}{cccc} \beta_0 = 1 & \beta_1 = \alpha_1 & \beta_2 = \sqrt{2}\alpha_2 & \beta_3 = \sqrt{2}\alpha_4 \\ \beta_4 = \sqrt{-1}\alpha_3 & \beta_5 = \sqrt{-1}\alpha_5 & \beta_6 = \sqrt{-2}\alpha_6 & \beta_7 = \sqrt{-2}\alpha_7 \end{array}.$$

Using these beta terms, we can begin constructing the integral basis using Chatelain's construction. Four of the basis terms are defined directly by these beta terms, and the other four are given by their conjugates with respect to the field $\mathbb{Q}\left(\sqrt{2}\alpha_2, \sqrt{-1}\alpha_3\right)$. The four explicit terms are defined as follows:

$$\gamma_0 := \frac{1}{2}(\beta_0+\beta_1) = \frac{1}{2}(1+\alpha_1),$$

$$\gamma_1 := \frac{1}{2}(\beta_2+\beta_3) = \frac{1}{2}\left(\sqrt{2}\alpha_2+\sqrt{2}\alpha_4\right),$$

$$\gamma_2 := \frac{1}{2} \cdot \sum_{k=4}^{6} \beta_k = \frac{1}{2}\left(\sqrt{-1}\alpha_3 + \sqrt{-1}\alpha_5 + \sqrt{-2}\alpha_6\right),$$

$$\gamma_3 := \frac{1}{4}\left(\sum_{j\in\{2,3,6,7\}} \beta_j\right) = \frac{1}{4}\left(\sqrt{2}\alpha_2 + \sqrt{2}\alpha_4 + \sqrt{-2}\alpha_6 + \sqrt{-2}\alpha_7\right).$$

Taking the conjugates of these elements with respect to $\mathbb{Q}\left(\sqrt{2}\alpha_2, \sqrt{-1}\alpha_3\right)$, we get

$$\gamma_4 := \frac{1}{2}(1 - \alpha_1),$$

$$\gamma_5 := \frac{1}{2}\left(\sqrt{2}\alpha_2 - \sqrt{2}\alpha_4\right),$$

$$\gamma_6 := \frac{1}{2}\left(\sqrt{-1}\alpha_3 - \sqrt{-1}\alpha_5 + \sqrt{-2}\alpha_6\right),$$

$$\gamma_7 := \frac{1}{4}\left(\sqrt{2}\alpha_2 - \sqrt{2}\alpha_4 + \sqrt{-2}\alpha_6 - \sqrt{-2}\alpha_7\right).$$

Using Chatelain's results, ([1], Theorem 10) $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ is an integral basis for $\mathcal{O}_k$.

We wish to simplify this basis to a more compact form; and in the remainder of this proof we simplify the above integral basis to be the basis stated in this proposition. We begin by adding conjugates, and replacing certain basis elements with these sums. To do this define $b'_{2i} = \gamma_i + \gamma_{i+4}$ and $b'_{2i+1} = \gamma_{i+4}$ for $0 \le i \le 3$. In particular,

$$b'_0 = \gamma_0 + \gamma_4 = 1,$$

$$b'_1 = \gamma_0 = \frac{1+\alpha_1}{2},$$

$$b'_2 = \gamma_1 + \gamma_5 = \sqrt{2}\alpha_2,$$

$$b'_3 = \gamma_1 = \frac{1}{2}\left(\sqrt{2}\alpha_2 + \sqrt{2}\alpha_4\right),$$

$$b'_4 = \sqrt{-1}\alpha_3 + \sqrt{-2}\alpha_6,$$

$$b'_5 = \gamma_2 = \frac{1}{2}\left(\sqrt{-1}\alpha_3 + \sqrt{-1}\alpha_5 + \sqrt{-2}\alpha_6\right),$$

$$b'_6 = \gamma_3 + \gamma_7 = \frac{1}{2}\left(\sqrt{2}\alpha_2 + \sqrt{-2}\alpha_6\right),$$

$$b'_7 = \gamma_3 = \frac{1}{4}\left(\sqrt{2}\alpha_2 + \sqrt{2}\alpha_4 + \sqrt{-2}\alpha_6 + \sqrt{-2}\alpha_7\right).$$

For $i \in \{0,1,2,3,5,6,7\}$, let $b_i := b'_i$ and let

$$\begin{aligned} b_4 &:= b'_4 - 2b'_6 + b'_2 \\ &= \sqrt{-1}\alpha_3 + \sqrt{-2}\alpha_6 - \sqrt{2}\alpha_2 - \sqrt{-2}\alpha_6 + \sqrt{2}\alpha_2 \\ &= \sqrt{-1}\alpha_3. \end{aligned}$$

Then $\{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ is an integral basis for $\mathcal{O}_K$, thus establishing the proposition. $\square$

Example: Let $K = \mathbb{Q}\left(\sqrt{-15}, \sqrt{-6}, \sqrt{7}\right) = \mathbb{Q}\left(\sqrt{-15}, \sqrt{2}\sqrt{-3}, \sqrt{-1}\sqrt{-7}\right)$. Then, $\alpha_1 = \sqrt{-15}$, $\alpha_2 = \sqrt{-3}$, and $\alpha_3 = \sqrt{-7}$. Using the definitions above, an integral basis for $\mathcal{O}_K$ is

$$\left\{1, \sqrt{-6}, \sqrt{7}, \frac{1}{2}\left(1+\sqrt{15}\right), \frac{1}{2}\left(\sqrt{-6}+\sqrt{10}\right), \frac{1}{2}\left(\sqrt{-6}+\sqrt{-42}\right), \right.$$
$$\left. \frac{1}{2}\left(\sqrt{7}+\sqrt{42}+\sqrt{-105}\right), \frac{1}{4}\left(\sqrt{-6}+\sqrt{10}+\sqrt{-42}+\sqrt{70}\right)\right\}.$$

2.2 An Integral Basis for $K = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$

Here we will calculate the general integral basis for the ring of integers of triquadratic fields of the form $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$, where $(\alpha_1)^2$, $(\alpha_2)^2$, and $(\alpha_3)^2$ are congruent to $1 \pmod 4$.

Proposition 2. *Take α_i, $1 \le i \le 7$ to be as defined above. Let K be any triquadratic number field that can be written in the form $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Then an integral basis for $\mathcal{O}_K$ is*

$$\left\{1, \frac{1}{2}(1+\alpha_1), \frac{1}{2}(1+\alpha_2), \frac{1}{2}(1+\alpha_3), \frac{1}{4}(1+\alpha_2+\alpha_3+\alpha_6), \frac{1}{4}(1+\alpha_1+\alpha_3+\alpha_5), \right.$$
$$\left. \frac{1}{4}(1+\alpha_1+\alpha_2+\alpha_4), \frac{1}{8}(1+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7)\right\}.$$

Proof. Theorem 9b of [1] gives us the integral basis of the field. A normal integral basis is guaranteed for these fields. The basis elements of the normal basis are given by the conjugates of

$$\gamma_0 = \frac{1}{8} + \sum_{i=1}^{7} \frac{\alpha_i}{8} = \frac{1}{8}\left(1+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\right)$$

with respect to $\mathbb{Q}$. Its conjugates are:

$$\gamma_1 = \frac{1}{8}\left(1-\alpha_1+\alpha_2+\alpha_3-\alpha_4-\alpha_5+\alpha_6-\alpha_7\right),$$

$$\gamma_2 = \frac{1}{8}\left(1+\alpha_1-\alpha_2+\alpha_3-\alpha_4+\alpha_5-\alpha_6-\alpha_7\right),$$

$$\gamma_3 = \frac{1}{8}\left(1+\alpha_1+\alpha_2-\alpha_3+\alpha_4-\alpha_5-\alpha_6-\alpha_7\right),$$

$$\gamma_4 = \frac{1}{8}\left(1-\alpha_1-\alpha_2+\alpha_3+\alpha_4-\alpha_5-\alpha_6+\alpha_7\right),$$

$$\gamma_5 = \frac{1}{8}\left(1-\alpha_1+\alpha_2-\alpha_3-\alpha_4+\alpha_5-\alpha_6+\alpha_7\right),$$

$$\gamma_6 = \frac{1}{8}\left(1+\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5+\alpha_6+\alpha_7\right),$$

$$\gamma_7 = \frac{1}{8}\left(1-\alpha_1-\alpha_2-\alpha_3+\alpha_4+\alpha_5+\alpha_6-\alpha_7\right).$$

Then, $\{\gamma_0,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7\}$ forms a normal integral basis for $\mathcal{O}_K$. We can further simplify this basis. Define b_i, $0 \le i \le 7$ as follows:

$$b_0 = \sum_{i=0}^{7} \gamma_i = 1$$

$$b_1 = \gamma_0+\gamma_2+\gamma_3+\gamma_6 = \frac{1}{2}(1+\alpha_1),$$

$$b_2 = \gamma_0+\gamma_1+\gamma_3+\gamma_5 = \frac{1}{2}(1+\alpha_2),$$

$$b_3 = \gamma_0+\gamma_1+\gamma_2+\gamma_4 = \frac{1}{2}(1+\alpha_3),$$

$$b_4 = \gamma_0+\gamma_1 = \frac{1}{4}(1+\alpha_2+\alpha_3+\alpha_6),$$

$$b_5 = \gamma_0+\gamma_2 = \frac{1}{4}(1+\alpha_1+\alpha_3+\alpha_5),$$

$$b_6 = \gamma_0+\gamma_3 = \frac{1}{4}(1+\alpha_1+\alpha_2+\alpha_4),$$

$$b_7 = \gamma_0 = \frac{1}{8}(1+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7).$$

Then $\{b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7\}$ forms an integral basis for $\mathcal{O}_K$. $\square$

Example: Let $K = \mathbb{Q}\left(\sqrt{5},\sqrt{13},\sqrt{-3}\right)$. Then $\alpha_1 = \sqrt{5}$, $\alpha_2 = \sqrt{13}$, and $\alpha_3 = \sqrt{-3}$. Using the definitions above, an integral basis for $\mathcal{O}_K$ is

$$\left\{1, \frac{1}{2}(1+\sqrt{-3}), \frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(1+\sqrt{13}), \frac{1}{4}(1+\sqrt{-3}+\sqrt{13}+\sqrt{-39}), \frac{1}{4}(1+\sqrt{-3}+\sqrt{5}+\sqrt{-15}),\right.$$
$$\left.\frac{1}{4}(1+\sqrt{5}+\sqrt{13}+\sqrt{65}), \frac{1}{8}(1+\sqrt{-3}+\sqrt{5}+\sqrt{13}+\sqrt{-15}+\sqrt{-39}+\sqrt{65}+\sqrt{-195})\right\}.$$

2.3 An Integral Basis for $K = \mathbb{Q}(\alpha_1, \alpha_2, \delta\alpha_3)$ for $\delta \in \{\sqrt{2}, \sqrt{-1}\}$

Here we will calculate the integral basis for the ring of integers of triquadratic fields of the form $\mathbb{Q}\left(\alpha_1, \alpha_2, \sqrt{2}\alpha_3\right)$ or the form $\mathbb{Q}\left(\alpha_1, \alpha_2, \sqrt{-1}\alpha_3\right)$, where $(\alpha_1)^2$, $(\alpha_2)^2$, and $(\alpha_3)^2$ are congruent to 1 (mod 4).

Proposition 3. *Take α_i, $1 \leq 7$ to be as defined above. Let K be any triquadratic number field that can be written in the form $\mathbb{Q}(\alpha_1, \alpha_2, \delta\alpha_3)$ for $\delta \in \{\sqrt{2}, \sqrt{-1}\}$. Then an integral basis for $\mathcal{O}_K$ is*

$$\left\{1, \frac{1}{4}(1+\alpha_1), \frac{1}{4}(1+\alpha_2), \frac{1}{4}(1+\alpha_4), \frac{1}{2}(\alpha_3+\delta\alpha_5),\right.$$
$$\left.\frac{1}{2}(\alpha_3+\delta\alpha_6), \frac{1}{4}(\alpha_4+\delta\alpha_5+\delta\alpha_6+\delta\alpha_7), \frac{1}{8}(1+\alpha_1+\alpha_2+\alpha_4)\right\}.$$

Proof. Theorem 11 of [1] shows us how to find an integral basis for $\mathcal{O}_K$. First, we find beta terms:

$$\beta_0 = 1,$$
$$\beta_1 = \alpha_1,$$
$$\beta_2 = \alpha_2,$$
$$\beta_3 = \alpha_4,$$
$$\beta_4 = \delta\alpha_3,$$
$$\beta_5 = \delta\alpha_5,$$
$$\beta_6 = \delta\alpha_6,$$
$$\beta_7 = \delta\alpha_7.$$

To find the integral basis, we calculate γ_0 and γ_1:

$$\gamma_0 = \sum_{i=0}^{3} \frac{\beta_i}{8} = \frac{1}{8}\left(1+\alpha_1+\alpha_2+\alpha_4\right),$$

$$\gamma_1 = \sum_{j=4}^{7} \frac{\beta_i}{4} = \frac{1}{4}\left(\alpha_3+\delta\alpha_5+\delta\alpha_6+\delta\alpha_7\right).$$

These two elements with their conjugates with respect to $\mathbb{Q}(\delta\alpha_3)$ form the integral basis of the ring of integers of fields of this form. The conjugates of these elements with respect to $\mathbb{Q}(\delta\alpha_3)$ are:

$$\gamma_2 := \frac{1}{8}\left(1-\alpha_1+\alpha_2-\alpha_4\right),$$
$$\gamma_3 := \frac{1}{8}\left(1+\alpha_1-\alpha_2-\alpha_4\right),$$

$$\gamma_4 := \frac{1}{8}\left(1 - \alpha_1 - \alpha_2 + \alpha_4\right),$$
$$\gamma_5 := \frac{1}{4}\left(\alpha_3 - \delta\alpha_5 + \delta\alpha_6 - \delta\alpha_7\right),$$
$$\gamma_6 := \frac{1}{4}\left(\alpha_3 + \delta\alpha_5 - \delta\alpha_6 - \delta\alpha_7\right),$$
$$\gamma_7 := \frac{1}{4}\left(\alpha_3 - \delta\alpha_5 - \delta\alpha_6 + \delta\alpha_7\right).$$

Then, $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ forms an integral basis for $\mathcal{O}_K$. We can further simplify this basis. For b_i, $0 \leq i \leq 7$, let

$$b_0 := \gamma_1 + \gamma_5 + \gamma_6 + \gamma_7 = 1,$$
$$b_1 := \gamma_1 + \gamma_5 = \frac{1}{2}\left(\alpha_3 + \delta\alpha_6\right),$$
$$b_2 := \gamma_1 + \gamma_6 = \frac{1}{2}\left(\alpha_3 + \delta\alpha_5\right),$$
$$b_3 := \gamma_0 + \gamma_3 = \frac{1}{4}\left(1 + \alpha_1\right),$$
$$b_4 := \gamma_0 + \gamma_2 = \frac{1}{4}\left(1 + \alpha_2\right),$$
$$b_5 := \gamma_0 + \gamma_4 = \frac{1}{4}\left(1 + \alpha_4\right),$$
$$b_6 := \gamma_0 = \frac{1}{8}\left(1 + \alpha_1 + \alpha_2 + \alpha_4\right),$$
$$b_7 := \gamma_1 = \frac{1}{4}\left(\alpha_3 + \delta\alpha_5 + \delta\alpha_6 + \delta\alpha_7\right).$$

Then, $\{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ forms an integral basis for $\mathcal{O}_K$. $\square$

Example 1: Let $K = \mathbb{Q}\left(\sqrt{-3}, \sqrt{-7}, \sqrt{26}\right) = \mathbb{Q}\left(\sqrt{-3}, \sqrt{-7}, \sqrt{2}\sqrt{13}\right)$. Then, $\alpha_1 = \sqrt{-3}$, $\alpha_2 = \sqrt{-7}$, $\alpha_3 = \sqrt{13}$, and $\delta = \sqrt{2}$. Using the definitions above, an integral basis for $\mathcal{O}_K$ is

$$\left\{1, \frac{1}{2}(\sqrt{13} + \sqrt{-78}), \frac{1}{2}(\sqrt{13} + \sqrt{-182}), \frac{1}{4}(1 + \sqrt{-3}), \frac{1}{4}(1 + \sqrt{-7}),\right.$$
$$\left.\frac{1}{4}(1 + \sqrt{21}), \frac{1}{4}(\sqrt{13} + \sqrt{-78} + \sqrt{-182} + \sqrt{546}), \frac{1}{8}(1 + \sqrt{-3} + \sqrt{-7} + \sqrt{21})\right\}.$$

Example 2: Let $K = \mathbb{Q}\left(\sqrt{-7}, \sqrt{-15}, \sqrt{-13}\right) = \mathbb{Q}\left(\sqrt{-7}, \sqrt{-15}, \sqrt{-1}\sqrt{13}\right)$. Then, $\alpha_1 = \sqrt{-7}$, $\alpha_2 = \sqrt{-15}$, $\alpha_3 = \sqrt{13}$, and $\delta = \sqrt{-1}$. By the definitions above, an integral basis for $\mathcal{O}_K$ is

$$\left\{1, \frac{1}{2}\left(\sqrt{13}+\sqrt{91}\right), \frac{1}{2}\left(\sqrt{13}+\sqrt{195}\right), \frac{1}{4}\left(1+\sqrt{-7}\right), \frac{1}{4}\left(1+\sqrt{-15}\right),\right.$$
$$\left.\frac{1}{4}\left(1+\sqrt{105}\right), \frac{1}{4}\left(\sqrt{13}+\sqrt{91}+\sqrt{195}+\sqrt{-1365}\right), \frac{1}{8}\left(1+\sqrt{-7}+\sqrt{-15}+\sqrt{105}\right)\right\}.$$

References

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