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an appreciation of euler's formula

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Abstract. For many mathematicians, a certain characteristic about an area of mathematics will lure him/her to study that area further. That characteristic might be an interesting conclusion, an intricate implication, or an appreciation of the impact that the area has upon mathematics. The particular area that we will be exploring is Euler's Formula, $e^{ix} = \cos x + i \sin x$, and as a result, Euler's Identity, $e^{i\pi} + 1 = 0$. Throughout this paper, we will develop an appreciation for Euler's Formula as it combines the seemingly unrelated exponential functions, imaginary numbers, and trigonometric functions into a single formula. To appreciate and further understand Euler's Formula, we will give attention to the individual aspects of the formula, and develop the necessary tools to prove it. We will also try to gain a small understanding of the impact that it has had on mathematics.

1 Introduction

Leonhard Euler (1703-1783) is commonly regarded as one of the greatest mathematicians of all time, and the content of this paper stems largely from his work [1]. Euler was born in Switzerland during a time that professorships were scarce, so he quickly made his way to Russia in hope of career progression. In Russia, at the age of 20, he became the chief mathematician of the Academy of St. Petersburg. After 15 years of work in Russia, Euler moved to Germany to direct the Berlin Academy. He hoped to find freedom in Germany from the political oppression that was hindering his work. While in Germany, Euler remained in correspondence with the Academy of St. Petersburg, which would prove beneficial later in his career. After 25 years in Germany, this correspondence resulted in another job at the Academy of St. Petersburg. It was here where he eventually lost his sight, a fact that didn't slow him down in his publishing. By the end of his life, he had published over 700 works that covered topics in a range of mathematical fields including complex analysis, graph theory, number theory, and calculus [1].

In the book *Introductio in Analysin Infinitorum* published in 1748, Euler gives us a lot of the notation crucial to mathematics. Before this publication, there had been varying uses of notation for concepts such as series, trigonometric functions, exponential functions, logarithmic functions, continued fractions, and number theory. After this work was published, the notation Euler used became generally accepted. It is from this book that we gain the notation π to denote the ratio of a circle's circumference to its diameter, e to denote the base of the natural logarithm, the modern notation for the trigonometric functions, as well as the expression of trigonometric functions as ratios rather than lengths. This book also gives us Euler's Identity ($e^{i\pi} + 1 = 0$) stemming from Euler's Formula ($e^{ix} = \cos x + i \sin x$), which is itself presented in *Introductio in Analysin Infinitorum* [1]. This brief acknowledgement of Leonhard Euler's work does not do justice to the influence that he had on mathematics, but an understanding of the historical background of the formula will help motivate our work. In this paper, we will attempt to develop a further appreciation of the impact that Euler's Formula left on mathematics.

In Section 2, we give a brief introduction to complex numbers in order to understand some of the inner parts to Euler's Formula. Section 3 outlines another type of mathematical construct found in Euler's Formula, trigonometric and exponential functions. In this section we will provide series definitions for the main trigonometric and exponential functions that appear in Euler's Formula, and develop the framework used to prove the formula. In Section 4 we will prove Euler's Formula, and following, in Section 5, we will provide some applications that make use of the formula. Finally, in Section 6, we make some closing remarks regarding Euler's Formula.

2 Complex Numbers

Complex numbers play an important role in Euler's Formula, so some background about the imaginary unit number i is in order. The important property of i is that it satisfies

$i^2 = -1$, giving us the ability to solve equations such as $x^2 + 1 = 0$. The implementation of this imaginary unit allows us to move from the real numbers, denoted $\mathbb{R}$, to the complex numbers, denoted $\mathbb{C}$. Complex numbers are of the form $a + bi$ where a is called the real part and b is called the imaginary part [5]. The relationship between the real numbers and the complex numbers will come into play later, so we now note that $\mathbb{R} \subset \mathbb{C}$ and that we can write any $x \in \mathbb{R}$ as a complex number by writing $x + 0i$.

Now we define complex numbers formally. We will first look at complex numbers through the lens of an ordered pair. So, we define a *complex number* as an ordered pair of real numbers (a, b) such that:

I. For complex numbers (a, b) and (c, d), $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
II. Multiplication and Addition of complex numbers are defined as follows [3]:

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \cdot (c, d) = (ac - bd, bc + ad).$$

We will denote the complex number i as $i = (0, 1)$. Commonly we see complex numbers written as $a + bi$. By definition, any complex number $z = (a, b)$ can be represented uniquely as $z = a + bi$ where a, b are real numbers and the relation $i^2 = -1$ holds [3]. Let us look back at our definition of $i = (0, 1)$. If we now consider our relation $i^2 = -1$, we see $(0, 1)^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1$, using our definition of multiplication for complex numbers. We see our way of representing a complex number is justified because $(a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1) \cdot (b, 0) = a + bi$. Now using this, we may write the definition for the complex numbers in general, being $\mathbb{C} = \{a + bi | a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1\}$ [2].

Another way to view complex numbers is through the lens of polar coordinates. We see a point, as described before, $P = (a, b) = a + bi$. If we plot the point on the complex plane (look at a coordinate plane where the x-axis refers to points $(x, 0)$ and the y-axis refers to points $(0, y)$, or in a more elementary way, the x-axis is the real axis and the y-axis is the imaginary axis), we can look at the vector $\vec{OP}$ where O is the origin and P is our original point. Let θ be the angle between $\vec{OP}$ and the x-axis, so our polar representation of (a, b) looks like $(r \cos \theta, r \sin \theta)$. We then see that [2]

$$(a, b) = (r \cos \theta, r \sin \theta) = r \cos \theta + ir \sin \theta.$$

We notice here that there is a strong connection between this polar representation of complex numbers and the unit circle, where points on the unit circle are $(\cos \theta, \sin \theta)$. Euler's Formula will provide insight in how to view concepts such as multiplying by i.

3 Trigonometry and Exponentiation

A conventional approach to proving Euler's Formula is to do so in terms of infinite series, a topical favorite of Leonhard Euler. Appropriately, that is how we will prove the formula, but in order to do so, we need to secure the series representation of the sine, cosine, and exponential functions that occur in Euler's Formula. In a general sense, we know that a Taylor series for a function can be built by considering where we want our series centered and by observing the derivatives. The Taylor Series for a function $f(x)$, with derivatives of all orders existing (and being continuous) on an interval containing c, is defined to be [12]

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

For our application, we are interested in the Maclaurin series for our functions, which is a Taylor series centered at 0. We observe the fact that for $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, and so on. The reader may want to take the time to understand why $\frac{d}{dx}e^x = e^x$, and a discussion on this can be found on page 342 of the text by Rogawski [10]. Thus, $f^{(n)}(x) = e^x$ for any n. Also, we see that for $g(x) = \sin x$ and $h(x) = \cos x$, the derivatives become cyclic. It will suffice to show the cyclic nature of the derivatives of sine, as it is very similar for cosine. For $g(x) = \sin x$, $g'(x) = \cos x$, $g''(x) = -\sin x$, $g'''(x) = -\cos x$, and $g^{(4)}(x) = \sin x$, where the pattern begins to repeat. Hence, for $f(x) = e^x$, $g(x) = \sin x$, and $h(x) = \cos x$, the derivatives of all orders exist and are continuous on an interval containing $c = 0$. So, when building our series (in this case, f, g, and h are all **equal** to their series representation, which is proved using the Taylor Inequality, see page 781 of Stewart [11]), we see that for $x \in \mathbb{R}$ [7]

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$$

and [8]

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

It is worth mentioning how similar the series are to each other. Just as we view the graphs of sine and cosine and see a similar pattern, we see a pattern appear here in the series, and we may even see a connection between the exponential and the trigonometric functions from this preliminary look at their series. It's also once again worth mentioning the amount of work that Leonhard Euler did in this field. The concept of infinite series was relatively new to mathematicians during the time of Euler, with the pioneering work being done in the early 18th century by mathematicians Brook Taylor and James Gregory [5]. Having a great interest in their work, Euler used these ideas often to justify his discoveries. For instance, he used the concepts of sine and cosine as series as well as the representation of sine and cosine as ratios to justify his formula.

The complication in understanding arises when we dare to put i in the exponent, as is the case in Euler's Formula. In order to formally understand what the complex input would yield in these cases, we define for $z \in \mathbb{C}$, $F(z) = e^z$, $G(z) = \sin z$, and $H(z) = \cos z$ as

$$F(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1}$$

$$G(z) = \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \tag{2}$$

$$H(z) = \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \tag{3}.$$

To ensure that our functions are well-defined, we need to check that the series converge over the whole complex plane. To do this, we will use the ratio test, which holds in $\mathbb{C}$ the same way it holds in $\mathbb{R}$. For $F(z) = e^z$, we see if $z = 0$, $F(0)$ is defined by properties of the exponential function. Hence, fix a non-zero value $z_0 \in \mathbb{C}$. Then, let $a_n = \frac{{z_0}^n}{n!}$. It follows [13]

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{{z_0}^n}{n!}}{\frac{{z_0}^{n+1}}{(n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{(z_0)n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{z_0} \right| = \infty.$$

Since $z_0 \in \mathbb{C}$ was arbitrary, $F(z) = e^z$ converges for all $z \in \mathbb{C}$.

For $G(z) = \sin z$ we see if $z = 0$, $G(0)$ is defined by properties of the sine function. Hence, fix a non-zero value $z_1 \in \mathbb{C}$. Then, let $a_n = \frac{z_1^{2n+1}}{(2n+1)!}$. It follows

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{z_1^{2n+1}}{(2n+1)!} \cdot \frac{(2n+3)!}{z_1^{2n+3}} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)(2n+3)}{z_1^2} \right| = \infty.$$

Since $z_1 \in \mathbb{C}$ was arbitrary, $G(z) = \sin z$ converges for all $z \in \mathbb{C}$.

For $H(z) = \cos z$, we see if $z = 0$, $H(0)$ is defined by properties of the cosine function. Hence, fix a non-zero value $z_2 \in \mathbb{C}$. Then, let $a_n = \frac{z_2^{2n}}{(2n)!}$. It follows

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{z_2^{2n}}{(2n)!} \cdot \frac{(2n+2)!}{z_2^{2n+2}} \right| = \lim_{n\to\infty} \left| \frac{(2n+1)(2n+2)}{z_2^2} \right| = \infty.$$

Since $z_2 \in \mathbb{C}$ was arbitrary, $H(z) = \cos z$ converges for all $z \in \mathbb{C}$.

Our three functions converge for all values in the complex plane, so the definitions are well-defined.

4 Proofs

Proofs are mathematical tools that answer the question of "why?" for a lot of mathematical concepts. Proofs may be intensive, requiring clever methods to prove an understood concept, or nearly trivial, justifying a fact that is almost obvious. In order to fully appreciate the impact of Euler's Formula and Euler's Identity, we claimed that we need to fully understand it. In typical mathematical fashion, this complete understanding will come at the mercy of proof.

As previously stated, we will prove Euler's Formula using the series (1), (2), and (3) from the previous section.

Proof of Euler's Formula. We wish to show that $e^{ix} = \cos x + i \sin x$.

Consider the case where $z = ix$ [6]. Then, by substituting ix in for z in series (1) [8],

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + ... \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + ... \\ &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...) \\ &= \cos x + i \sin x. \end{aligned}$$

□

This proof relies on computing the powers of i, which follows easily from the definition of i and i^2. The simplicity of this proof may be one of its greatest displays of beauty. At a superficial glance, the proof seems almost elementary, but as we've noticed, there is a lot more to this formula than meets the eye.

Now that we have proved Euler's Formula, we can derive Euler's Identity by substituting $x = \pi$. Consider Euler's Formula

$$e^{ix} = \cos x + i \sin x.$$

Choose $x = \pi$. We see that

$$e^{i\pi} = \cos \pi + i \sin \pi.$$

Since $\sin \pi = 0$ and $\cos \pi = -1$,

$$e^{i\pi} = -1.$$

This is one way to write Euler's Identity, but it is more conventionally written [9]

$$e^{i\pi} + 1 = 0.$$

Now we see that Euler's Identity is simply one case of Euler's Formula, so we must ask ourselves why Euler's Identity is seen as one of the most beautiful equations in mathematics, when there are many other cases we could choose to look at.

One of the reasons Euler's Identity is so heavily appreciated for its beauty is it relates so many important mathematical concepts using one simple equation. For one, it is aesthetically pleasing. We see five numbers (e, i, π, 0, and 1) that we are familiar with as well as three simple operations (exponentiation, multiplication, and addition) that we use frequently in mathematics. Euler's Identity takes many concepts that are seemingly unrelated and puts them together in one equation that we can make sense of.

5 Applications

Euler's Formula has some interesting applications within mathematics. The first application we'll consider is the verification of the addition formulas for the sine and cosine functions. Consider the addition formulas for sine and cosine,

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

and

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Using Euler's Formula, we look at $e^{i(x+y)}$. We see that

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

and

$$e^{i(x+y)} = e^{ix}e^{iy}.$$

Now,

$$\begin{aligned}
e^{ix}e^{iy} &= (\cos x + i\sin x)(\cos y + i\sin y) \\
&= \cos x\cos y + i\sin x\cos y + i\sin y\cos x + i^2\sin x\sin y, \\
&= \cos x\cos y - \sin x\sin y + i\sin x\cos y + i\sin y\cos x, \\
&= \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \sin y\cos x).
\end{aligned}$$

Thus,

$$\cos(x+y) + i\sin(x+y) = \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \sin y\cos x).$$

Equating the real and imaginary parts yields the addition formulas as desired [9].

Another application Euler's Formula lends itself to is rewriting trigonometric functions in terms of complex exponential functions to make them more feasible to work with. To this end, consider the system of equations built from Euler's Formula:

$$\begin{aligned}
e^{ix} &= \cos x + i\sin x \\
e^{-ix} &= \cos -x + i\sin -x = \cos x - i\sin x.
\end{aligned}$$

Here, we see if we solve for $\cos x$ using elementary operations we have $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, and if we solve for $\sin x$ in a similar way we have $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$.

To justify the statement that writing the sine functions as a combination of complex exponential functions is an application, consider the following integration technique.

Application: Find a closed-form formula for

$$\int_0^{2\pi} \sin^n x dx.$$

Solution: The binomial theorem tells us [11]

$$(e^{ix} - e^{-ix})^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j e^{-ijx} \cdot e^{(n-j)ix} = \sum_{j=0}^{n} \binom{n}{j} (-1)^j e^{(n-2j)ix}.$$

Thus, from our sine identity derived above and the binomial theorem, we have

$$\begin{aligned}\int_0^{2\pi} \sin^n x dx &= \frac{1}{(2i)^n} \int_0^{2\pi} (e^{ix} - e^{-ix})^n dx \\ &= \frac{1}{(2i)^n} \int_0^{2\pi} \sum_{j=0}^{n} \binom{n}{j} (-1)^j e^{(n-2j)ix} dx \\ &= \frac{1}{(2i)^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \int_0^{2\pi} e^{(n-2j)ix} dx.\end{aligned}$$

We note here that if m is any integer, $e^{2\pi mi} = \cos 2\pi m + i \sin 2\pi m = 1$. Then, using antidifferentiation, which holds in $\mathbb{C}$ as it does in $\mathbb{R}$, we have

$$\int_0^{2\pi} e^{imx} dx = \begin{cases} 2\pi & m = 0 \\ 0 & \text{otherwise} \end{cases}$$

In this case, if n is odd, there exists no index j for which $n - 2j = 0$ in our above formula. However, if n is even, the index $j = \frac{n}{2}$ will satisfy this. In this case,

$$\begin{aligned}\int_0^{2\pi} \sin^n x dx &= \frac{1}{(2i)^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \int_0^{2\pi} e^{(n-2j)ix} dx \\ &= \frac{1}{2^n} \cdot \binom{n}{\frac{n}{2}} (2\pi).\end{aligned}$$

Hence,

$$\int_0^{2\pi} \sin^n x dx = \begin{cases} \binom{n}{\frac{n}{2}} \cdot \frac{\pi}{2^{n-1}} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

So, as a result, instead of using a typical integration by parts approach (requiring induction), we can substitute our definition of sine as given by Euler's Formula and use that to evaluate a closed-form for the integral.

6 Conclusion and Closing Remarks

When we look at Euler's Identity $e^{i\pi} + 1 = 0$, we see a way to relate several of important mathematical concepts stemming from a specific case of Euler's Formula. Superficially, we see an equation relating the numbers e, π, i, 0, and 1, which are all extremely important mathematical concepts. Similarly, Euler's Formula surprisingly relates exponential functions to trigonometric functions. We proved this using Taylor series, which breaks down Euler's Formula into infinitely many terms and brings it all back together to justify our formula. The applications that we discussed also elude to the impact of Euler's Formula.

If the reader seeks further work in this area, the reader could try to use Euler's Formula to prove the subtraction formulas for sine and cosine, or prove that

$$\cos x \cos y = \frac{\cos(x+y) + \cos(x-y)}{2}$$

using the definition of cosine. These can be proved in a simple manner involving Euler's Formula.

Perhaps our appreciation of Euler's Formula and Euler's Identity may best be illustrated by a quote. Benjamin Peirce, a Harvard mathematician in the 19th century, said of Euler's Identity "Gentlemen, that is surely true, it is absolutely paradoxical, we can't understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth" [4].

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