Super-walk Formulae for Even and Odd Laplacians in Finite Graphs

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Abstract. The number of walks from one vertex to another in a finite graph can be counted by the adjacency matrix. In this paper, we prove two theorems that connect the graph Laplacian with two types of walks in a graph. By defining two types of walks and giving orientation to a finite graph, one can easily count the number of the total signs of each kind of walk from one element to another of a fixed length.

1   Introduction

In graph theory and combinatorics, an interesting problem is that of finding the number of different ways a certain operation can be performed on a given graph. For example, given two vertices v_i and v_j, a natural question is: How many different walks of length k connect v_i and v_j? The answer to this question has interesting applications to random walks, spectral graph theory, and surprisingly, discrete versions of quantum mechanics, in which the number of such walks appears in the partition function for graph quantum mechanics [3]. Such physical interpretation can be adapted to the case of super-symmetric quantum mechanics, for which both edges and vertices span the space of quantum states of the system.

The particular aim of this paper is to study generalized walks associated to a finite graph that we call **super-walks** and **edge super-walks**. Super-walks differ from conventional walks between vertices by the fact that staying in a vertex after choosing an edge is allowed, and that such walks come equipped with a natural orientation. Edge super-walks are in some way dual to super-walks, in the sense that now the walks are considered to be between edges.

While powers of the adjacency matrix of a graph can be used to determine the number of conventional walks between vertices, we will show that another matrix associated with a graph, the even Laplacian Δ^+, provides a counting formula for super-walks (Theorem 3.3). Similar expressions were proved via Feynman-type expansion of the graph Laplacian in the case of regular graphs by Mnëv [3, 4], and for general finite graphs by del Vecchio [2]. We provide a combinatorial argument for such formulae, using induction on the length of the super-walks.

Edge super-walks appear naturally while studying the super-symmetric version of graph quantum mechanics [1, 2, 3, 4], for which the walks are considered to be between edges rather than vertices. Theorem 4.5 provides a counting formula for edge super-walks in terms of the odd Laplacian Δ^-. It turns out that the number of such walks depends on the orientation of the graph, and in Section 4 we will explain how.

This paper is organized as follows. In Section 2, we introduce the different notions of walks and some examples. In Section 3, we introduce super-walks for non-oriented graphs, and we prove the corresponding counting formula (Theorem 3.3). In Section 4, we introduce edge super-walks as a super-symmetric version of conventional walks, and we determine their counting formula (Theorem 4.5). In the last section, we give an outlook of further directions and generalizations.

2   Walks

We start by recalling the usual notion of walk in a finite graph. In this section, Theorem 2.3 shows that the number of walks from vertices to vertices of fixed length can be counted by the corresponding adjacency matrix and its powers.

A *graph* Γ is an ordered pair (V, E) comprising a set V of vertices together with a set E of edges, which are 2-element subsets of V (that is, an edge is associated with two vertices,

and the association takes the form of the unordered pair of the vertices). If the set V and the set E are finite sets, we say Γ is a *finite graph*. A finite graph Γ is often indicated by the *adjacency matrix*, which is defined by

$$A_\Gamma := A(i, j) = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j \\ 0 & \text{otherwise} \end{cases}.$$

The *incidence matrix* I, the *even graph Laplacian* Δ_Γ^+, and the *odd graph Laplacian* Δ_Γ^- are also used to describe a graph. A graph is *oriented* if a starting and ending vertex are specified for each edge. Let Γ be an oriented graph, with vertices $v_1, v_2, \ldots, v_{|V|}$, and edges $e_1, e_2, \ldots, e_{|E|}$. The *incidence matrix* I of Γ is a $|V| \times |E|$ matrix given by

$$I(i, j) = \begin{cases} 1 & \text{if } e_j \text{ ends at } v_i \\ -1 & \text{if } e_j \text{ starts at } v_i \\ 0 & \text{otherwise} \end{cases}.$$

The even graph Laplacian is defined by $\Delta_\Gamma^+ := II^t$. The odd graph Laplacian is defined by $\Delta_\Gamma^- := I^t I$.

Definition 2.1. Given a graph Γ with vertices $v_1, v_2, \ldots, v_{|V|}$, and edges $e_1, e_2, \ldots, e_{|E|}$:
    (1) A *walk* of length 1 is a walk which starts at v_i and goes to one of its neighbors v_j.
    (2) A *walk* of length k is a walk which starts at v_i, repeats (1) k times, and ends at v_j.

Example 2.2. In Figure 1: (1) $v_1 \xrightarrow{e_2} v_4$ is a walk from v_1 to v_4 of length 1; (2) $v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_3} v_4 \xrightarrow{e_4} v_5 \xrightarrow{e_7} v_3 \xrightarrow{e_6} v_6$ and $v_1 \xrightarrow{e_2} v_4 \xrightarrow{e_4} v_5 \xrightarrow{e_5} v_6 \xrightarrow{e_6} v_3 \xrightarrow{e_6} v_6$ are two walks from v_1 to v_6 of length 5.
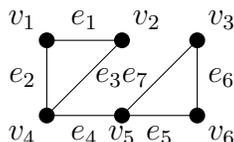

Figure 1: The graph used in Example 2.2 and Example 3.2

Theorem 2.3. *Let A_Γ be the adjacency matrix of a graph Γ. Then $A_\Gamma^k(i, j)$ is the number of* walks *of length k in Γ from v_i to v_j.*

3 Super-Walks

Theorem 2.3 gives the combinatorial interpretation of the entries of powers of the adjacency matrix. It is also natural to consider the combinatorial interpretations of entries of graph Laplacians. In this section, we formally define super-walks, and Theorem 3.3 shows that the total signs of super-walks of fixed length can be calculated by the corresponding even graph Laplacian and its powers.

Definition 3.1. Given a graph Γ with vertices $v_1, v_2, \ldots, v_{|V|}$, and edges $e_1, e_2, \ldots, e_{|E|}$, we have:

A *super-walk* of length 1 is a walk which starts at v_{i_1}, and:

(1) goes to one of its neighbor v_{i_2}; or

(2) goes towards one of its neighbors v_{i_2}, but does not reach it, and goes back to v_{i_1}.

In case (1), we define the super-walk to have a negative sign, and in case (2), we define the super-walk to have a positive sign.

A *super-walk* of length k from v_i to v_j is formed by repeating (1) and/or (2) k times, starting at v_i and ending at v_j, where we determine the sign by multiplying together the signs of the steps.

Example 3.2. In Figure 1: (1) $v_1 \overset{e_1}{\to} v_2$ is a super-walk from v_1 to v_2 of length 1; (2) $v_1 \overset{e_1}{\to} v_1$ and $v_1 \overset{e_2}{\to} v_1$ are super-walks from v_1 to v_1 of length 1; (3) $v_5 \overset{e_4}{\to} v_5$, $v_5 \overset{e_5}{\to} v_5$, and $v_5 \overset{e_7}{\to} v_5$ are super-walks from v_5 to v_5 of length 1. Furthermore, we say (1) is a super-walk with negative sign, while (2) and (3) are super-walks with positive sign.

Moreover, (4) $v_1 \overset{e_1}{\to} v_2 \overset{e_1}{\to} v_2$, $v_1 \overset{e_1}{\to} v_2 \overset{e_3}{\to} v_2$, $v_1 \overset{e_1}{\to} v_1 \overset{e_1}{\to} v_2$, $v_1 \overset{e_2}{\to} v_1 \overset{e_1}{\to} v_2$, and $v_1 \overset{e_2}{\to} v_4 \overset{e_3}{\to} v_2$ are super-walks from v_1 to v_2 of length 2. Furthermore, $v_1 \overset{e_1}{\to} v_2 \overset{e_1}{\to} v_2$, $v_1 \overset{e_1}{\to} v_2 \overset{e_3}{\to} v_2$, $v_1 \overset{e_1}{\to} v_1 \overset{e_1}{\to} v_2$, $v_1 \overset{e_2}{\to} v_1 \overset{e_1}{\to} v_2$ have one negative sign and one positive sign, thus we define each them with a negative sign. While $v_1 \overset{e_2}{\to} v_4 \overset{e_3}{\to} v_2$ has two negative signs, thus we define it with a positive sign.

We now present one of our two main results, relating the number of super-walks with sign, to entries of power of the even Laplacian.

Theorem 3.3. *Let Δ_Γ^+ be the even Laplacian of a graph Γ. Then $(\Delta_\Gamma^+)^k(i,j) = \sum_{\gamma, i \to j, k} sgn(\gamma)$ where $\gamma, i \to j, k$ is a super-walk γ that starts at vertex i, ends at vertex j, and has length k.*

Proof. Proof by induction on k. When $k = 1$, we have: $(\Delta_\Gamma^+)^k = (\Delta_\Gamma^+)^1$.

Note that the even graph Laplacian can also be calculated by

$$\Delta_\Gamma^+ := \Delta_\Gamma^+(i,j) = \begin{cases} \text{val}(i) & \text{if } i = j & (1) \\ -1 & \text{if } i \text{ is adjacent to } j & (2) \\ 0 & \text{otherwise} & (3) \end{cases}$$

where $\text{val}(i)$ is the *valence* of a vertex v, which equals to the number of neighbors of v.

In case (1), $(\Delta_\Gamma^+)(i,i) = \text{val}(i) = $ the number of neighbors of $i = $ the number of super-walks of length 1 goes from vertex i back to vertex i. Thus, the claim holds in case (1).

In case (2), $(\Delta_\Gamma^+)(i,j) = -1$ if i is adjacent to j. In this case, there is only one super-walk from vertex i to one of its neighbors vertex j, and the sign is negative. Thus, the claim holds in case (2).

In case (3), $(\Delta_\Gamma^+)(i,j) = 0$ if vertex i is not adjacent to vertex j. Since no super-walk of length 1 can go from vertex i to vertex j, the claim holds in case (3).

Therefore, the claim holds when $k = 1$.

Suppose the claim holds for some natural number k. Consider $k + 1$. We have

$$(\Delta_\Gamma^+)^{k+1} = (\Delta_\Gamma^+)^k \cdot (\Delta_\Gamma^+)^1$$

$$\text{So, } (\Delta_\Gamma^+)^{k+1}(i,j) = \sum_{q=1}^{n}(\Delta_\Gamma^+)^k(i,q)(\Delta_\Gamma^+)(q,j)$$

$$= (\Delta_\Gamma^+)^k(i,j) \cdot (\Delta_\Gamma^+)(j,j) + \overbrace{\sum_{q=1}^{n}(\Delta_\Gamma^+)^k(i,q) \cdot (-1)}^{\text{where } q \text{ and } j \text{ adjacent}} + \overbrace{\sum_{q=1}^{n}(\Delta_\Gamma^+)^k(i,q) \cdot 0}^{\text{where } q \text{ and } j \text{ non-adjacent}}$$

$$= \sum_{\gamma,i\to j,k} \text{sgn}(\gamma) \cdot \text{val}(j) + \sum_{\gamma,i\to j,k} \text{sgn}(\gamma) \cdot (-1) + 0.$$

Let $\sum_{\gamma,i\to j,k} \text{sgn}(\gamma) \cdot \text{val}(j) = X$. In X, we first count the super-walks from vertex i to vertex j of length k by calculating $\sum_{\gamma,i\to j,k} \text{sgn}(\gamma)$. Now we stand at vertex j, and have to go from vertex j to vertex j of length 1. The number of the last step equals to $\text{val}(j)$. Since $\text{val}(j)$ is always non-negative, X calculates $\sum_\gamma \text{sgn}(\gamma)$ where γ is one of the super-walks from vertex i to vertex j of length $k+1$, and the last step going from vertex j to vertex j.

Let $\sum_{\gamma,i\to j,k} \text{sgn}(\gamma) \cdot (-1) = Y$. In Y, we first count the super-walks from vertex i to vertex q of length k by calculating $\sum_{\gamma,i\to q,k} \text{sgn}(\gamma)$. Now we stand at vertex q, a neighbor of vertex j, and have to go from vertex q to vertex j of length 1. There is only one way to go from vertex q to vertex j of length 1, and its sign is negative. That is $\sum_{\gamma,q\to j,1} \text{sgn}(\gamma) = -1$. Hence, Y calculates $\sum_\gamma \text{sgn}(\gamma)$ where γ is one of the super-walks from vertex i to vertex j of length $k+1$, and the last step going from vertex q to vertex j.

Therefore, $X+Y$ calculates $\sum_{\gamma,i\to j,k+1} \text{sgn}(\gamma)$. Hence, the claim holds when k is replaced by $k+1$.

This is the end of the proof.

$\square$

4   Edge Super-Walks

In this section, we define edge super-walks. Similar to super-walks, an edge super-walk is a sign-sensitive edge-to-edge walk in an oriented graph. Particularly, the sign of an edge super-walk depends on not only the way each step goes, but also the orientation of the graph, which is defined in Definition 4.1 in this section. Theorem 4.5 shows that the total signs of edge super-walks of fixed length can be calculated by the corresponding odd graph Laplacian and its powers.

Definition 4.1. Given an oriented graph Γ, let i be an edge in Γ which starts at vertex A and ends at vertex B. We say *the sign of A in i* is negative, and *the sign of B in i* is positive.

Example 4.2. In Figure 2, we say that the sign of v_5 in e_4 is positive; the sign of v_5 in e_5 in negative; the sign of v_5 in e_7 is positive.
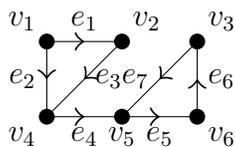
Figure 2: The graph used in Example 4.2 and Example 4.4

Definition 4.3. Given an oriented graph Γ with vertices $v_1, v_2, \ldots, v_{|V|}$ and edges $e_1, e_2, \ldots, e_{|E|}$, we have:

An *edge super-walk* of length 1 is a walk starts at e_{i_1}, and:

(1) goes to one of its neighbor edges e_{i_2} through the vertex v_i, where e_{i_1} and e_{i_2} intersect at v_i; or

(2) goes through one of the end points and back to itself.

In case (1), if the sign of v_i in e_{i_1} and the sign of v_i in e_{i_2} are the same, we define this walk to have a positive sign; if not, we define this walk to have a negative sign. In case (2), the walk is defined to have a positive sign.

An *edge super-walk* of length k from e_i to e_j is formed by repeating (1) and/or (2) k times, starting at e_i and ending at e_j, where we determine the sign by multiplying together the signs of the steps.

Example 4.4. In Figure 2: (1) $e_1 \overset{v_1}{\to} e_2$ is an edge super-walk from e_1 to e_2 of length 1; (2) $e_1 \overset{v_1}{\to} e_1$ and $e_1 \overset{v_2}{\to} e_1$ are edge super-walks from e_1 to e_1 of length 1. Furthermore, we say (1) is an edge super-walk with a negative sign, while both walks in (2) have a positive sign.

Moreover, $e_1 \overset{v_1}{\to} e_1 \overset{v_1}{\to} e_2$, $e_1 \overset{v_2}{\to} e_1 \overset{v_1}{\to} e_2$, $e_1 \overset{v_1}{\to} e_2 \overset{v_1}{\to} e_2$, $e_1 \overset{v_1}{\to} e_2 \overset{v_4}{\to} e_2$, and $e_1 \overset{v_2}{\to} e_3 \overset{v_4}{\to} e_2$ are edge super-walks from e_1 to e_2 of length 2. Furthermore, all of the first four edge super-walks have a positive sign, while the last one has a negative sign.

We now present the other main result, relating the number of edge super-walks with sign, to entries of power of the odd Laplacian.

Theorem 4.5. *Let* Δ_Γ^- *be the odd Laplacian of a graph* Γ. *Then* $(\Delta_\Gamma^-)^k = \sum_{\gamma, i \to j, k} sgn(\gamma)$ *where* $\gamma, i \to j, k$ *is an edge super-walk* γ *that starts at edge* i, *ends at edge* j, *and has length* k.

Proof. Prove by induction on k. When $k = 1$, we have: $(\Delta_\Gamma^-)^k = (\Delta_\Gamma^-)^1 = I^t I$.

Therefore, $\Delta_\Gamma^-(i, j) = \sum_{q=1}^n I_{qi} I_{qj}$.

By the definition of incidence matrix, $I_{ij} = 0$ or 1 or -1.

Case (1), if $i = j$, we have $\Delta_\Gamma^-(i, i) = \sum_{q=1}^n I_{qi} I_{qi} = 2$.

Meanwhile, we have two edge super-walks of length 1 going from one edge to itself, and both of them are positive. Hence, $\Delta_\Gamma^-(i, i) = 2$ follows the claim. The claim holds in case (1).

Case (2), if $i \neq j$, and edge i is not incident to edge j. It follows that if I_{qi} is non-zero, then I_{qj} must be zero. Hence, $\Delta_\Gamma^-(i, j) = \sum_{q=1}^n I_{qi} I_{qj} = 0$.

Meanwhile, we cannot go from edge i to edge j of length 1 if two edges have no intersection. Hence $\Delta_\Gamma^-(i,j) = 0$ follows the claim. The claim holds in case (2).

Case (3), if $i \neq j$, and edge i intersects edge j at vertex A. We have: $\Delta_\Gamma^-(i,j) = \sum_{q=1}^n I_{qi} I_{qj} = \pm 1$.

Since only non-zero items contribute to the result, if the sign of A in i is the same as sign of A in j, then we get $(+1)$; if the sign of A in i is different from sign of A in j, then we get (-1).

Meanwhile, if the sign of A in i and the sign of A in j are the same, then both edges end at A or start at A. We have one edge super-walk of length 1 going from edge i to edge j, and it is positive. Hence, $\Delta_\Gamma^-(i,j) = 1$ follows the claim. Also, if the sign of A in i and the sign of A in j are different, then one of the edges ends at A, and the other one starts at A. We have one edge super-walk of length 1 going from edge i to edge j, and it is negative. Hence, $\Delta_\Gamma^-(i,j) = -1$ follows the claim. The claim holds in case (3).

Therefore, the claim holds when $k = 1$.

Suppose the claim holds for some natural number k. Consider $k + 1$.

$$(\Delta_\Gamma^-)^{k+1} = (\Delta_\Gamma^-)^k \cdot (\Delta_\Gamma^-)^1$$

So, $(\Delta_\Gamma^-)^{k+1}(i,j) = \sum_{q=1}^n (\Delta_\Gamma^-)^k(i,q)(\Delta_\Gamma^-)(q,j)$

$$= (\Delta_\Gamma^-)^k(i,j) \cdot (\Delta_\Gamma^-)(j,j) + \overbrace{\sum_{q=1}^n (\Delta_\Gamma^-)^k(i,q) \cdot (\Delta_\Gamma^-)(q,j)}^{\text{where } q \text{ and } j \text{ adjacent}} + \overbrace{\sum_{q=1}^n (\Delta_\Gamma^-)^k(q,j) \cdot 0}^{\text{where } q \text{ and } j \text{ non-adjacent}}$$

$$= \sum_{\gamma, i \to j, k} \mathrm{sgn}(\gamma) \cdot (\Delta_\Gamma^-)(j,j) + \sum_{\gamma, i \to j, k} \mathrm{sgn}(\gamma) \cdot (\Delta_\Gamma^-)(q,j) + 0.$$

Let $\sum_{\gamma, i \to j, k} \mathrm{sgn}(\gamma) \cdot (\Delta_\Gamma^-)(j,j) = X$. In X, we first count the edge super-walks from edge i to edge j of length k by calculating $\sum_{\gamma, i \to j, k} \mathrm{sgn}(\gamma)$. Now we stand at edge j, and have to go from edge j to edge j of length 1. The number of the last step is 2, because every edge has two ends. Since $(\Delta_\Gamma^-)(j,j)$ is 2 for all j, X calculates $\sum_\gamma \mathrm{sgn}(\gamma)$ where γ is the edge super-walks from edge i to edge j of length $k + 1$, and the last step going from edge j to edge j.

Let $\sum_{\gamma, i \to j, k} \mathrm{sgn}(\gamma) \cdot (\Delta_\Gamma^-)(q,j) = Y$. In Y, we first count the edge super-walks from edge i to edge q of length k by calculating $\sum_{\gamma, i \to q, k} \mathrm{sgn}(\gamma)$. Now we stand at edge q, an edge connected to the edge j, and have to go from edge q to edge j of length 1. There is only one way to go from edge q to edge j of length 1, and its sign can be negative or positive. That is $\sum_{\gamma, q \to j, 1} \mathrm{sgn}(\gamma) = -1 = (\Delta_\Gamma^-)(q,j)$, or $\sum_{\gamma, q \to j, 1} \mathrm{sgn}(\gamma) = 1 = (\Delta_\Gamma^-)(q,j)$. Hence, Y calculates $\sum_\gamma \mathrm{sgn}(\gamma)$ where γ is the edge super-walks from edge i to edge j of length $k + 1$, and the last step going from edge q to edge j.

Therefore, $X + Y$ calculates $\sum_{\gamma, i \to j, k+1} \mathrm{sgn}(\gamma)$. Hence, the claim holds when k is replaced by $k + 1$.

This is the end of the proof.

$\square$

5   Conclusion and Perspectives

The combinatorial interpretation of the Laplacian has a physical counterpart. In quantum mechanics, we have the evolution of quantum states $\psi \in \mathcal{H}$ which is calculated via the so called *Schrödinger equation:* $\frac{\partial}{\partial t}\psi = \mathbb{H}\psi$, where $\mathbb{H}$ is the Schrödinger operator. One can also calculate it in another way by looking at the wave equation:

$$\psi(y, t_1) = \int_{\mathbb{R}} k(t_1 - t_0, x, y)\psi(x, t_0)dx.$$

This integral is the *heat integral kernel*, where k is *a propagator of the quantum system*. In order to compute such heat kernel, Feynman proposed the following description: $k(t_1 - t_0, x, y) = \int_{\gamma} \mathcal{D}(\Gamma)e^{i/\hbar}S_{cl}(\gamma)$, where the integral is over paths γ in $\mathbb{R}$, and $\mathcal{D}(\Gamma)$ is a measure on the space of walks. The two types of super-walks and their theorems given by this paper show the discrete form of the walk expansion for the partition function of the super-symmetric version of graph quantum mechanics. In this combinatorial model of quantum mechanics, we know that $\psi_{t_1} = exp((t_0 - t_1)\Delta)\psi_{t_0}$, where Δ is the super-Laplacian (the direct sum of the even and odd graph Laplacians). Thus we can calculate ψ by summing all ψ_t's: $\psi = \int_{\mathbb{R}} \sum_{\gamma} \frac{t^{|\gamma|}}{|\gamma|!}e^{-kt}\psi(x, t_0)dx$, where γ are all the super-walks defined in this paper.

The formulae in this paper also suggest a natural extension of walks to CW-complexes, with a given orientation, and such expressions would lead to a combinatorial interpretation of the partition function of super-symmetric quantum mechanics for CW-complexes. In del Vecchio's work, walk sum formulae and gluing formulae are derived for the case of graph quantum mechanics , and it is conjectured that this approach is still useful in the more general setting of Laplacians for CW-complexes [2]. In this case the graph Laplacian is replaced by the Hodge Laplacian $\langle u|(\Delta_X)^k|v\rangle = \langle u|(d_X^* d_X)^k|v\rangle + \langle u|(d_X d_X^*)^k|v\rangle$, where X is the CW-complex such that $\Delta_X : \bigoplus_{i=0}^{n} Span_{\mathbb{C}}(C_i(X)) \to Span_{\mathbb{C}}(C_i(X))$. Since any graph is a 1-dimensional CW-complex, one can extend the two types of super-walks defined in this paper to CW-complexes so that $\langle u|(\Delta_X)^k|v\rangle = \sum_{\gamma} \text{sgn}(\gamma)$, where X is the CW-complex, u and v are cells, γ are the super-walks from u to v of length k. The super-walks for CW-complexes correspond to sequences of cells with alternating parity in dimension.

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