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Abstract. A word $\bar{w} = \bar{u}\bar{u}$ is called a *long square* if $\bar{u}$ is of length at least 3; a word $\bar{w}$ is called *long-square-free* if $\bar{w}$ contains no sub-word that is a long square. If there exists a k-coloring of the vertices of a graph G such that, for any path P in G, the word generated by the coloring of P is long-square-free, then G is called *long-repetition-free k-colorable*. We show that every rooted tree of radius $r \leq 7$ is long-repetition-free 2-colorable. We also show that there exists a class of trees which are not long-repetition-free 2-colorable.

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1 Introduction

The aim of this paper is to introduce the concepts of "long square" and "long repetition," and to prove two main results regarding the long-repetition-free 2-colorability of trees. Most terminology relevant to this paper is provided, but the reader is encouraged to reference M. Lothaire [3] and Bondy and Murty [1] for more information regarding combinatorics on words and graph theory, respectively.

The investigation of words that avoid specific letter patterns (such as, for example, squares, cubes, or palindromes) has been a topic of interest in the mathematical sciences and the biological sciences since the early 20th century, beginning largely with the papers of Axel Thue [4, 5]. Thue proved that it is impossible to define an infinitely-long, square-free word over a binary alphabet; however, Thue proved that it *is* possible to define an infinitely-long, square-free word over a ternary alphabet, and Thue gave a definition for such a word (which today is called the Thue-Morse word). One can use such a word to color the vertices of a graph and, in so doing, create what we call repetition-free 3-colorings.

We say a word $\bar{w}$ is a "long square" if $\bar{w} = \bar{u}\bar{u}$ for some word $\bar{u}$ where $|\bar{u}| \geq 3$. Much is known regarding words that avoid squares generally (and graph colorings that avoid repetitions generally) - see Grytczuk [2] for a survey of such results; however, little is known about words that allow for squares of lengths 2 and 4.

In this article, we begin in Section 2 by defining terms used throughout the paper. In Section 3, we prove our first main result, Proposition 3.1, which asserts that every tree of radius $r \leq 7$ is long-repetition-free 2-colorable. The proof of this proposition relies on a computer program which inputs a word and returns whether or not the word is long-square-free. Finally, in Section 4, we prove two lemmas which lead to the proof of our second main result, Proposition 4.4, which asserts that there exists a class of trees which are not long-repetition-free 2-colorable.

2 Terminology and Notation

We begin by providing several useful definitions and detailing the notation that is used throughout this paper.

Definition 2.1. An *alphabet* is a finite set of elements called *letters*, and we typically denote alphabets by a single capital letter (for example, A). The set of all words over alphabet A is denoted A^*. We typically denote words of arbitrary length by a single lower-case letter with an overhead bar (for example, $\bar{w}$), and we typically denote letters (words of length 1) by a single lower-case letter (for example, a). We denote the concatenation of words $\bar{w}$ and $\bar{v}$ by $\bar{w}\bar{v}$ or, if we wish to stress the significance of concatenation, by $\bar{w} \cdot \bar{v}$. If a word $\bar{u}$ contains sub-word $\bar{w}$, we say $\bar{w} \leq \bar{u}$.

Definition 2.2. For word $\bar{w} = a_1 \ldots a_n$ over alphabet A, we define the *reversal* of $\bar{w}$ to be $\overleftarrow{w} = a_n \ldots a_1$. Let $\bar{w}$ be a word. We say $\bar{w}$ is a *palindrome* if $\bar{w} = \overleftarrow{w}$. Similarly we say $\bar{w}$ is a *long palindrome* when $\bar{w}$ is of at least length 4. We call $\bar{w}$ a *square* if there exists some non-empty word $\bar{u}$ such that $\bar{w} = \bar{u}\bar{u}$. Furthermore, $\bar{w}$ is *square-free* if $\bar{w}$ contains no sub-word that is a square. Similarly, word $\bar{w} = \bar{u}\bar{u}$ is a *long square* if $|\bar{u}| \geq 3$. Word $\bar{w}$ is *long-square-free* if $\bar{w}$ contains no sub-word that is a long square.

Definition 2.3. Let $G = (V, E)$ be a simple, connected graph, and let $A = \{1, \ldots, k\} \subset \mathbb{N}$. Graph G is said to be *long-repetition-free k-colorable* if there exists a coloring $\chi : V \to A$ such that, for every path $P = u_1 e_1 \ldots e_{n-1} u_n \subseteq G$, the word $\chi(u_1) \ldots \chi(u_n)$ is long-square-free.

Definition 2.4. Let $G = (V, E)$ be a connected graph. The *distance* between vertices $u, v \in V$ is defined to be the number of edges contained in the shortest path from u to v. The *eccentricity* of vertex v is the maximum distance between v and any other vertex in G. The *radius* of graph G is the minimum eccentricity of its vertices, and the *center* of graph G is the sub-graph induced by the vertices of minimum eccentricity.

Definition 2.5. A tree $T = (V, E)$ is called a *rooted tree* if one vertex has been distinguished to be called the *root* of T. The edges of a rooted tree are described with an implicit orientation away from the root vertex. In a rooted tree, the *parent* of a vertex v is the vertex adjacent to v on the path to the root vertex. A *child* (plural, *children*) of vertex v is a vertex of which v is the parent. A *descendant* of a vertex v is any vertex which is either a child of v or is, recursively, the descendant of any of the children of v. A *sibling* to a vertex v is any other vertex in V which has the same parent as v. The k^{th} *generation* of root r of tree T, denoted $G_k(T)$, is the set of descendant vertices of distance k from r. The *height* of T is the number of generations in T from r, with the convention that $G_0(T) = \{r\}$.

In particular, it is of note that if $T = (V, E)$ is a rooted tree, then for root $r \in T$ and vertex $v \in V$ there exists a unique paths from r to v and from v to r. As a matter of notation, it is perhaps noteworthy that while the k^{th} generation from r is of course dependent on our choice of r, we simply notate this $G_k(T)$ since, for this paper, the particular vertex distinguished as the root is not particularly relevant and furthermore does not change once the root has been chosen.

Definition 2.6. The *Tyler tree of height n*, denoted $T_n = (V_n, E_n)$, is the rooted tree with root r in which each vertex in the j^{th} generation from r has $2^{n-j} + 1$ children, for every $0 \leq j \leq n-1$. Computationally, $|V_n| = 1 + \sum_{j=0}^{n-1} \prod_{k=n-j}^{n} (2^k + 1)$. A *binary tree* is a rooted tree in which each vertex has at most two children. A full binary tree with height n is denoted B_n. The reader can verify that the Tyler tree of height n contains $2^n + 1$ Tyler trees of height $n-1$, and it is of note that all vertices in generation n are leaves.

3 A Class of Trees that is Long-Repetition-Free 2-Colorable

The goal of the present section is to prove the first of our two main results: Proposition 3.1, which asserts that every rooted tree of radius less than or equal to 7 can be long-repetition-free 2-colored.

Proposition 3.1. *Every rooted tree of radius less than or equal to 7 is long-repetition-free 2-colorable.*

Proof. Let $T = (V, E)$ be a tree of radius $R \leq 7$, and choose a root vertex r in the center of T; that is, choose r to have minimal eccentricity. Let $g = \max\{k \in \mathbb{N} \mid G_k(r) \neq \varnothing\}$ be the number of generations in T from vertex r. First, note that $g \leq 7$. Indeed, if $g \geq 8$, then there exists a vertex $w \in G_8(r)$, so $d(r, w) = 8$, and so the eccentricity of r is at least 8. However, since T has radius $R \leq 7$, there exist vertices of eccentricity less than 8, contradicting our choice of r; therefore, $g \leq 7$. It suffices to show the proposition holds for $g = 7$, so assume without loss of generality that $g = 7$. Assume also that at least two vertices belong to each generation $G_k(r)$ for all $1 \leq k \leq 7$.

Color the vertices of T in the following way. Let $\bar{a} = 00010111$, and define $\chi : V \to \{0, 1\}$ by $\chi(v) = a_j$ whenever $v \in G_j(r)$, for $0 \leq j \leq 7$. See Figure 3.2 for an example of a tree of radius 7 whose vertices have been colored according to χ.

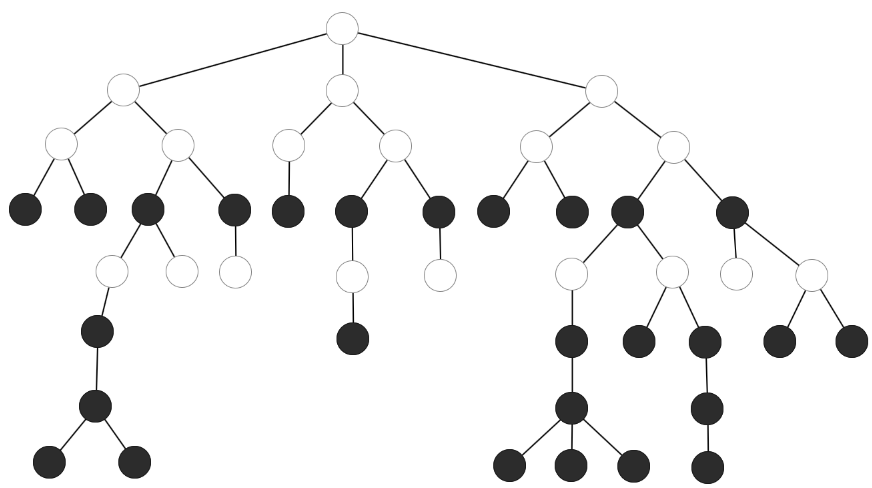

Figure 3.2. *The coloring χ as defined in Proposition 3.1 applied to a tree of radius 7.*

We claim that χ is a long-repetition-free 2-coloring of T. Suppose, by way of contradiction, that there exists a path $P \subseteq T$ such that $\chi(P)$ is a long repetition. Let $P = v_1 e_1 v_2 e_2 \ldots v_n e_n v_{n+1} \ldots v_{2n-1} e_{2n-1} v_{2n}$. Clearly $n \geq 3$ in order for P to be a long repetition, and since $g \leq 7$, it follows that $n \leq 7$. Since T is a tree, two adjacent vertices in P cannot belong to the same generation; moreover, if $v_i, v_{i+1} \in P$ are adjacent and if $v_i \in G_j(r)$ (for some $0 \leq j \leq 7$), then either $v_{i+1} \in G_{j+1}(r)$ or $v_{i+1} \in G_{j-1}(r)$. Furthermore, since T is a tree, the sequence of generations $g(v_1), \ldots, g(v_{2n})$ of vertices $v_1, \ldots, v_{2n}$ is either strictly increasing, strictly decreasing, increases and then decreases, or decreases and then increases. Consequently, it follows that P is a subgraph of some path P'_ℓ, where P'_ℓ is a path starting at a vertex in $G_7(r)$, ascending to a vertex in $G_\ell(r)$, and descending to another vertex in $G_7(r)$, for some $0 \leq \ell \leq 4$. Hence there are five cases to consider for $\chi(P)$: (1) $\chi(P) \leq 1110111$, (2) $\chi(P) \leq 111010111$, (3) $\chi(P) \leq 11101010111$, (4) $\chi(P) \leq 1110100010111$, and (5) $\chi(P) \leq 111010000010111$.

To prove the statement of the proposition, it suffices to check that the words 1110111, 111010111, 11101010111, 1110100010111, and 111010000010111 are long-square-free. For this, we use a computer program (written in the language Python, given in the Appendix). In brief, the computer program inputs a word and checks each subword of length 6 or more contained in the word. If a long square is found, then the program prints the detected long square and returns 'True'; if not, then the program returns 'False'. Using this program, we find the words 1110111, 111010111, 11101010111, 1110100010111, and 111010000010111 to be long-square-free.

Thus, we find the coloring $\chi(P)$ of path P to be long-repetition-free, and so the coloring of every path in T is long-repetition-free, as desired. □

4 A Class of Trees that is Not Long-Repetition-Free 2-Colorable

One may ask whether or not every tree is long-repetition-free 2-colorable. In this section, we show that a certain class of trees require a minimum of 3 colors for a long-repetition-free coloring, making 3 colors the lowest bound for long-repetition-free colorings of trees. In order to gain this result, we first prove Lemma 4.1 and Lemma 4.2.

Lemma 4.1. *Every binary word of length at least* 9 *contains a long palindrome.*

Proof. It is clear that we need only consider binary words of length 9, for any binary words of length greater than 9 contains a binary word of length 9. Let $a_1a_2\dots a_9$ be a word over alphabet $A = \{0, 1\}$. Define a_i^c to be the binary complement of a_i; more precisely, $a_i^c \equiv a_i + 1 \mod 2$ for every $i \leq 9$. The reader can verify that for $a_1 \dots a_9$ to be long-palindrome-free, the following conditions must hold:

(i) If $a_i = a_{i+3}$, then $a_{i+1} = a_{i+2}^c$, for $1 \leq i \leq 6$;

(ii) If $a_i = a_{i+4}$, then $a_{i+1} = a_{i+3}^c$, for $1 \leq i \leq 5$;

(iii) If $a_i = a_{i+1}$, then $a_{i+1} = a_{i+2}^c$, for $2 \leq i \leq 6$.

Suppose without loss of generality that $a_1 = 0$. We consider four cases. First, suppose $a_1 = a_4 = a_5 = 0$. By (i), it follows that $a_2 = a_3^c$. Then $a_2 = 1$ and $a_3 = 0$, for otherwise we have long palindrome 00100. However, we now have no choice for a_6, for $a_6 = 0$ yields long palindrome 0000, and $a_6 = 1$ yields long palindrome 10001.

Suppose next that $a_1 = a_4 = a_5^c = 0$. By (i), it follows that $a_2 = a_3^c$. Then $a_2 = 0$ and $a_3 = 1$, for otherwise we have long palindrome 1001. Then $a_6 = 1$, for otherwise we have long palindrome 01010, and so $a_7 = 1$, for otherwise we have long palindrome 0110. However, we now have no choice for a_8, for $a_8 = 0$ yields long palindrome 01110, and $a_8 = 1$ yields long palindrome 1111.

Suppose that $a_1 = a_4^c = a_5^c = 0$. By (iii), $a_6 = 0$, and it follows that $a_3 = 1$ for otherwise we have long palindrome 0110. However, we now have no choice for a_2, for $a_2 = 0$ yields long palindrome 01110, and $a_2 = 1$ yields long palindrome 1111.

Finally, suppose that $a_1 = a_4^c = a_5 = 0$. By (ii), $a_2 = 0$, and it follows that $a_3 = 0$ for otherwise we have long palindrome 0110. Then $a_6 = 1$, for otherwise we have long palindrome 00100; $a_7 = 1$, for otherwise we have long palindrome 01010; and $a_8 = 1$, for otherwise we have 0110. However, we now have no choice for a_9, for $a_9 = 0$ yields long palindrome 01110, and $a_9 = 1$ yields long palindrome 1111. □

Lemma 4.2. *Let T_n denote the Tyler tree of height n with root r, and apply an arbitrary 2-coloring ϕ to T_n. Then T_n contains a binary sub-tree of height n, B_n, with root r such that, for every $u, v \in G_i(B_n)$, $\phi(u) = \phi(v)$, for all $0 \leq i \leq n$. In other words, the vertices in the same generation of B_n share the same color.*

Proof. We prove the result by induction on the height, n, of the Tyler tree.

Tyler tree T_1 contains exactly four vertices: root r and 3 children vertices adjacent to r. By the pigeonhole principle, at least 2 of these 3 children vertices must be colored with the same color. Thus, T_1 contains a full binary tree of height 1 whose root is r and where all the vertices on the same level have the same color. See Figure 4.3 for an example of T_1 and T_2 each with an embedded full binary tree colored by generations.

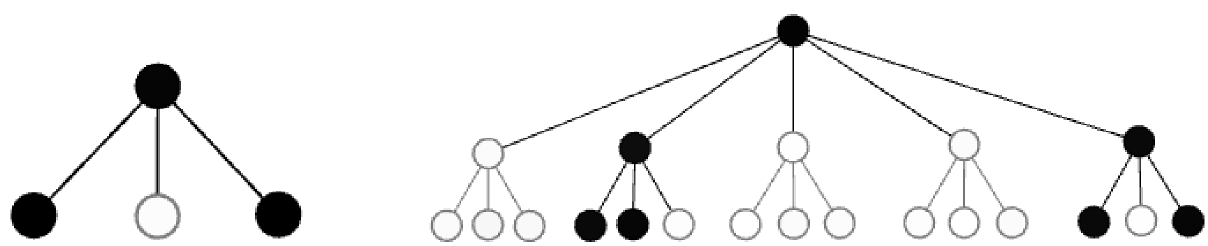

Figure 4.3. *The Tyler tree of height 1 (left) and the Tyler tree of height 2 (right) each contain a binary sub-tree in which every vertex in each generation from the root has the same color.*

Suppose that every Tyler tree of height $k \geq 1$ contains a binary sub-tree such that, for every $u, v \in G_i(B_k)$, we have $\phi(u) = \phi(v)$, for every $0 \leq i \leq k$. Let T_{k+1} denote the Tyler tree of height $k+1$ with root r, and apply arbitrary 2-coloring ϕ to T_{k+1}. By definition, r has $2^{k+1} + 1$ children, and each of these children are the root of a Tyler tree of height k. Therefore, T_{k+1} contains $2^{k+1} + 1$ Tyler trees of height k as sub-trees, and, by the inductive hypothesis, it follows that each of these Tyler trees of height k contain a binary sub-tree of height k such that, for every $u, v \in G_i(B_k)$, we have $\phi(u) = \phi(v)$, for every $0 \leq i \leq k$.

Each of these $2^{k+1}+1$ binary sub-trees has k generations, and, given the coloring constraints provided in the inductive hypothesis, there are exactly 2^k possible 2-colorings of these binary sub-trees. Attach to these binary sub-trees the root r. As there are 2 possible 2-colorings of r, there are a total of 2^{k+1} possible 2-colorings of this binary sub-tree of height $k+1$ in which each generation is similarly colored. Since there are $2^{k+1} + 1$ binary sub-trees of height k and only 2^{k+1} possible 2-colorings in which each generation is similarly colored, it follows that at least two binary sub-trees of height k must be colored identically. The sub-tree induced by these two identically-colored full binary trees of height k, when attached to r, form a full binary tree of height $k+1$ in which, for every $u, v \in G_i(B_{k+1})$, we have $\phi(u) = \phi(v)$, for all $0 \leq i \leq k+1$, thereby completing the proof. □

Proposition 4.4. *If $n \geq 8$, then T_n is not long-repetition-free 2-colorable.*

Proof. Let T_8 denote the Tyler tree of height 8 with root r. We need only consider T_8 to prove the result. Apply an abritrary 2-coloring ϕ to T_8. By Lemma 4.2, T_8 contains a binary sub-tree B_8 in which, for each $u, v \in G_i(B_8)$, we have $\phi(u) = \phi(v)$, for all $0 \leq i \leq 8$.

Pick vertices $u_0, \ldots, u_8$ such that $u_i \in G_i(B_8)$ for each $0 \leq i \leq 8$, and such that $u_0, \ldots, u_8$ is a simple path. Let b_i denote the letter associated with $\phi(u_i)$, and let $\bar{b} = b_0 \ldots b_8$. By Lemma 4.1, $\bar{b}$ contains a palindrome $\bar{p}$ of length 4 or of length 5. Suppose $\bar{p} = b_j \ldots b_{j+m}$, where $0 \leq j \leq 4$ and where $m = 3$ or $m = 4$. Consider the path $P = u_{j+m} \ldots u_{j+1} u_j u'_{j+1} \ldots u'_{j+m}$, where $u'_i \in G_i(B_8)$ for each $0 \leq i \leq j+m$.

Let $\bar{c}$ be the word associated with $\phi(P) = \phi(u_{j+m} \ldots u_{j+1} u_j u'_{j+1} \ldots u'_{j+m})$. If $|\bar{p}| = 4$, then either $\bar{c} = c_1c_1c_1c_1c_1c_1c_1$, or $\bar{c} = c_1c_2c_2c_1c_2c_2c_1$, where $c_i \in \{0, 1\}$. In either case, $\bar{c}$ contains a long square. If $|\bar{p}| = 5$, then there are four possibilities: (1) $\bar{c} = c_1(c_1c_1c_1c_1)(c_1c_1c_1c_1)$, (2) $\bar{c} = c_1(c_2c_1c_2c_1)(c_2c_1c_2c_1)$, (3) $\bar{c} = c_1(c_1c_2c_1c_1)(c_1c_2c_1c_1)$ or (4) $\bar{c} = c_1(c_2c_2c_2c_1)(c_2c_2c_2c_1)$. In any of these four cases, $\bar{c}$ contains a long square. Since $\bar{c}$ contains a long square regardless of the length of $\bar{p}$, it follows that P contains a long repetition, and thus T_8 is not long-repetition-free 2-colorable. □

5 Appendix

The following computer program, written in the language Python, inputs a word (or a string) and checks every subword of length 6 or more contained in the given word. For each long square contained in the word, the program prints the long square and returns 'True', indicating that the word does contain a long square. If no long square is found in the word, then the program returns 'False', indicating that the word does not contain a long square and is therefore long-square-free.

```
def longSquareChecker(string):
    for length in range(3,len(string)//2+1):
        for index in range(0,len(string)-length*2+1):
            if string[index:index+length]...
            ==string[index+length:index+2*length]:
                print(string[index:index+length],index)
                return True
    return False
```

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