Rose-Hulman Undergraduate Mathematics Journal

Volume 18
Issue 1

Article 14

Scott Sentences in Uncountable Structures

Brian Tyrrell
Trinity College Dublin

Rose-Hulman Undergraduate Mathematics Journal

SCOTT SENTENCES IN UNCOUNTABLE STRUCTURES

Brian Tyrrell[a]

Volume 18, No. 1, Spring 2017

Sponsored by

Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
mathjournal@rose-hulman.edu
scholar.rose-hulman.edu/rhumj

[a]Trinity College Dublin

Rose-Hulman Undergraduate Mathematics Journal
Volume 18, No. 1, Spring 2017

SCOTT SENTENCES IN UNCOUNTABLE STRUCTURES

Brian Tyrrell

Abstract. Using elementary first order logic we can prove many things about models and theories, however more can be gleamed if we consider sentences with countably many conjunctions and disjunctions, yet still have the restriction of using only finitely many quantifiers. A logic with this feature is $L_{\omega_1,\omega}$. In 1965 Scott proved by construction the existence of an $L_{\omega_1,\omega}$ sentence that could describe a countable model up to isomorphism. This type of infinitary sentence is now known as a Scott sentence. Given an infinitary cardinal κ, we wish to find a set of conditions such that if a countable model satisfies (or can be expanded to satisfy) these conditions, a Scott sentence of it will have a model of cardinality κ.

Acknowledgements: I'd like to thank Professor Julia Knight of the University of Notre Dame for supervising me, providing me guidance and teaching me for the duration of this project, and always being open to questions long after. Thanks to the University of Notre Dame for providing the facilities and structure that allowed me to work at the university this summer and thank you to the Naughton Fellowship Programme for awarding me the fellowship that enabled me to pursue this topic. Finally, thank you to the RHUMJ referee and Professor Langley for their suggestions and corrections to this paper.

1 Introduction

When one wishes to prove something about a group, one uses group theory. However if one wishes to prove something about group theory, or set theory, or algebra in general one can step to a higher level of abstraction and use model theory. On the first page of their 1990 book, Chang and Keisler [1] explain this branch of mathematics using the equation

$$\text{universal algebra} + \text{logic} = \text{model theory}.$$

In more precise terms, model theory is the study of all mathematical structures using mathematical logic. A *model* is a set containing the interpretations of a formal language. One can create logical sentences that can be evaluated to be true or false in any model. The type of sentences which a large portion of model theory is devoted to studying, known as elementary first order sentences, contain finitely many quantifiers (ranging over elements of the set), conjunctions and disjunctions. The collection of all sentences true in a model is known as the *theory* of the model, and the theorems of model theory generally concern themselves with proving things about models and their theories. Note that that the term *structure* is often used in concert with the term *model*.

If we remove the restriction of using only finitely many conjunctions and disjunctions in our sentences, and instead restrict ourselves to countably many, more opportunities and proofs are available to us. In particular, every countable model has an infinitary sentence that identifies it up to isomorphism. This sentence is known as *a Scott sentence*. We construct a particular Scott sentence of a countable model $\mathfrak{A}$ in Section 3, which we refer to as "the" Scott sentence for $\mathfrak{A}$. At the end of this section we present examples of infinitary sentences and a Scott sentence of a vector space.

The Upward Löwenheim Skolem Tarski theorem ensures that every theory with a countable model has an uncountable one. For Scott sentences, a natural question to ask is, given a Scott sentence of a countable model, is this sentence satisfied in an uncountable model? Suppose $\mathfrak{B}$ is some special countable structure (the nature of 'special' we will determine) and ϕ is a Scott sentence of $\mathfrak{B}$. The goal of this project is to find a set of conditions $\mathfrak{B}$ must satisfy in order for ϕ to have an uncountable model of a specified cardinality.

In Section 2 we introduce the notion of a *back-and-forth family*, a family of partial isomorphisms that is useful in showing two countable structures are isomorphic. In Section 4 we introduce concepts that will be crucial to the rest of the paper; similarly in Section 5 we define an *atomic model* and provide proofs that are referenced throughout the paper.

In Section 6 we outline conditions on a countable model $\mathfrak{A}$ with a Scott sentence φ that will guarantee the existence of a structure of size $\aleph_1$ satisfying φ. In **Main Theorem I** (*Theorem 8.1*, Section 8) we prove these conditions are sufficient and necessary in order for φ to have an $\aleph_1$-sized model.

Before this, in Section 7, we outline the proof of *Vaught's Two-Cardinal theorem* as the style of this proof has many similarities as to how we tackle the proof of *Theorem 8.1*.

The conditions of *Theorem 8.1* are further expanded upon in Section 12 to guarantee the existence of a structure of size $\aleph_2$ satisfying φ. This exploration culminates in the proof of

Main Theorem II (*Theorem 12.1*, Section 12) which determines the Section 12 conditions are sufficient for φ to have an $\aleph_2$-sized model, drawing on many of the proofs in Sections 6 & 8.

In Section 9 we outline more restrictive conditions involving built-in Skolem functions and indiscernible sets which guarantee the existence of a structure of any infinite cardinality satisfying φ. In Section 10 we prove there is a countable elementary first order theory Z (related to $\mathfrak{A}$) with added predicates in the language L of $\mathfrak{A}$ such that another model $\mathfrak{B}$ will satisfy the Scott sentence of $\mathfrak{A}$ if and only if $\mathfrak{B}$ can be expanded to an atomic model of Z.

Finally in Section 11 we explore examples of structures where a Scott sentence of a countable model can and cannot have models of size $\aleph_2$.

2 Back-and-forth families

In this section we detail a method used to construct isomorphisms between two countable structures. Notes on this method were obtained from Simmons' notes for a model theory course [3].

Definition 2.1. Let $\mathfrak{A}$, $\mathfrak{B}$ be countable L-structures. A partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ is a bijection $f : U \to V$ on subsets U, V of $\mathfrak{A}$, $\mathfrak{B}$ which itself is an isomorphism.

Definition 2.2. A *back-and-forth family* P on countable L-structures $\mathfrak{A}, \mathfrak{B}$ is a nonempty set of partial isomorphisms $f : U \to V$ with the properties that:

(1) For each $f \in P$ and $x \in \mathfrak{A}$ there is a $y \in \mathfrak{B}$ and $f^+ \in P$ such that $f^+ : U \cup \{x\} \to V \cup \{y\}$ and $f^+(x) = y$.

(2) For each $f \in P$ and $y \in \mathfrak{B}$ there is an $x \in \mathfrak{A}$ and $f^+ \in P$ such that $f^+ : U \cup \{x\} \to V \cup \{y\}$ and $f^+(x) = y$.

Theorem 2.3. *If there exists a back-and-forth family P on two countable structures $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. We may write $\mathfrak{A} = \{a_i : i < \omega\}$ and $\mathfrak{B} = \{b_i : i < \omega\}$. Given some $f = f_0 \in P$ we create a sequence of functions $f_0, f_1, f_2, \cdots$ where $f_i \in P$, $f_i : U_i \to V_i$, $U_i \subseteq U_{i+1}$ and $V_i \subseteq V_{i+1}$ and specifically $a_i \in U_{i+1}$ and $b_i \in V_{i+1}$ - that is, $a_0 \in U_1$, $a_1 \in U_2$, etc. Then $F = \bigcup_{i<\omega} f_i$ is an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as required. ∎

When we make reference to making a 'back-and-forth argument' we mean we are constructing a back-and-forth family to prove two countable structures are isomorphic.

3 Scott's Isomorphism Theorem

We begin with the following definitions:

Definition 3.1. Let $\mathfrak{A}$ be a countable L-structure. A sentence φ of $L_{\omega_1,\omega}$ is called a *Scott sentence* if for all countable L-structures $\mathfrak{B}$,

$$\mathfrak{B} \models \varphi \qquad \Leftrightarrow \qquad \mathfrak{B} \cong \mathfrak{A}.$$

Definition 3.2. Let $\mathfrak{A} = (A, \dots)$ be a countable L-structure, and let $a_1, ..., a_n = \vec{a} \in A$ be some n-tuple. Suppose $\alpha < \omega_1$ is some ordinal. Define the formula $\varphi^{\alpha}_{\vec{a}}(\vec{x})$ inductively:
$\alpha = 0:$ $\varphi^{\alpha}_{\vec{a}}(\vec{x})$ is the conjunction of all quantifier free formulae true of $\vec{a}$.
$\alpha = \beta + 1:$

$$\varphi^{\beta+1}_{\vec{a}}(\vec{x}) := \varphi^{\beta}_{\vec{a}}(\vec{x}) \wedge \forall y \bigvee_{b \in A} \varphi^{\beta}_{\vec{a},b}(\vec{x}, y) \wedge \bigwedge_{b \in A} \exists y \varphi^{\beta}_{\vec{a},b}(\vec{x}, y).$$

If β is a limit ordinal:

$$\varphi^{\beta}_{\vec{a}}(\vec{x}) := \bigwedge_{\gamma<\beta} \varphi^{\gamma}_{\vec{a}}(\vec{x}).$$

We explain this definition as follows; for $\alpha = 0$ if for some $\vec{b} \in A$, $\mathfrak{A} \models \varphi^{\beta}_{\vec{a}}(\vec{b})$ then $\vec{a}, \vec{b}$ satisfy the same quantifier free formulae. From here, if $\alpha = \beta + 1$ then similarly $\vec{a}$ and $\vec{b}$ are 'β equivalent' with the added condition that, if we lengthen $\vec{a}$ by one element, there is a corresponding element to lengthen $\vec{b}$ by for $\vec{a}$ and $\vec{b}$ to remain β equivalent, and vice versa. This idea extends naturally to limit ordinals as well.

Note that $\mathfrak{A} \models \varphi^{\beta}_{\vec{a}}(\vec{a})$ and for $\gamma < \beta < \omega_1$

$$\mathfrak{A} \models \forall \vec{x}(\varphi^{\beta}_{\vec{a}}(\vec{x}) \rightarrow \varphi^{\gamma}_{\vec{a}}(\vec{x}))$$

by definition. As $\mathfrak{A}$ is a countable model, for every $\vec{a} \in A$, there exists $\alpha < \omega_1$ such that for all $\beta \geq \alpha$

$$\mathfrak{A} \models \forall \vec{x}(\varphi^{\alpha}_{\vec{a}}(\vec{x}) \leftrightarrow \varphi^{\beta}_{\vec{a}}(\vec{x})).$$

α is known as the *Scott rank of* $\vec{a}$. Expanding this idea further:

Definition 3.3. The *Scott rank of* $\mathfrak{A}$ is the smallest $\alpha < \omega_1$ such that for all $\beta \geq \alpha$, for all $\vec{a} \in A$,

$$\mathfrak{A} \models \forall \vec{x}(\varphi^{\alpha}_{\vec{a}}(\vec{x}) \leftrightarrow \varphi^{\beta}_{\vec{a}}(\vec{x})).$$

Lemma 3.4. *In a countable model $\mathfrak{A}$, the Scott rank of $\vec{a} \in \mathfrak{A}$ exists.*

Proof. As $\mathfrak{A}$ is countable, there are at most a countable number of tuples $\vec{b}$ the same length as $\vec{a}$. Suppose for each $\vec{b}$ there is some β such that $\mathfrak{A} \not\models \varphi^{\beta}_{\vec{a}}(\vec{b})$ - then there is a *smallest* β such that $\mathfrak{A} \not\models \varphi^{\beta}_{\vec{a}}(\vec{b})$ - call it $\beta_{\vec{b}}$. (If $\vec{b}$ satisfies $\varphi^{\beta}_{\vec{a}}$ for all β, we do not worry about that $\vec{b}$). Let $\alpha = sup_{\vec{b}}(\beta_{\vec{b}})$ - we claim this is the Scott rank of $\vec{a}$.
Let γ be the Scott rank of $\vec{a}$ as defined originally above. Suppose $\mathfrak{A} \models \varphi^{\alpha}_{\vec{a}}(\vec{b})$. Then by definition, for all $\delta < \alpha$, $\mathfrak{A} \models \varphi^{\delta}_{\vec{a}}(\vec{b})$. Suppose $\mathfrak{A} \not\models \varphi^{\beta}_{\vec{a}}(\vec{b})$ for $\beta > \alpha$. Then $\beta_{\vec{b}} > \alpha$, a contradiction. Thus for **all** β, $\mathfrak{A} \models \forall \vec{x}(\varphi^{\alpha}_{\vec{a}}(\vec{x}) \rightarrow \varphi^{\beta}_{\vec{a}}(\vec{x}))$ meaning $\gamma \leq \alpha$.

Fix $\vec{b} \in \mathfrak{A}$ with a corresponding β which is the *smallest* β such that $\mathfrak{A} \not\models \varphi_{\vec{a}}^{\beta}(\vec{b})$. Thus if $\gamma < \beta$ then $\mathfrak{A} \models \varphi_{\vec{a}}^{\gamma}(\vec{b})$ and thus by modus ponens $\mathfrak{A} \models \varphi_{\vec{a}}^{\beta}(\vec{b})$; a contradiction. Thus $\gamma \geq \beta$ - taking the supremum over $\vec{b}$, we get $\gamma \geq \alpha$.
Thus we conclude $\gamma = \alpha$; in particular since α can be constructed, γ, the Scott rank of $\vec{a}$, exists, as required. ■

In the next theorem, *Theorem 3.6*, we prove every countable structure has a Scott sentence. We show this by proving every countable L-structure satisfies the following sentence, known as *the* Scott sentence (for a structure):

Definition 3.5. The *Scott sentence* φ *for* $\mathfrak{A}$ is the following sentence:

$$\varphi := \varphi_0^{\alpha} \ \wedge \bigwedge_{n<\omega, \vec{a} \in A} \forall \vec{x}(\varphi_{\vec{a}}^{\alpha}(\vec{x}) \rightarrow \varphi_{\vec{a}}^{\alpha+1}(\vec{x})).$$

Here the conjunction ranges over all $n < \omega$ and all tuples $\vec{a} \in A$ of length n, and α is the Scott rank of $\mathfrak{A}$.

Note that we have not yet shown the Scott sentence for $\mathfrak{A}$ is indeed a Scott sentence of $\mathfrak{A}$. The proof of this fact follows in *Theorem 3.6.*
The following proof originates with Scott [4, pp. 329-341].

Theorem 3.6. *If* $\mathfrak{A}$ *is a countable* L*-structure, then it has a Scott sentence.*

Proof. Let φ be the Scott sentence for $\mathfrak{A}$, as defined in *Definition 3.5.* We shall see $\mathfrak{A} \models \varphi$ by definition (note the reason we chose α to be the *Scott rank of* $\mathfrak{A}$ was so $\mathfrak{A} \models \forall \vec{x}(\varphi_{\vec{a}}^{\alpha}(\vec{x}) \rightarrow \varphi_{\vec{a}}^{\alpha+1}(\vec{x}))$ for all $\vec{a} \in A$.
We can prove $\mathfrak{A} \models \varphi_0^{\alpha}$ inductively;
For the base case,$0 \leq \alpha$:

$$\varphi_0^0 \text{ is the conjunction of all quantifier free sentences, so } \mathfrak{A} \models \varphi_0^0$$

For successor ordinals $\beta + 1 \leq \alpha$:

$$\mathfrak{A} \models \left(\forall x \bigvee_b \varphi_b^{\beta}(x)\right) \& \left(\bigwedge_b \exists x \ \varphi_b^{\beta}(x)\right) \text{ naturally so } \mathfrak{A} \models \varphi_0^{\beta} \Rightarrow \mathfrak{A} \models \varphi_0^{\beta+1}$$

For limit ordinals $\beta \leq \alpha$:

$$\varphi_0^{\beta} = \bigwedge_{\gamma<\beta} \varphi_0^{\gamma} \text{ and if for all } \gamma < \beta,\, \mathfrak{A} \models \varphi_0^{\gamma} \text{ then } \mathfrak{A} \models \varphi_0^{\beta}$$

Suppose $\mathfrak{B} = (B, ...)$ is countable and $\mathfrak{B} \models \varphi$. We show $\mathfrak{A} \cong \mathfrak{B}$ by a back-and-forth argument (thus proving φ is indeed a Scott sentence for $\mathfrak{A}$).

Let $\vec{a} \in A$ and $\vec{b} \in B$ with $\mathfrak{B} \models \varphi^{\alpha}_{\vec{a}}(\vec{b})$. Since $\mathfrak{B} \models \varphi$, $\mathfrak{B} \models \varphi^{\alpha+1}_{\vec{a}}(\vec{b})$ by modus ponens. Thus $\mathfrak{B} \models (\exists x_{n+1})\varphi^{\alpha}_{\vec{a},a_{n+1}}(\vec{b}, x_{n+1})$ for all $a_{n+1} \in A$ by definition so

$$(\forall a_{n+1} \in A)(\exists b_{n+1} \in B)\mathfrak{B} \models \varphi^{\alpha}_{\vec{a},a_{n+1}}(\vec{b}, b_{n+1}) \tag{1}$$

Also note since $\mathfrak{B} \models \varphi^{\alpha+1}_{\vec{a}}(\vec{b})$ then $\mathfrak{B} \models (\forall x_{n+1}) \bigvee_{a_{n+1}\in A} \varphi^{\alpha}_{\vec{a},a_{n+1}}(\vec{b}, x_{n+1})$ so for some $a_{n+1} \in A$, $\mathfrak{B} \models \varphi^{\alpha}_{\vec{a},a_{n+1}}(\vec{b}, b_{n+1})$ which is to say

$$(\forall b_{n+1} \in B)(\exists a_{n+1} \in A)\mathfrak{B} \models \varphi^{\alpha}_{\vec{a},a_{n+1}}(\vec{b}, b_{n+1}) \tag{2}$$

These two conditions (1), (2) give $\mathfrak{A} \cong \mathfrak{B}$ by induction on the indices n as we can find a $b_n \in B$ to map an $a_n \in A$ to, and then find an $a_{n+1} \in A$ to map a $b_{n+1} \in B$ to, and so on, by a back-and-forth argument. Note the base case for this induction is $n = 0$, i.e. $\mathfrak{B} \models \varphi^{\alpha}_{0}$ which is immediately true as $\mathfrak{B} \models \varphi$.

For the other direction, note that isomorphism of models preserves truth; so if $\mathfrak{B} \cong \mathfrak{A}$ then $\mathfrak{B} \models \varphi$ as required.

We conclude by definition φ is a Scott sentence, and every countable structure thus has a Scott sentence, as required. ■

Remark 3.7. For a model $\mathfrak{A}$, $\mathrm{Th}(\mathfrak{A})$ is the collection of elementary first order sentences true in $\mathfrak{A}$, which means a Scott sentence φ of $\mathfrak{A}$ is not necessarily in its theory. So if $\mathfrak{A}$ and $\mathfrak{B}$ are two countable structures, $\mathfrak{A} \equiv \mathfrak{B} \not\Rightarrow \mathfrak{A} \cong \mathfrak{B}$. An example of this;
Let $\mathfrak{A} = (\omega, +, S, 0)$ and $\Gamma = \mathrm{Th}(\mathfrak{A}) \bigcup \{c \neq 0, c \neq S0, \cdots\}$ - using the Compactness Theorem (we are working with elementary first order sentences here) we can construct a model $\mathfrak{B}$ of Γ, which is a nonstandard model of arithmetic. A Scott sentence φ of $\mathfrak{A}$ can include an infinitary sentence

$$\forall x(x = 0 \vee x = S0 \vee x = S^2 0 \vee \cdots)$$

saying every element of $\mathfrak{A}$ is some successor of 0; however this will not be true in $\mathfrak{B}$ as some element in it will witness the negation of this, by design. So since $\varphi \notin \mathrm{Th}(\mathfrak{A})$, it will still be true $\mathfrak{A} \equiv \mathfrak{B}$ but $\mathfrak{A} \not\cong \mathfrak{B}$. ♢

Lemma 3.8. *Let $\vec{a}$ be a tuple in $\mathfrak{A}$ and suppose α is the Scott rank of $\mathfrak{A}$. Then $\varphi^{\alpha}_{\vec{a}}(\vec{x})$ defines the orbit of $\vec{a}$ in $\mathfrak{A}$; that is, for all elementary first order formulae $\phi(\vec{x})$ true of $\vec{a}$, $\mathfrak{A} \models \forall \vec{x}(\varphi^{\alpha}_{\vec{a}}(\vec{x}) \to \phi(\vec{x}))$.*

Proof. Let $\vec{b} \in \mathfrak{A}$ such that $\mathfrak{A} \models \varphi^{\alpha}_{\vec{a}}(\vec{b})$. We want to create an automorphism of $\mathfrak{A}$ taking $\vec{a}$ to $\vec{b}$; we will do so by a back-and-forth argument.

Let $c \in \mathfrak{A}$. From the definition of Scott rank, $\mathfrak{A} \models \varphi^{\alpha+1}_{\vec{a}}(\vec{b})$.

In particular $\mathfrak{A} \models \forall y \bigvee_{d\in A} \varphi^{\alpha}_{\vec{a},d}(\vec{b}, y)$, thus if $y = c$, for some $c' \in \mathfrak{A}$, $\mathfrak{A} \models \varphi^{\alpha}_{\vec{a},c'}(\vec{b}, c)$. Therefore

$$\forall c\ \exists c'\ \mathfrak{A} \models \varphi^{\alpha}_{\vec{a},c'}(\vec{b}, c)$$

i.e. condition (2).

Note that as $\mathfrak{A} \models \bigwedge_{b \in A} \exists y \varphi^{\alpha}_{\vec{a},b}(\vec{b}, y)$, in particular for some d, $\mathfrak{A} \models \exists y \varphi^{\alpha}_{\vec{a},d}(\vec{b}, y)$. Let d' witness this. Therefore

$$\forall d \; \exists d' \; \mathfrak{A} \models \varphi^{\alpha}_{\vec{a},d}(\vec{b}, d')$$

i.e. condition (1).

As $\mathfrak{A}$ is countable, together these give conditions (1) and (2) mean there is an automorphism of $\mathfrak{A}$ taking $\vec{a}$ to $\vec{b}$ by a back-and-forth argument. Thus $\mathfrak{A} \models \phi(\vec{b})$ (as the isomorphism will preserve truth) so we can conclude $\varphi^{\alpha}_{\vec{a}}(\vec{x})$ defines the orbit of $\vec{a}$, as required. ■

Example 3.9. A vector space of dimension n**.** Let $\mathfrak{A} = (V, 0, +, -, (*)_{q \in \mathbb{Q}})$ be a structure. Let ϕ be a sentence encapsulating the axioms of a vector space (the behaviour of addition, scalar multiplication, 0, distribution, closure, etc); $\phi \in \mathrm{Th}(\mathfrak{A})$.

To say there are n linearly independent vectors;

$$\psi(x_1, \ldots, x_n) = \left(\bigwedge_{q_1, \ldots, q_n \in \mathbb{Q}} q_1 * x_1 + \cdots + q_n * x_n = 0 \leftrightarrow (q_1 = 0 \wedge \cdots \wedge q_n = 0) \right).$$

To say n vectors span the space;

$$\xi(x_1, \ldots, x_n) = \left(\forall y \bigvee_{q_1, \ldots, q_n \in \mathbb{Q}} y = q_1 * x_1 + \cdots + q_n * x_n \right).$$

All together,

$$\varphi = \phi \wedge \exists x_1, \ldots, x_n (\psi(x_1, \ldots, x_n) \wedge \xi(x_1, \ldots, x_n))$$

captures $\mathfrak{A}$ up to isomorphism. Therefore φ is a Scott sentence of $\mathfrak{A}$. ◊

We can begin to explore the problem of Scott sentences in larger models with the following examples:

1 *Is it necessary that every Scott sentence has an uncountable model?*

1A No. Let φ be the sentence $\exists x_1, x_2 (x_1 \neq x_2) \wedge \neg \exists y (y \neq x_1 \wedge y \neq x_2))$. If $\mathfrak{A} \models \varphi$, it must be finite and thus its Scott sentence *cannot* have an uncountable model.

2 *Can a Scott sentence have arbitrarily large models?*

2A Yes. Let φ be the sentence describing a dense linear order. If $\mathfrak{A} \models \varphi$ then a Scott sentence of $\mathfrak{A}$ will have arbitrary large models, as required.

For a sentence ϕ of $L_{\omega_1,\omega}$ in general we can still place restrictions hampering the size of models of ϕ.

3 *Is it necessary for ϕ to have arbitrarily large models?*

3A No - consider the following example of $\phi \in L_{\omega_1,\omega}$ with a model of size $\aleph_1$ but no larger model.

Let $\mathfrak{A} = (A, <^*, F, (c)_{i<\omega}, <, S, +)$ be a structure, where A is a set of cardinality $\aleph_1$. Suppose there is an infinitary sentence giving conditions for the constants $(c)_{i<\omega}$ to form a copy of $(\omega, <, S, +)$ - call it U - in $\mathfrak{A}$.

Suppose the rest of the elements not named by constants form a dense linear order under $<^*$, completely separate to U. Denote this ordered set by V. Let F be a binary function taking $b \in V$ and a *predecessor* of b to U where F maps distinct predecessors to distinct elements of U. Note that for any arbitrary $b \in V$ there are a countable number of elements before it in the ordering. Let φ be the $L_{\omega_1,\omega}$ sentence describing U, V and F; $\mathfrak{A}$ is a model of φ of size $\aleph_1$.

Suppose there is a model of φ of size $\aleph_2$; then $\|V\| = \aleph_2$. However this implies there exists some $b \in V$ with $\aleph_1$ many predecessors, so here F cannot map distinct predecessors to distinct elements of U. Therefore φ cannot have a model of size $\aleph_2$, as required.

Remark 3.10. We will try avoid structures such as this in Main Theorem II (*Theorem 12.1*) where we construct a model of size $\aleph_2$ for a Scott sentence of a countable model. $\Diamond$

3A Consider the following sentence of $L_{\omega_1,\omega}$ with a model of size $2^{\aleph_0}$ but no larger model.

Let $\mathfrak{B} = (B, U, (c_i)_{i<\aleph_0}, \in^*)$ be a structure, with domain B, predicate U, constants $(c_i)_{i<\aleph_0}$ and binary relation $\in^*$. We will make U a copy of ω inside $\mathfrak{B}$, then consider all subsets of U. Suppose $\mathfrak{B}$ is a model of the following sentences:

$$\bigwedge_i \left(\bigwedge_{j \neq i} c_i \neq c_j \right) \tag{3}$$

$$\forall x \left(Ux \leftrightarrow \left(\bigvee_i x = c_i \right) \right) \tag{4}$$

$$\forall x, y (x \in^* y \rightarrow Ux \wedge \neg Uy) \tag{5}$$

$$\forall x, y \left((x \neq y \wedge \neg Ux \wedge \neg Uy) \rightarrow \exists u (u \in^* x \leftrightarrow u \notin^* y) \right) \tag{6}$$

where

(3) means "the constants c_i are all distinct".

(4) means "every element of U is given by a constant".

(5) describes how $\in^*$ holds between two elements.

(6) means "if x, y not in U are nonequal, they disagree on some U-element".

Let ϕ be the conjunction of (3)-(6).

Every element of $\mathfrak{B}$ not in U corresponds to a subset of ω. Since $\|P(\omega)\| = 2^{\aleph_0}$, ϕ has a model of size $2^{\aleph_0}$ but no larger, as required.

4 Elementary substructures, homogeneity and types

In this section we give the background information and some definitions used throughout this paper.

Definition 4.1. Let $\mathfrak{M}$ and $\mathfrak{N}$ be L-structures. A map $h : |\mathfrak{M}| \to |\mathfrak{N}|$ is called *elementary* if it preserves the validity of elementary first order formulae $\varphi(\vec{x})$, that is;

$$\forall \vec{a} \in \mathfrak{M}, \mathfrak{M} \models \varphi(\vec{a}) \Leftrightarrow \mathfrak{N} \models \varphi(\vec{a})$$

Definition 4.2. Let $\mathfrak{M}$ be a substructure of $\mathfrak{N}$. Then $\mathfrak{M}$ is an *elementary substructure* of $\mathfrak{N}$ (written $\mathfrak{M} \preceq \mathfrak{N}$) if the inclusion map is elementary. Here, $\mathfrak{N}$ is called an elementary extension of $\mathfrak{M}$.

The following theorem is a criterion for elementary substructures:

Theorem 4.3. Tarski's Criterion. *$\mathfrak{M} \preceq \mathfrak{N}$ if and only if for every L-formula $\varphi(\vec{u}, x)$ and for all $\vec{m} \in \mathfrak{M}$, if there exists $a \in \mathfrak{N}$ such that $\mathfrak{N} \models \varphi(\vec{m}, a)$ then there exists $b \in \mathfrak{M}$ such that $\mathfrak{N} \models \varphi(\vec{m}, b)$.*

Remark 4.4. An elementary substructure is not the same as having a substructure elementarily equivalent to its superstructure.

For example, take a dense linear ordering with one endpoint, e.g. let $\mathfrak{A} = \mathbb{Q}$ with a point 1^* smaller than every element, have a substructure $\mathfrak{B} = \{a \in \mathfrak{A} : a > 1\} \cup \{1\}$.

$\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$ (same theory) however $\mathfrak{B}$ is not an *elementary* substructure of $\mathfrak{A}$ as $\mathfrak{A} \models \exists x(x < 1)$ however $\mathfrak{B} \not\models \exists x(x < 1)$ by definition. ♢

Definition 4.5. Let $\mathfrak{A}$ be an L-structure. For $\vec{a} \in \mathfrak{A}$, the *type* of $\vec{a}$ (denoted $\mathrm{tp}(\vec{a})$) is the set of all formulae $\varphi(\vec{x})$ with $\mathfrak{A} \models \varphi(\vec{a})$. Furthermore,

$$\vec{x} \equiv \vec{y} \Leftrightarrow \mathrm{tp}(\vec{x}) = \mathrm{tp}(\vec{y}).$$

Definition 4.6. An n-type (of $\mathfrak{A}$) is a set of formulae $p(x_1, \ldots, x_n)$, each having free variables only occurring amongst $x_1, \ldots, x_n$ such that for every finite subset $p_0(x_1, \ldots, x_n)$ there exists $\vec{b} = (b_1, \ldots, b_n) \in \mathfrak{A}$ such that $\mathfrak{A} \models p_0(\vec{b})$.

Definition 4.7. A complete type $p(\vec{x})$ in variables $\vec{x} = (x_1, \ldots, x_n)$ contains $\varphi(\vec{x})$ or $\neg\varphi(\vec{x})$ for every elementary first order formula $\varphi(\vec{x})$ in the variables $x_1, \ldots, x_n$.

Definition 4.8. A countable structure $\mathfrak{A}$ is (ω-)homogeneous if for any $\vec{a}$, $\vec{b} \in \mathfrak{A}$ such that $\vec{a}, \vec{b}$ satisfy the same elementary first order formulae there is an automorphism of $\mathfrak{A}$ taking $\vec{a}$ to $\vec{b}$.

Definition 4.9. Let $(\mathfrak{A}, U)$ be a pair where $\mathfrak{A}$ is a countable structure and U is a predicate. $(\mathfrak{A}, U)$ is *pair*-homogeneous if, given $\vec{a}, \vec{b}, c$ such that $\vec{a}$ and $\vec{b}$ realise the same type in $(\mathfrak{A}, U)$, there exists $d \in \mathfrak{A}$ such that $(\vec{a}, c)$ and $(\vec{b}, d)$ realise the same type in $(\mathfrak{A}, U)$.

Remark 4.10. *Definition 4.9* is similar to *Definition 4.8* as, by a back-and-forth argument on *Definition 4.9* there is an automorphism of $(\mathfrak{A}, U)$ taking $\vec{a}$ to $\vec{b}$. $\Diamond$

5 Atomic models

In this section we define and prove many useful results about atomic models which will be key in later parts of this paper. First, we define a *principal* type:

Definition 5.1. A type $p(\vec{x})$ is *principal* (with respect to a theory T) if there is a formula $\gamma(\vec{x}) \in p(\vec{x})$ such that

$$\forall \alpha(\vec{x}) \in p(\vec{x}), \qquad T \vdash \forall \vec{x}(\gamma(\vec{x}) \to \alpha(\vec{x})).$$

$\varphi(\vec{x})$ is known as a *generating* formula for $p(\vec{x})$.

Definition 5.2. An *atomic* model is one where the complete type of every tuple is principle.

Lemma 5.3. *Atomic models are homogeneous.*

Proof. Note that as $T = \mathrm{Th}(\mathfrak{A})$ is complete and has an atomic model, for all formulae $\varphi(\vec{x})$,

$$T \vdash \forall \vec{x}(\gamma(\vec{x}) \to \varphi(\vec{x})) \qquad \text{or} \qquad T \vdash \forall \vec{x}(\gamma(\vec{x}) \to \neg\varphi(\vec{x}))$$

where $\gamma(\vec{x})$ is a type generator. Let $\vec{a}, \vec{b} \in \mathfrak{A}$ satisfy the same elementary first order formulae; thus, they satisfy the same types. Suppose the type of $\vec{a}$ and $\vec{b}$ is generated by $\gamma_0(\vec{x})$. Let c be an element of $\mathfrak{A}$.
The type of $(\vec{b}, c)$ is principal and generated by $\gamma_1(\vec{x}, y)$ (say). As $\mathfrak{A} \models \gamma_0(\vec{b})$ and $\mathfrak{A} \models \gamma_0(\vec{a})$ and $T \vdash \exists y(\gamma_1(\vec{b}, y))$, then

$$T \vdash \forall \vec{x}(\gamma_0(\vec{x}) \to \exists y(\gamma_1(\vec{x}, y)))$$

so in particular $T \vdash \exists y(\gamma_1(\vec{a}, y))$ and there exists $d \in \mathfrak{A}$ witnessing this.
We map $\vec{a}$ to $\vec{b}$, then $(\vec{b}, c)$ to $(\vec{a}, d)$ and continue. By a back-and-forth argument we can conclude $\mathfrak{A}$ is homogeneous, as required. ■

Lemma 5.4. *Two countable structures that realise the same types and are each homogeneous are isomorphic.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two such structures. We may write $\mathfrak{A} = \{a_i : i < \omega\}$ and $\mathfrak{B} = \{b_i : i < \omega\}$. We wish to create a back-and-forth family to prove $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Suppose we are given a partial isomorphism

$$f_m : \{a_1, \ldots, a_m\} \to \{b_1, \ldots, b_m\} \qquad m \geq 1.$$

Consider $a_{m+1} \in \mathfrak{A}$. By the hypothesis there exists a tuple $(d_1, \ldots, d_{m+1}) \in \mathfrak{B}$ realising the same type as $(a_1, \ldots, a_{m+1})$. As $(d_1, \ldots, d_m)$ realises the same type as $\vec{b}$, and $\mathfrak{B}$ is homogeneous, there is a (truth preserving) isomorphism h such that $h(d_1, \ldots, d_m) = \vec{b}$.

Let $b_{m+1} = h(d_{m+1})$. We conclude $(a_1, \ldots, a_{m+1})$ and $(b_1, \ldots, b_{m+1})$ realise the same type thus the partial isomorphism f_m can be extended to

$$f_m^+ : \{a_1, \ldots, a_{m+1}\} \to \{b_1, \ldots, b_{m+1}\}$$

as desired.

Similarly given $b_{m+1} \in \mathfrak{B}$, using the homogeneity of $\mathfrak{A}$ we can extend

$$g_m : \{b_1, \ldots, b_m\} \to \{a_1, \ldots, a_m\}$$

to

$$g_m^+ : \{b_1, \ldots, b_{m+1}\} \to \{a_1, \ldots, a_{m+1}\}.$$

Therefore $\mathfrak{A} \cong \mathfrak{B}$ by *Theorem 2.3*, as required. ∎

Theorem 5.5. *If T is a complete, elementary first order theory with a countable atomic model, then that model is unique (up to isomorphism).*

Proof. Let $\mathfrak{A}$ be an atomic model with $\mathrm{Th}(\mathfrak{A}) = T$. By *Lemma 5.3*, $\mathfrak{A}$ is homogeneous. Let $\mathfrak{B}$ be another atomic model of T. Then $\mathfrak{A} \equiv \mathfrak{B}$ (through T). If $p(\vec{x})$ is a type realised by $\vec{a} \in \mathfrak{A}$, generated by $\phi(\vec{x})$, then $\mathfrak{A} \models \exists \vec{x}(\phi(\vec{x}))$. Thus $\mathfrak{B} \models \exists \vec{x}(\phi(\vec{x}))$ so $\phi(\vec{x})$ is realised by some $\vec{b} \in \mathfrak{B}$. Therefore $\vec{b}$ realises $p(\vec{x})$. Therefore $\mathfrak{B}$ realises all the types realised in $\mathfrak{A}$.

Note $\mathfrak{B}$ realises no new types as it is elementarily equivalent to $\mathfrak{A}$.

We can conclude $\mathfrak{B}$ realises the same types as $\mathfrak{A}$, and vice versa. Thus $\mathfrak{A} \cong \mathfrak{B}$, by *Lemma 5.4*. ∎

Corollary 5.6. *If $\mathfrak{A}$, $\mathfrak{B}$ are countable, atomic and elementary equivalent, they are isomorphic.* ∎

We have seen that atomic models have many properties relating to when they are isomorphic. We will now introduce Scott sentences into the mix:

Remark 5.7. Note that if $\mathfrak{A}$ is a countable atomic model, then it has a Scott sentence φ that is the conjunction of $\mathrm{Th}(\mathfrak{A})$ and sentences saying

$$\forall \vec{x} \left(\bigvee_i \gamma_i(\vec{x}) \right) \tag{7}$$

where the γ_i are the generators for the complete atomic types consistent with T.

(Recall a Scott sentence was an infinitary sentence that could identify a countable structure up to isomorphism. If $\mathfrak{B}$ is another countable structure with $\mathfrak{B} \models \varphi$, then $\mathfrak{B} \cong \mathfrak{A}$ by *Theorem 5.5*. Thus φ is indeed a Scott sentence of $\mathfrak{A}$.) $\Diamond$

Theorem 5.8. *Suppose $\mathfrak{A}$, $\mathfrak{B}$ are atomic models for the same countable elementary first order theory, where $\mathfrak{A}$ is countable. Then $\mathfrak{B} \models \varphi$, where φ is a Scott sentence of $\mathfrak{A}$.*

Proof. Let φ be the Scott sentence in *Remark 5.7.* This is a Scott sentence of $\mathfrak{A}$. Since $\mathfrak{B}$ is atomic and $\mathfrak{B} \models \mathrm{Th}(\mathfrak{A})$, we can conclude $\mathfrak{B} \models \varphi$, as required. ■

Theorem 5.8 can be used in conjunction with the following theorem to gain a more complete picture of atomic models:

Theorem 5.9. *Let $\mathfrak{A}$ be a countable atomic model of its theory. If $\mathfrak{B}$ satisfies* some *Scott sentence of $\mathfrak{A}$, then $\mathfrak{B}$ is an atomic model of* $\mathrm{Th}(\mathfrak{A})$.

Proof. Let ψ be the Scott sentence whose models are atomic models of $\mathrm{Th}(\mathfrak{A})$. Let φ be another Scott sentence of $\mathfrak{A}$, with $\mathfrak{B} \models \varphi$.

If $\mathfrak{B}$ is countable, by the definition of a Scott sentence, $\mathfrak{B} \cong \mathfrak{A}$, meaning $\mathfrak{B} \models \psi$. Thus $\mathfrak{B}$ is an atomic model of $\mathrm{Th}(\mathfrak{A})$, as required.

Suppose $\mathfrak{B}$ is uncountable. Let L be the language of $\mathfrak{A}$. Expand $\mathfrak{B}$, adding to L predicates for all (first order) subformulae of φ and ψ. Let $\mathfrak{B}^*$ be this expansion. By the Downward Löwenheim Skolem Tarksi Theorem $\mathfrak{B}^*$ has a countable, elementary substructure. Call this substructure $\mathfrak{B}_0^*$. Note as $\mathfrak{B}^* \models \varphi$, we have $\mathfrak{B}_0^* \models \varphi$. The reduct of $\mathfrak{B}_0^*$ to L, which we call $\mathfrak{B}_0$, then satisfies φ. As φ is a Scott sentence of $\mathfrak{A}$, $\mathfrak{B}_0 \cong \mathfrak{A}$. Therefore $\mathfrak{B}_0 \models \psi$, i.e. $\mathfrak{B}_0$ is an atomic model of $\mathrm{Th}(\mathfrak{A})$.

If $\mathfrak{B}^*$ failed to satisfy ψ, then $\mathfrak{B}_0^*$ would fail to satisfy ψ. Thus if $\mathfrak{B}$ failed to satisfy ψ, then $\mathfrak{B}_0$ would fail to satisfy ψ. By the contrapositive, $\mathfrak{B} \models \psi$. Therefore $\mathfrak{B}$ is an atomic model of $\mathrm{Th}(\mathfrak{A})$, as required. ■

Remark 5.10. If $\mathfrak{A}$ and $\mathfrak{B}$ are not atomic, then it is not necessarily true that $\mathfrak{A} \equiv \mathfrak{B}$ means $\mathfrak{A} \cong \mathfrak{B}$. A counterexample;

Consider $\mathrm{Th}(\omega, <)$. This has models of the form $\omega + \mathbb{Z} * \rho$, which is ω followed by ρ many $\mathbb{Z}$ chains. Let $\mathfrak{A} = \omega + \mathbb{Z} * \omega$ and $\mathfrak{B} = \omega + \mathbb{Z} * \omega^*$, where ω^* is ω but in reverse;

$$\cdots < 3^* < 2^* < 1^* < 0^*.$$

Then $\mathfrak{A} \equiv \mathfrak{B}$ and they realise the same types, however they are not isomorphic. $\Diamond$

6 Conditions for Main Theorem I

Let T^* be a set of elementary first order sentences in the language of $\mathfrak{A}$ with an added unary predicate symbol U, saying the following:

(a) $(\exists x)\neg Ux$

(b) For each formula $\varphi(\vec{u}, x)$ with parameters $\vec{u} = (u_1, \ldots, u_n)$ such that Uu_i for $1 \leq i \leq n$, we have
$$\exists x(\varphi(\vec{u}, x)) \to \exists x(Ux \wedge \varphi(\vec{u}, x)).$$

Main Theorem I (*Theorem 8.1*) states if $\mathfrak{A}$ is a countable model which can be expanded to a model of T^*, then a Scott sentence of $\mathfrak{A}$ will have a model $\mathfrak{M}$ of size $\aleph_1$.

We will prove this by building a chain of models (*Construction 6.3*), the union of which will be the desired $\mathfrak{M}$. *Theorem 6.1* will provide us with the first step in this proof:

Theorem 6.1. *Suppose $\mathfrak{A}$ is a countable atomic model of its theory in a language L and has an expansion $\mathfrak{A}^* = (\mathfrak{A}, U^{\mathfrak{A}^*})$ satisfying T^*. The substructure formed by restricting $\mathfrak{A}^*$ to $U^{\mathfrak{A}^*}$ is isomorphic to $\mathfrak{A}$.*

Proof. Let $\mathfrak{B}$ be the substructure with $|\mathfrak{B}| = U^{\mathfrak{A}^*}$. From condition (b) of T^*, it is an elementary substructure of $\mathfrak{A}$ by Tarski's Criterion (*Theorem 4.3*). Therefore $\mathfrak{B}$ is atomic. If $T = \mathrm{Th}(\mathfrak{A})$, then as $\mathfrak{A}$ and $\mathfrak{B}$ are countable atomic models of T, they are isomorphic by *Theorem 5.5*. Thus $\mathfrak{B} \cong \mathfrak{A}$ as required. ■

The following definition will be crucial in our construction:

Definition 6.2. For L-structures $\mathfrak{C} \subset \mathfrak{B} \subset \mathfrak{A}$, define $(\mathfrak{A}, \mathfrak{B}) \cong (\mathfrak{B}, \mathfrak{C})$ if and only if there is an isomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $f \restriction_{\mathfrak{B}}$ is an isomorphism from $\mathfrak{B} \subset \mathfrak{A}$ to $\mathfrak{C} \subset \mathfrak{B}$.

We will now outline the process of building the aforementioned chain of models. The following construction is pivotal to the rest of the paper and will be referenced many times:

Construction 6.3. Let $D_0 = \mathfrak{B}$ and $D_1 = \mathfrak{A}$. As $D_1 \cong D_0$ we wish to construct D_2 with D_1 a substructure and $(D_2, D_1) \cong (D_1, D_0)$.

Let f be the isomorphism $f : D_1 \to D_0$. Let D be a set with $\|D\| = \|D_1\|$ and $D_1 \subsetneq D$. Let D_2 be a structure with the same language as D_1 and $|D_2| = D$. As f is a bijection from D_1 to D_0, we can extend this to a bijection g of D_2 to D_1. Impose the following on D_2; for all $\vec{a}, \vec{d} \in D_1$, for all formulae φ,
$$D_2 \models \varphi(g^{-1}(\vec{a}), g^{-1}(\vec{d})) \qquad \Leftrightarrow \qquad D_1 \models \varphi(\vec{a}, \vec{d})$$

As relations, functions and constants are now preserved (by defining the interpretation of constants, relations and functions on D_2 to be whatever works under g^{-1}) $D_2 \cong D_1$ under g. Also since g is an extension of f, $g \restriction_{D_1} = f$ so $(D_2, D_1) \cong (D_1, D_0)$ as required.
Continuing on like this, we get a chain of structures $D_0 \subset D_1 \subset D_2 \subset \cdots$ with $D_\omega = \bigcup_{n<\omega} D_n$. ◊

We want to ensure that the union of our chain still has nice properties, as the following results show:

Lemma 6.4. *The union of a countable elementary chain of countable homogeneous structures is homogeneous.*

Proof. Let $(A_\alpha)_{\alpha<\omega}$ be a chain of countable homogeneous structures with union A_ω. Note that A_ω is countable. Let $\vec{a}, \vec{b}$ be n-tuples in A_ω with $tp(\vec{a}) = tp(\vec{b})$.
Let c be an element of A_ω. There is some k large enough such that $\vec{a}, \vec{b}, c \in A_k$ and since A_k is homogeneous, there is a $d \in A_k \subset A_\omega$ with $tp(\vec{a}, c) = tp(\vec{b}, d)$. Thus by a back-and-forth argument there is an automorphism of A_ω taking $\vec{a}$ to $\vec{b}$. Therefore A_ω is homogeneous, as required. ∎

Theorem 6.5. $D_\omega \cong D_0$.

Proof. D_ω is homogeneous immediately from *Lemma 6.4*. Note that for any tuple $\vec{b} \in D_\omega$ there is an $n < \omega$ with $\vec{b} \in D_n$ and $D_n \cong D_0$. Suppose D_ω does not realise the same types as D_0 - it then realises some extra type. But the realisation of this type is in some D_n, isomorphic to D_0 - a contradiction.
Thus by *Lemma 5.4*, $D_\omega \cong D_0$ as required. ∎

Remark 6.6. *Lemma 6.4* and *Theorem 6.5* did not use any properties of T^* or any properties of any D_α. ♢

Definition 6.7. Define a chain $(A_i)_{i\in I}$ to be *elementary* if for some ordering $\leq$ on I,

$$i \leq j \Rightarrow A_i \preceq A_j.$$

Lemma 6.8. *Suppose $(A_\alpha)_{\alpha<\gamma}$ is an elementary chain of atomic structures, where γ is a limit ordinal. Then $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ is atomic.*

Proof. Assume A_γ is not atomic - then there is some complete type $p(\vec{x})$ not principal. Then there is some β large enough such that A_β realises $p(\vec{x})$. However A_β is atomic, so $p(\vec{x})$ must be principal - a contradiction. Thus A_γ is atomic, as required. ∎

Lemma 6.9. *If $(A_i)_{i\in I}$ is an elementary chain of structures and $\mathfrak{A} = \bigcup_{i\in I} A_i$, then $A_i \preceq \mathfrak{A}$ for all $i \in I$.*
This implies that $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(A_i)$ *for all i.*

Proof. We want to show by induction on formulae $\varphi(\vec{x})$ that for $i \in I$, $\vec{a} \in A_i$,

$$\mathfrak{A} \models \varphi(\vec{a}) \qquad \Leftrightarrow \qquad A_i \models \varphi(\vec{a}). \tag{8}$$

Fix $i \in I$ and $\vec{a} \in A_i$.

If $\varphi(\vec{a})$ is atomic, then $\mathfrak{A} \models \varphi(\vec{a}) \Leftrightarrow A_i \models \varphi(\vec{a})$ immediately.

If $\varphi(\vec{a}) = \phi(\vec{a}) \wedge \psi(\vec{a})$, where $\phi(\vec{a})$ and $\psi(\vec{a})$ satisfy (8), then

$$\mathfrak{A} \models \varphi(\vec{a}) \Leftrightarrow \mathfrak{A} \models \phi(\vec{a}) \wedge \psi(\vec{a}) \Leftrightarrow \mathfrak{A} \models \phi(\vec{a}) \ \& \ \mathfrak{A} \models \psi(\vec{a})$$

$$\Leftrightarrow A_i \models \phi(\vec{a}) \ \& \ A_i \models \psi(\vec{a}) \Leftrightarrow A_i \models \phi(\vec{a}) \wedge \psi(\vec{a}) \Leftrightarrow A_i \models \varphi(\vec{a})$$

If $\varphi(\vec{a}) = \neg\phi(\vec{a})$ where $\phi(\vec{a})$ satisfies (8), then

$$\mathfrak{A} \models \varphi(\vec{a}) \Leftrightarrow \mathfrak{A} \models \neg\phi(\vec{a}) \Leftrightarrow \mathfrak{A} \not\models \phi(\vec{a})$$

$$\Leftrightarrow A_i \not\models \phi(\vec{a}) \Leftrightarrow A_i \models \neg\phi(\vec{a}) \Leftrightarrow A_i \models \varphi(\vec{a})$$

If $\varphi(\vec{a}) = \exists x(\phi(\vec{a}, x))$, suppose $A_i \models \exists x(\phi(\vec{a}, x))$. Then there is a $b \in A_i$ such that $A_i \models \phi(\vec{a}, b)$ - by the induction hypothesis $\mathfrak{A} \models \phi(\vec{a}, b)$ so $\mathfrak{A} \models \exists x(\phi(\vec{a}, x))$. Conversely, if $\mathfrak{A} \models \exists x(\phi(\vec{a}, x))$ then there is some $j \in I$, $j \geq i$, $b \in A_j$ such that $A_j \models \phi(\vec{a}, b)$, thus $A_i \models \exists x(\phi(\vec{a}, x))$ since $A_i \preceq A_j$.

Thus, by induction on the complexity of formulae, we can conclude $A_i \preceq \mathfrak{A}$ for all i, as required. ■

7 Vaught's Two-Cardinal theorem

After setting up many crucial ideas in the previous section, in this section we present the ideas behind and proof of Vaught's Two-Cardinal theorem. This theorem and its proof will have many similarities to Main Theorem I in Section 8.

Definition 7.1. A structure $\mathfrak{A}$ in a language with a unary predicate symbol U is said to have type (κ, λ) if and only if $||\mathfrak{A}|| = \kappa$ and $||U|| = \lambda$.
A set of sentences T is said *to admit* (κ, λ) if and only if T has a model of type (κ, λ).
T has a *Vaughtian pair* if T admits (κ, λ) for some $\kappa > \lambda \geq \aleph_0$.

Definition 7.2. A structure $\mathfrak{A}$ is *recursively saturated* provided that for any computably enumerable set $\Gamma(\vec{a}, \vec{x})$ of formulae $\phi(\vec{a}, \vec{x})$ with parameters $\vec{a}$ in $\mathfrak{A}$, if every finite subset $\Gamma'(\vec{a}, \vec{x})$ is satisfied in $\mathfrak{A}$, then some tuple satisfies the whole set $\Gamma(\vec{a}, \vec{x})$.

We will use the following theorem without proof [1, Theorem 2.4.1];

Theorem 7.3. *Given a countable structure $\mathfrak{A}$ there is a countable recursively saturated elementary extension $\mathfrak{A}^+$.* ■

Theorem 7.4. Vaught's Two-Cardinal Theorem. *If Z is a countable elementary first order theory in a computable language L which admits (κ, λ) for some $\kappa > \lambda \geq \aleph_0$, then it admits $(\aleph_1, \aleph_0)$.*

Proof. Let $\mathfrak{A}$ be a model of Z of type (κ, λ) where V is the predicate of size λ. Let $\mathfrak{B}$ be an elementary substructure of $\mathfrak{A}$ of size λ with $V^{\mathfrak{B}} = V^{\mathfrak{A}}$. Define $\mathfrak{A}^* = (\mathfrak{A}, U)$ where U is a predicate such that $U^{\mathfrak{A}^*} = |\mathfrak{B}|$. By the Downward Löwenheim Skolem Tarski Theorem, there is a countable elementary substructure $\mathfrak{A}_0^* = (\mathfrak{A}_0, U^{\mathfrak{A}_0^*})$. If $U^{\mathfrak{A}_0^*} = |\mathfrak{B}_0|$ then the substructure $\mathfrak{B}_0 \preceq \mathfrak{A}_0$ and $V^{\mathfrak{B}_0} = V^{\mathfrak{A}_0}$ is countable.
Let ${\mathfrak{A}_0^*}^+ = (\mathfrak{A}_0^+, \mathfrak{B}_0^+)$ be a countable recursively saturated elementary extension of $\mathfrak{A}_0^*$.

Lemma 7.5. *$\mathfrak{A}_0^+, \mathfrak{B}_0^+$ realise the same types and are homogeneous.*

Proof. Let $\vec{a}$ be a tuple in $\mathfrak{A}_0^+$. Define

$$\Gamma(\vec{a}, \vec{x}) = \{\phi(\vec{a}) \leftrightarrow \phi(\vec{x}) \wedge U\vec{x} : \phi \text{ is a formula over } \mathfrak{A}_0^+\}.$$

This is computably enumerable. We show it is finitely satisfied;

Let $\Gamma'(\vec{a}, \vec{x})$ be a finite subset of $\Gamma(\vec{a}, \vec{x})$ involving $\phi_1, \ldots, \phi_k$. Let $\psi(\vec{x})$ be the conjunction of formulae $\pm\phi_i(\vec{x})$ true of $\vec{a}$. Since $\mathfrak{A}_0^+ \models \psi(\vec{a})$ then $\mathfrak{A}_0^+ \models \exists\vec{x}(\psi(\vec{x}))$ so $\mathfrak{B}_0^+ \models \exists\vec{x}(\psi(\vec{x}))$ (as $\mathfrak{B}_0^+ \preceq \mathfrak{A}_0^+$) so $\mathfrak{A}_0^+ \models \exists\vec{x}(\psi(\vec{x}) \wedge U\vec{x})$, as required.
Thus $\mathfrak{A}_0^+$ satisfies $\Gamma(\vec{a}, \vec{x})$ meaning there is some $\vec{b} \in \mathfrak{B}_0^+$ satisfying the same type as $\vec{a}$. As $\mathfrak{B}_0^+$ realises no new types, we can conclude $\mathfrak{A}_0^+, \mathfrak{B}_0^+$ realise the same types.

Let $\vec{a}, \vec{b}$ be tuples in $\mathfrak{A}_0^+$ realising the same type. Let c be an element in $\mathfrak{A}_0^+$. In order to prove homogeneity we need to show there is an element $d \in \mathfrak{A}_0^+$ such that $(\vec{a}, c)$ and $(\vec{b}, d)$ satisfy the same types. Define

$$\Gamma(\vec{a}, \vec{b}, c, x) = \{\phi(\vec{a}, c) \leftrightarrow \phi(\vec{b}, x) : \phi \text{ is a formula over } \mathfrak{A}_0^+\}.$$

Let $\Gamma'(\vec{a}, \vec{b}, c, x)$ be a finite subset of $\Gamma(\vec{a}, \vec{b}, c, x)$ involving $\phi_1, \ldots, \phi_k$. Let $\psi(\vec{u}, x)$ be the conjunction of formulae $\pm\phi_i(\vec{u}, x)$ true of $(\vec{a}, c)$. Since $\mathfrak{A}_0^+ \models \psi(\vec{a}, c)$ then $\mathfrak{A}_0^+ \models \exists x(\psi(\vec{a}, x))$ so $\mathfrak{A}_0^+ \models \exists x(\psi(\vec{b}, x))$, thus $\Gamma(\vec{a}, \vec{b}, c, x)$ is finitely satisfied. We can conclude $\mathfrak{A}_0^+$ satisfies $\Gamma(\vec{a}, \vec{b}, c, x)$ meaning there an element d such that $(\vec{a}, c)$ and $(\vec{b}, d)$ satisfy the same type. Therefore $\mathfrak{A}_0^+$ is homogeneous, as required.

Note that by choosing $\vec{a}, \vec{b}$ in $\mathfrak{B}_0^+$ realising the same type, and $c \in \mathfrak{B}_0^+$, and defining

$$\Gamma(\vec{a}, \vec{b}, c, x) = \{\phi(\vec{a}, c) \leftrightarrow \phi(\vec{b}, x) \wedge Ux : \phi \text{ is a formula over } \mathfrak{B}_0^+\}$$

by the same proof we can conclude $\mathfrak{B}_0^+$ is homogeneous.
Therefore $\mathfrak{A}_0^+, \mathfrak{B}_0^+$ realise the same types and are homogeneous, as required. ∎

Returning to the proof of Theorem 7.4:
By *Lemma 7.5* and *Lemma 5.4*, we can conclude $\mathfrak{A}_0^+ \cong \mathfrak{B}_0^+$. Note that $V^{\mathfrak{B}_0^+} = V^{\mathfrak{A}_0^+}$.

Set $D_0 = \mathfrak{B}_0^+$, $D_1 = \mathfrak{A}_0^+$ and construct the chain $(D_\alpha)_{\alpha<\omega_1}$ as in *Construction 6.3*; where at limit ordinals γ, $D_\gamma = \bigcup_{\beta<\gamma} D_\beta$ and at successor ordinals, $(D_{\alpha+1}, D_\alpha) \cong (D_1, D_0)$. Note for all $\alpha < \omega_1$, $D_\alpha \cong D_0$ (*Theorem 6.5*).
Finally set $\mathfrak{M} = \bigcup_{\alpha<\omega_1} D_\alpha$ which has cardinality $\aleph_1$ and

$$V^{\mathfrak{M}} = V^{D_\alpha} = V^{\mathfrak{A}_0^+} = V^{\mathfrak{B}_0^+},$$

making $V^{\mathfrak{M}}$ countable. Thus Z admits $(\aleph_1, \aleph_0)$, as required. ■

8 Main Theorem I

Recall from Section 6;

Let T^* be a set of elementary first order sentences in the language of $\mathfrak{A}$ with an added unary predicate symbol U, saying the following:

(a) $(\exists x)\neg Ux$

(b) For each formula $\varphi(\vec{u}, x)$ with parameters $\vec{u} = (u_1, \dots, u_n)$ such that Uu_i for $1 \leq i \leq n$, we have

$$\exists x(\varphi(\vec{u}, x)) \to \exists x(Ux \wedge \varphi(\vec{u}, x)).$$

In this section, we will prove that this set of conditions allows us to perform *Construction 6.3*, which, as in the proof of *Vaught's Two-Cardinal theorem* (*Theorem 7.4*) leads to an $\aleph_1$-sized model.

Theorem 8.1. Main Theorem I. *Let $\mathfrak{A}$ be a countable atomic model. Then a Scott sentence of $\mathfrak{A}$ has a model of cardinality $\aleph_1$ if $\mathfrak{A}$ can be expanded to a model of T^*.*

Proof. Let φ be a Scott sentence of $\mathfrak{A}$ and suppose $\mathfrak{A}$ can be expanded to a model of T^*. Let $\mathfrak{B}$ be a substructure of $\mathfrak{A}$ with $|\mathfrak{B}| = U^{\mathfrak{A}}$. By *Theorem 6.1*, $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

Set $D_0 = \mathfrak{B}$ and $D_1 = \mathfrak{A}$ and construct the chain $(D_\alpha)_{\alpha<\omega_1}$ by *Construction 6.3*, where at limit ordinals γ, $D_\gamma = \bigcup_{\beta<\gamma} D_\beta$ and at successor ordinals, $(D_{\alpha+1}, D_\alpha) \cong (D_1, D_0)$. Note for all $\alpha < \omega_1$, $D_\alpha \cong D_0$ (*Theorem 6.5*).

Set $\mathfrak{M} = \bigcup_{\alpha<\omega_1} D_\alpha$ which has cardinality $\aleph_1$. By *Lemma 6.8*, $\mathfrak{M}$ is atomic and by *Lemma 6.9*, $\mathfrak{M}$ is a model of $\mathrm{Th}(\mathfrak{A})$. Therefore by *Theorem 5.8 & Theorem 5.9*, $\mathfrak{M} \models \varphi$, as required. ■

We can also prove the converse of Main Theorem I:

Theorem 8.2. *Let $\mathfrak{A}$ be a countable atomic model. If a Scott sentence of $\mathfrak{A}$ has a model of cardinality $\aleph_1$ then $\mathfrak{A}$ can be expanded to a model of T^*.*

Proof. Let φ be a Scott sentence of $\mathfrak{A}$. Let $\mathfrak{M}$ be an $\aleph_1$ sized model of φ. By *Theorem 5.9*, $\mathfrak{M}$ is an atomic model of $\mathrm{Th}(\mathfrak{A})$. We will essentially 'reverse' *Construction 6.3*.

We may write $\mathfrak{M} = \{m_i : i < \omega_1\}$. Let M_0 be a countable elementary substructure of $\mathfrak{M}$ given by the Downward Löwenheim Skolem Tarski Theorem. Define the chain $(M_\alpha)_{\alpha<\omega_1}$ inductively; given M_α let $M_{\alpha+1}$ be an elementary substructure of $\mathfrak{M}$ containing M_α and m_i, for the first i with the property that $m_i \notin M_\alpha$. As each M_α is countable, we can find such an m_i. At limit ordinals β, let $M_\beta = \cup_{\alpha<\beta} M_\alpha$.

As $\mathfrak{M}$ is an atomic model of $\mathrm{Th}(\mathfrak{A})$, each M_α is an atomic model of $\mathrm{Th}(\mathfrak{A})$. As each M_α is countable, by *Theorem 5.5* they are all isomorphic to $\mathfrak{A}$.

Let $f : M_1 \to \mathfrak{A}$ be such an isomorphism. Define a new predicate U by

$$Ux \Leftrightarrow x \in f(M_0).$$

By construction, this expansion $\mathfrak{A}^* = (\mathfrak{A}, U)$ satisfies (a) of T^*. By the isomorphism, condition (b) is also satisfied. Therefore $\mathfrak{A}^*$ is a model of T^*, meaning $\mathfrak{A}$ can be expanded to a model of T^*, as required. ■

9 Skolem functions & Indiscernible sets

In this section we define Skolem functions and indiscernible sets. We then prove that a Scott sentence of a countable, atomic model with built-in Skolem functions and an infinite set of indiscernibles has models of any infinite cardinality. In essence we are generalising Main Theorems I (and Main Theorem II, to come later) with more restrictive conditions in place.

Definition 9.1. Let $\mathfrak{A}$ be a model and $\varphi(\vec{u}, x)$ be a formula in the language of $\mathfrak{A}$. A Skolem function for $\varphi(\vec{u}, x)$ is a function f_φ such that

$$\mathfrak{A} \models \forall \vec{u} \left(\exists x (\varphi(\vec{u}, x) \to \varphi(\vec{u}, f_\varphi(\vec{u})) \right).$$

Definition 9.2. A set of *indiscernibles* in $\mathfrak{A}$ is a linearly ordered set $(X, <)$ such that $X \subset |\mathfrak{A}|$ and for any two finite increasing sequences

$$\vec{x} = x_1 < \cdots < x_n, \quad \vec{y} = y_1 < \cdots < y_n \quad \text{in } (X, <)$$

$\vec{x}$ and $\vec{y}$ satisfy the same type.

Finally, we define:

Definition 9.3. The Skolem Hull of I^* in $\mathfrak{A}^*$ (denoted $Sk(I^*)$) is the set of all $f_\varphi(\vec{a})$ where $\vec{a} \in I^*$ and φ is a formula over $\mathfrak{A}^*$.

Remark 9.4. In the above definition, $I^* \subset Sk(I^*)$ as there are formulae such as $\varphi(u, x) =$ "$x = u$" where $f_\varphi(u) = u$. ◊

We now prove the main theorem of this section:

Theorem 9.5. *Let $\mathfrak{A}$ be countable, atomic, have built-in Skolem functions and have an infinite set of indiscernibles $(I, <)$. For any infinite cardinal κ there is a structure $\mathfrak{B}$ of size κ that satisfies a Scott sentence φ of $\mathfrak{A}$.*

Proof. If we add κ many new constants, all in I, by the Compactness Theorem there is a model $(\mathfrak{A}^*, I^*, <^*)$ of $\mathrm{Th}(\mathfrak{A}, I, <)$ where I^* is of size κ. Let $\mathfrak{B} = Sk(I^*)$. Note $\|\mathfrak{B}\| = \|I^*\| = \kappa$. We show $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}^*$:

If $\mathfrak{A}^* \models \varphi(\vec{b}, c)$ with $\vec{b} \in \mathfrak{B} = Sk(I^*)$ and $c \in \mathfrak{A}^*$, then for some $d \in \mathfrak{B}$ by Skolem functions $\mathfrak{A}^* \models \varphi(\vec{b}, d)$. Thus by the Tarski Criterion (*Theorem 4.3*) $\mathfrak{B} \preceq \mathfrak{A}^* \equiv \mathfrak{A}$. Therefore $\mathrm{Th}(\mathfrak{B}) = \mathrm{Th}(\mathfrak{A})$. We claim $\mathfrak{B}$ realises no new types:

Let $\vec{b} \in \mathfrak{B}$ realise some type $p(\vec{x})$. Then there exists a Skolem function $\vec{f}$ such that $\vec{b} = \vec{f}(\vec{i})$ for $\vec{i} = i_1 <^* i_2 <^* \cdots <^* i_n \in I^*$. Since I^* is a set of indiscernibles, there is an increasing sequence $\vec{j} = j_1 < \cdots < j_n \in I \subset I^*$ realising the same type as $\vec{i}$. As $(\mathfrak{A}, I, <) \equiv (\mathfrak{A}^*, I^*, <^*)$, applying $\vec{f}$ to $\vec{j}$ we obtain $\vec{b'} = \vec{f}(\vec{j}) \in \mathfrak{A}$ realising the same type as $\vec{b} \in \mathfrak{B}$. Thus $\mathfrak{B}$ realises no new types. Finally, we claim $\mathfrak{B}$ is atomic:

Let $p(\vec{x})$ be a type realised in $\mathfrak{B}$. This structure realises no new types, so this type is realised in $\mathfrak{A}$ by some $\vec{a} = (a_0, \ldots, a_n)$. Suppose

$$\mathfrak{A} \models \gamma_0(a_0), \gamma_1(a_0, a_1), \ldots, \gamma_n(\vec{a})$$

where $\gamma_i(a_0, \ldots, a_i)$ generates $p(a_0, \ldots, a_i)$. Then

$$\mathfrak{A} \models \gamma_0(b_0), \gamma_1(b_0, b_1), \ldots, \gamma_n(\vec{b})$$

where $b_0 = f_{\gamma_0}()$, $b_1 = f_{\gamma_1}(b_0)$, etc. So

$$\mathfrak{B} \models \gamma_0(b_0), \gamma_1(b_0, b_1), \ldots, \gamma_n(\vec{b})$$

and so $\vec{b} = (b_0, \ldots, b_n)$ realises the generator $\gamma_n(\vec{x})$ making $p(\vec{x})$ principal. We can conclude $\mathfrak{B}$ is atomic.

Therefore, by *Theorem 5.8 & Theorem 5.9*, $\mathfrak{B} \models \varphi$ and $\|\mathfrak{B}\| = \kappa$ as required. ■

10 Conditions involving additional predicates

In this section we outline and prove for a set Z of more restrictive conditions the following theorem:

Theorem 10.1. *Let $\mathfrak{A}$ be a countable model in a countable language L. There is a countable elementary first order theory Z in L with added predicates such that for any L-structure $\mathfrak{B}$, $\mathfrak{B}$ satisfies the Scott sentence for $\mathfrak{A}$ if and only if $\mathfrak{B}$ can be expanded to an atomic model of Z.*

Remark 10.2. Note that in this theorem we are proving results about *the* Scott sentence of $\mathfrak{A}$; that is, the sentence from *Definition 3.5.* ♢

Proof of Theorem 10.1:
($\Rightarrow$). Recall from *Definitions 3.2 & 3.3* the formulae $\varphi^{\alpha}_{\vec{a}}(\vec{x})$ for $\vec{a} \in \mathfrak{A}$ and α the Scott rank of $\mathfrak{A}$. For each $\vec{a}$ in $\mathfrak{A}$, let $P_{\vec{a}}$ be a new predicate symbol, and expand $\mathfrak{A}$ to $\mathfrak{A}^* = (\mathfrak{A}, (P_{\vec{a}})_{\vec{a} \in \mathfrak{A}})$

such that $P_{\vec{a}}^{\mathfrak{A}^*} = \{\vec{b} : \mathfrak{A} \models \varphi_{\vec{a}}^{\alpha}(\vec{b})\}$.
Let $Z = \mathrm{Th}(\mathfrak{A}^*)$. We claim $\mathfrak{A}^*$ is atomic, where $P_{\vec{a}}$ generates $\mathrm{tp}(\vec{a})$:

Let $\vec{a} \in \mathfrak{A}^*$. We know $\mathfrak{A}^* \models P_{\vec{a}}(\vec{a})$ and we claim for all elementary first order formulae ϕ true of $\vec{a}$, $\mathfrak{A}^* \models \forall \vec{x}(P_{\vec{a}}(\vec{x}) \to \phi(\vec{x}))$. This is equivalent to proving $\mathfrak{A} \models \forall \vec{x}(\varphi_{\vec{a}}^{\alpha}(\vec{x}) \to \phi(\vec{x}))$ which is true by *Lemma 3.8*.

Now suppose $\mathfrak{B}$ satisfies φ, the Scott sentence of $\mathfrak{A}$. If $\mathfrak{B}$ is countable, by *Theorem 3.6* it is isomorphic to $\mathfrak{A}$, thus set $\mathfrak{B}^* = (\mathfrak{B}, (P_{\vec{a}})_{\vec{a} \in \mathfrak{A}})$ where $P_{\vec{a}}^{\mathfrak{B}^*} = \{\vec{b} : \mathfrak{B} \models \varphi_{\vec{a}}^{\alpha}(\vec{b})\} = \varphi_{\vec{a}}^{\alpha}$. Then $\mathfrak{B}^* \cong \mathfrak{A}^*$ so we can conclude $\mathfrak{B}$ can be expanded to an atomic model of Z.

Suppose $\mathfrak{B}$ is uncountable. Define $\mathfrak{B}^* = (\mathfrak{B}, (P_{\vec{a}})_{\vec{a} \in \mathfrak{A}})$ where $P_{\vec{a}}^{\mathfrak{B}^*} = \{\vec{b} : \mathfrak{B} \models \varphi_{\vec{a}}^{\alpha}(\vec{b})\}$. Let F be a countable fragment of $L_{\omega_1,\omega}$ including φ and be closed under subformulae of φ, and include all finitary formulae of L and be closed under $\wedge, \vee, \neg$ (note the language of $\mathfrak{A}$ is *countable*). Using the Infinitary Downward Löwenheim Skolem Tarski Theorem we can obtain a countable $\mathfrak{B}_0$ which satisfies φ. Thus $\mathfrak{B}_0 \cong \mathfrak{A}$ and $\mathfrak{B}_0^* \cong \mathfrak{A}^*$ therefore $\mathfrak{B}_0^*$ is an atomic model of Z.

Note that the predicates $P_{\vec{a}}^{\mathfrak{B}^*}$ are defined by $\varphi_{\vec{a}}^{\alpha}$ which are subformulae of φ, thus their truth is preserved from $\mathfrak{B}$ to $\mathfrak{B}_0$ by the fragment F. Therefore $P_{\vec{a}}^{\mathfrak{B}^*} = P_{\vec{a}}^{\mathfrak{B}_0^*} = \varphi_{\vec{a}}^{\alpha}$ and thus $\mathfrak{B}^*$ is an atomic model of Z, as required.
(Note that for any tuple $\vec{a}$ and any formula $\phi(\vec{x})$ true of $\vec{a}$,

$$\mathfrak{B}^* \models \forall x(P_{\vec{a}}(\vec{x}) \to \phi(\vec{x})) \Leftrightarrow \mathfrak{B} \models \forall x(\varphi_{\vec{a}}^{\alpha}(\vec{x}) \to \phi(\vec{x}))$$

$$\Leftrightarrow \mathfrak{B}_0 \models \forall x(\varphi_{\vec{a}}^{\alpha}(\vec{x}) \to \phi(\vec{x})) \Leftrightarrow \mathfrak{B}_0^* \models \forall x(P_{\vec{a}}(\vec{x}) \to \phi(\vec{x}))$$

so the predicates $P_{\vec{a}}^{\mathfrak{B}^*}$ generate the types.)

($\Leftarrow$). Suppose $\mathfrak{B}$ is an L-structure that can be expanded to an atomic model $\mathfrak{B}^*$ of $Z = \mathrm{Th}(\mathfrak{A}^*)$.

Suppose $\mathfrak{B}$ is countable. As $\mathfrak{B}^* \models \mathrm{Th}(\mathfrak{A}^*)$, $\mathfrak{B}^* \equiv \mathfrak{A}^*$ so by *Corollary 5.6*, $\mathfrak{B}^* \cong \mathfrak{A}^*$, and thus by taking the reduct back to L, $\mathfrak{B} \models \varphi$, as required.

Now suppose $\mathfrak{B}$ is uncountable. Let F be a fragment of $L_{\omega_1,\omega}$ including all finitary formulae of $L \cup \{P_{\vec{a} \in \mathfrak{A}}\}$ and be closed under $\wedge$, $\vee$, $\neg$. Note $\mathrm{Th}(\mathfrak{A}^*) \subseteq F$. Also F should preserve the truth of φ and all its subformulae. Then by applying the Infinitary Downward Löwenheim Skolem Tarski Theorem to $\mathfrak{B}^*$, we obtain a countable model $\mathfrak{B}_0^* \preceq_F \mathfrak{B}^*$ (in particular, $\mathfrak{B}_0^* \models \varphi \Leftrightarrow \mathfrak{B}^* \models \varphi$). As $\mathfrak{B}_0^* \models \mathrm{Th}(\mathfrak{A}^*)$ still, and since it remains atomic, by *Corollary 5.6* we can conclude $\mathfrak{B}_0^* \models \varphi$ thus $\mathfrak{B}^* \models \varphi$ so $\mathfrak{B} \models \varphi$ as required. ■

11 Examples for Main Theorem II

We now begin building towards Main Theorem II (*Theorem 12.1*). As we will discover, one of the conditions required for a Scott sentence φ of a countable model to have a model of size $\aleph_2$ is for φ to have *one* $\aleph_1$-sized model, up to isomorphism.

Definition 11.1. Let κ be an infinite cardinal and T a theory with models of size κ. T is *κ-categorical* if any two models of T of size κ are isomorphic.

We examine examples of structures that are, or are not, $\aleph_1$-categorical.

Examples.

(1) **An algebraically closed field of characteristic p.**
Marker [2, Proposition 2.2.5] gives an example that the theory of algebraically closed fields of characteristic p is $\aleph_1$-categorical.

(2) **$\mathbb{Z}$ chains and equivalence classes.**
Let $\mathfrak{A} = (A, \sim, <, S)$ be a countable structure with at least countably many equivalence classes (under $\sim$), where each class is a $\mathbb{Z}$ chain.

Formally, we can describe this as follows: let $\psi_n(x)$ be the formula

$$\exists y_1, \ldots, y_n (x \sim y_1 \wedge \cdots \wedge x \sim y_n).$$

Let ϕ_n be the sentence

$$\exists x_1, \ldots, x_n \left(\bigwedge_{i \neq j} x_i \not\sim x_j \right).$$

Finally let χ be the sentence

$$\forall x, y \left(x \sim y \rightarrow \left(\bigvee_{n<\omega} x = S^n(y) \vee \bigvee_{n<\omega} y = S^n(x) \vee x = y \right) \right).$$

This last sentence guarantees that there can only be one $\mathbb{Z}$ chain per equivalence class. $<$ is defined as usual, but only holds between elements of the same equivalence class. Altogether, the sentence

$$\varphi = \chi \wedge \bigwedge_{n<\omega} \phi_n \wedge \forall x \bigwedge_{n<\omega} \psi_n(x)$$

is a Scott sentence for a structure with at least countably many equivalence classes, where each class is a $\mathbb{Z}$ chain.

Mapping equivalence class to equivalence class and $\mathbb{Z}$ chain to $\mathbb{Z}$ chain, φ will have $\aleph_1$-categorical models.

(3) **Dense linear order.**
Let $\mathfrak{A} = \eta * \omega_1$, meaning $\mathfrak{A}$ is $\aleph_1$ copies of a dense linear order η (obtained by replacing the points in ω_1 by η). Let $\mathfrak{B} = \eta * (\omega_1 + 1^*)$, where '$+1^*$' symbolises adding a copy of the dense linear order η *after* $\eta * \omega_1$. Note that both $\mathfrak{A}$ and $\mathfrak{B}$ satisfy a Scott sentence φ of a dense linear order, and both are uncountable.

For any point in $\mathfrak{A}$, there are an uncountable number of points bigger than it. If we choose a point on the '$+1^*$' copy of the dense linear order in $\mathfrak{B}$, this is no longer true. Thus $\mathfrak{A} \not\cong \mathfrak{B}$ so φ does *not* have $\aleph_1$-categorical models.

Remark 11.2. This example is of a dense linear order *without* endpoints. If there were endpoints, the proof is unchanged. ♢

(4) **Equivalence classes.**
Let $\mathfrak{A} = (A, \sim)$ be a countable structure where $\sim$ is an equivalence relation with a unique class of size n for each $n \in \omega$, and infinitely many classes of infinite size.

Formally, we can describe this as follows: let ϕ be the sentence

$$\bigwedge_{n<\omega} \exists x_0, \ldots, x_n \left(\underset{i,j}{\wedge} x_i \sim x_j \wedge \forall z \left(\left(\underset{i<n}{\vee} z \sim x_i \right) \to \left(\underset{i<n}{\vee} z = x_i \right) \right) \right)$$

meaning "for each n, there is an equivalence class of size n". This can be modified to ϕ^* which will say "for each n, there is a *unique* equivalence class of size n".

Let $\chi(x)$ be the formula

$$\bigwedge_{n<\omega} \exists y_0, \ldots, y_n \left(\left(\underset{i<n}{\wedge} y_i \sim x \right) \wedge \left(\underset{i \neq j}{\wedge} y_i \neq y_j \right) \right)$$

and let ψ be the sentence

$$\bigwedge_{k<\omega} \exists x_0, \ldots, x_k \left(\left(\underset{i \neq j}{\wedge} x_i \neq x_j \right) \wedge \left(\underset{i<n}{\wedge} \chi(x_i) \right) \right)$$

meaning "there are infinitely many classes of infinite size". Then $\varphi = \phi^* \wedge \psi$ is a Scott sentence of $\mathfrak{A}$.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two uncountable models of φ. Suppose one of the infinite classes of $\mathfrak{M}$ is uncountable in size, and the rest are countable. Suppose all of the infinite classes of $\mathfrak{N}$ are uncountable in size. Then clearly $\mathfrak{M} \ncong \mathfrak{N}$ and thus φ does not have $\aleph_1$-categorical models.

12 Main Theorem II

In this section we outline and prove there is a sufficient set of conditions guaranteeing the existence of an $\aleph_2$ sized model for a Scott sentence of a countable structure. We will build on the conditions used in the proof of Main Theorem I (*Section 8*) and use results about these conditions from *Section 6.*

Suppose there are unary predicates U, V_1, V_2 on an L-structure $\mathfrak{A}$ and conditions such that:

(a) $Ux \Leftrightarrow V_1 x \wedge V_2 x$

(b) $\exists x(\neg V_1 x \wedge V_2 x) \wedge \exists y(V_1 y \wedge \neg V_2 y)$

(c) $(\mathfrak{A}, V_1) \cong (\mathfrak{A}, V_2) \cong (V_1, U) \cong (V_2, U) \cong (\mathfrak{A}, U)$

(d) $\mathfrak{A} \cong V_1 \cong V_2 \cong U$

(e) $\mathfrak{A}$ is a countable, atomic model where $\mathrm{Th}(\mathfrak{A})$ has a *unique* atomic model of size $\aleph_1$.

Theorem 12.1. *If $\mathfrak{A}$ is a model of conditions $(a)-(e)$, and φ is a Scott sentence of $\mathfrak{A}$, then φ has a model of size $\aleph_2$.*

Proof. See *FIGURE 1* on *PAGE 261* for a picture of what we want to construct.

As in *Theorem 8.1* (Main Theorem I) we wish to build a chain of models, that, at each step of the chain preserves some properties of $\mathfrak{A}$. We then take a union of all structures in the chain to obtain a model of φ of size $\aleph_1$. While this was enough for Main Theorem I, here we need to construct a *second* $\aleph_1$ sized model. Using (E) these two $\aleph_1$ sized models are isomorphic, which will allow us to run *Construction 6.3* a final time. We will obtain a chain of $\aleph_1$ sized models, the union of which is the desired $\aleph_2$ sized model of φ.

Let $\mathcal{B}_0^0 = \mathfrak{A}$. As in *Construction 6.3*, construct $\mathcal{B}_0^1$ such that $(\mathcal{B}_0^1, \mathcal{B}_0^0) \cong (\mathcal{B}_0^0, V_1)$. Continuing like this, build a chain such that

$$(\mathcal{B}_0^{\alpha+1}, \mathcal{B}_0^{\alpha}) \cong (\mathcal{B}_0^1, \mathcal{B}_0^0) \qquad \text{for successor ordinals}$$

$$\mathcal{B}_0^{\kappa} = \bigcup_{\alpha<\kappa} \mathcal{B}_0^{\alpha} \qquad \text{for limit ordinals}$$

Note that by *Theorem 6.5*, $\mathcal{B}_0^{\omega} \cong \mathcal{B}_0^0$. By *Lemma 6.8*, $\mathcal{B}_0^{\omega_1}$ is atomic and by *Lemma 6.9*, $\mathcal{B}_0^{\omega_1}$ is a model of $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathcal{B}_0^0)$. Therefore by *Theorem 5.8 & Theorem 5.9*, $\mathcal{B}_0^{\omega_1} \models \varphi$. By condition (E), this model is unique.

Now construct $\mathcal{B}_1^0$ such that $(\mathcal{B}_1^0, \mathcal{B}_0^0) \cong (\mathcal{B}_0^0, V_2)$. Note that

$$(\mathcal{B}_1^0, \mathcal{B}_0^0) \cong (\mathcal{B}_0^0, V_2) \cong (\mathcal{B}_0^0, V_1) \cong (\mathcal{B}_0^1, \mathcal{B}_0^0) \qquad \text{so} \qquad \mathcal{B}_1^0 \cong \mathcal{B}_0^1$$

where $(\mathcal{B}_0^0, V_2) \cong (\mathcal{B}_0^0, V_1)$ from (C). (As V_1 and V_2 are distinct, $\mathcal{B}_0^1$ and $\mathcal{B}_1^0$ are distinct too.)

Construct $\mathcal{B}_1^1$ such that $\mathcal{B}_0^1, \mathcal{B}_1^0 \subset \mathcal{B}_1^1$ and

$$(\mathcal{B}_1^1, \mathcal{B}_0^1, \mathcal{B}_1^0, \mathcal{B}_0^0) \cong (\mathfrak{A}, V_1, V_2, U).$$

By the way $\mathcal{B}_0^1$ and $\mathcal{B}_1^0$ were constructed and properties (C) and (D), this is made possible.

Continuing like this, construct $\mathcal{B}_1^{\alpha+1}$ to contain $\mathcal{B}_1^{\alpha}$, $\mathcal{B}_0^{\alpha+1}$ and to have the property $(\mathcal{B}_1^{\alpha+1}, \mathcal{B}_0^{\alpha+1}, \mathcal{B}_1^{\alpha}, \mathcal{B}_0^{\alpha}) \cong (\mathfrak{A}, V_1, V_2, U)$ for successor ordinals, and for limit ordinals $\mathcal{B}_1^{\kappa} = \bigcup_{\alpha<\kappa} \mathcal{B}_1^{\alpha}$.

Lemma 12.2. *The following structures are isomorphic;*

(1) For $\alpha < \omega$, $\mathcal{B}_1^{\alpha} \cong \mathcal{B}_0^{\alpha}$.

(2) $\mathcal{B}_1^\omega \cong \mathcal{B}_0^\omega$.

(3) $\mathcal{B}_1^{\omega_1} \cong \mathcal{B}_0^{\omega_1}$.

Proof. (1). For $\alpha < \omega$, note that by construction,

$$\mathcal{B}_1^\alpha \cong \mathcal{B}_1^0 \cong \mathcal{B}_0^1 \cong \mathcal{B}_0^\alpha$$

Thus $\mathcal{B}_1^\alpha \cong \mathcal{B}_0^\alpha$ as required.

(2). By construction $\mathcal{B}_0^\omega = \bigcup_{\alpha<\omega} \mathcal{B}_0^\alpha$ and $\mathcal{B}_1^\omega = \bigcup_{\alpha<\omega} \mathcal{B}_1^\alpha$. Then, by *Theorem 6.5*,

$$\mathcal{B}_1^\omega \cong \mathcal{B}_1^0 \cong \mathcal{B}_0^0 \cong \mathcal{B}_0^\omega.$$

So $\mathcal{B}_1^\omega \cong \mathcal{B}_0^\omega$ as required.

(3). Note that by taking (1) and (2) in general for successor or limit ordinals γ with $\omega < \gamma < \omega_1$, we obtain

$$\forall \alpha < \omega_1 \qquad \mathcal{B}_1^\alpha \cong \mathcal{B}_1^0 \cong \mathcal{B}_0^0$$

Since $\mathcal{B}_0^0$ is atomic, we then conclude for all $\alpha < \omega_1$, $\mathcal{B}_1^\alpha$ is atomic. Then $\mathcal{B}_1^{\omega_1}$ is atomic by *Lemma 6.8*. Since for all $\alpha < \omega_1$ $\mathcal{B}_1^\alpha \models \mathrm{Th}(\mathfrak{A})$, by *Lemma 6.9*, $\mathcal{B}_1^{\omega_1} \models \mathrm{Th}(\mathfrak{A})$.

Thus $\mathcal{B}_0^{\omega_1}$, $\mathcal{B}_1^{\omega_1}$ are both atomic models for $\mathrm{Th}(\mathfrak{A})$. However $\mathrm{Th}(\mathfrak{A})$ has a unique atomic model of size $\aleph_1$, so $\mathcal{B}_1^{\omega_1} \cong \mathcal{B}_0^{\omega_1}$, as required. ■

Returning to the proof of Theorem 12.1:
Now we have $\mathcal{B}_0^{\omega_1} \subset \mathcal{B}_1^{\omega_1}$ and $\mathcal{B}_0^{\omega_1} \cong \mathcal{B}_1^{\omega_1}$. Define $\mathcal{C}_0 = \mathcal{B}_0^{\omega_1}, \mathcal{C}_1 = \mathcal{B}_1^{\omega_1}$. Again by *Construction 6.3* we can construct $\mathcal{C}_2$ such that $(\mathcal{C}_2, \mathcal{C}_1) \cong (\mathcal{C}_1, \mathcal{C}_0)$. Form a chain $(\mathcal{C}_\alpha)_{\alpha<\omega_2}$, where at successor ordinals

$$(\mathcal{C}_{\alpha+1}, \mathcal{C}_\alpha) \cong (\mathcal{C}_1, \mathcal{C}_0)$$

and at limit ordinals $\gamma < \omega_2$, define

$$\mathcal{C}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{C}_\alpha.$$

$\mathcal{C}_\gamma$ is atomic by *Lemma 6.8*, and by *Lemma 6.9*, $\mathcal{C}_\gamma$ satisfies $\mathrm{Th}(\mathfrak{A})$. Thus $\mathcal{C}_\gamma \cong \mathcal{C}_0$.

Define $\mathcal{C}_{\omega_2} = \bigcup_{\alpha<\omega_2} \mathcal{C}_\alpha$. This is atomic and satisfies $\mathrm{Th}(\mathfrak{A})$ by *Lemma 6.8* and *Lemma 6.9* respectively. Thus by *Theorem 5.8* and *Theorem 5.9*, $\mathcal{C}_{\omega_2} \models \varphi$ and $||\mathcal{C}_{\omega_2}|| = \aleph_2$ (so $\mathcal{C}_{\omega_2} \ncong \mathcal{C}_0$).

Therefore under conditions (a)-(e) on $\mathfrak{A}$, there is a structure of size $\aleph_2$ satisfying φ, as required. ■

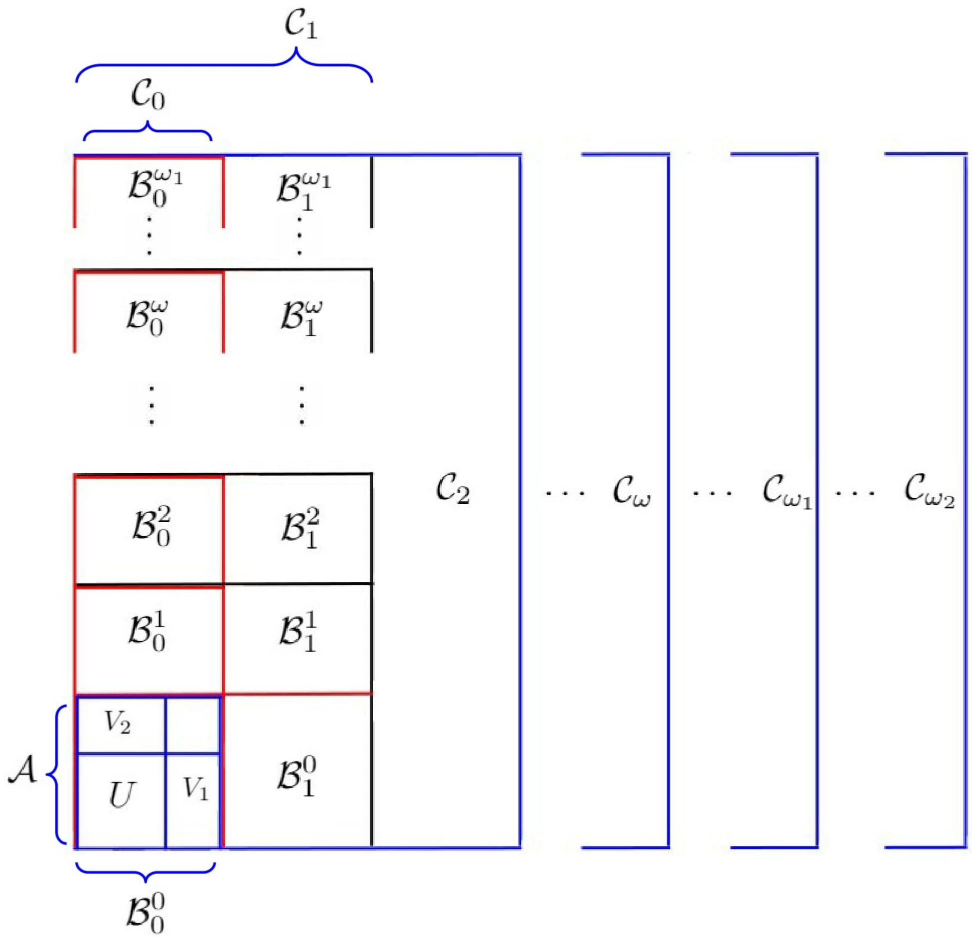

Figure 1: *Structure diagram.*

References

[1] Chang, C. C. and Keisler, H. J. *Model Theory*, Dover, 2012

[2] Marker, D. *Model theory: An introduction*, Springer, 2002

[3] Simmons, H. *Extracted from* `notes` *on a model theory course.*

[4] Scott, D. *Logic with denumerably long formulas and finite strings of quantifiers*, The Theory of Models, ed. by J. Addison, L. Henkin, and A. Tarski, North-Holland, Amsterdam, 1965