Existence and Uniqueness of Solutions of an Einstein-Maxwell PDE System

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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF AN EINSTEIN-MAXWELL PDE SYSTEM

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Abstract. We consider a nonlinear coupled system of partial differential equations with asymptotic boundary conditions which is relevant in the field of general relativity. Specifically, the PDE system relates the factors of a conformally flat spatial metric obeying the laws of gravity and electromagnetism to its charge and mass distributions. The solution to the system is shown to be existent, smooth, and unique. While the discussion of the PDE assumes knowledge of physics and differential geometry, the proof uses only the PDE theory of flat space.

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1 Introduction

In the Newtonian framework of gravity the gravitational potential, $\theta$, is described by the equation

$$\Delta \theta = -4\pi \rho_m$$  \hspace{1cm} (1)

where $\rho_m$ is the mass density. In general relativity, gravitational “force” is seen to be caused by curved spatial geometry which evolves in time. This geometry is governed by a metric, measuring distances across a spatial slice. An equation similar to (1) arises from an examination of the conformally flat spatial metric

$$g = \theta^4 \delta,$$

which arises naturally as a limit of spaces in which many, many Swarzschild solutions are superimposed (see Benko, Stavrov [1]). The mass density function is defined according to the metric as

$$R(g) = 16\pi \rho_m$$

where $R(g)$ signifies the scalar curvature of the metric $g$. We define the form

$$\omega = \omega(x^1, x^2, x^3) dx^1 dx^2 dx^3 = \rho_m dvol_g.$$

This form captures a metric independent notion of mass density. In the chosen coordinate system, $\omega(x^1, x^2, x^3) = \rho_m \theta^6$. Substituting this expression and evaluating the scalar curvature gives

$$-8\Delta \theta / \theta^5 = 16\pi \omega(x^1, x^2, x^3) / \theta^6$$

and so

$$\Delta \theta \cdot \theta = -2\pi \omega(x^1, x^2, x^3).$$

We move the $2\pi$ factor into the $\theta$ functions. Constant factors in the metric are irrelevant since they can be eliminated by a coordinate change. We are left with

$$\Delta \theta \cdot \theta = -\omega(x^1, x^2, x^3).$$  \hspace{1cm} (2)

When $\theta$ is required to approach 1 along the boundary (infinity), solutions to this equation are known to be existent and unique. Furthermore, it is not hard to see that this equation parallels (1) with the conformal factor $\theta$ standing in for the gravitational potential. Equation (1) also has analogies within the Newtonian model of gravity and electromagnetism and within a relativistic model incorporating both gravity and electromagnetism, as we will see.

In the presence of charge, the Newtonian gravitational and electric potentials may be described by two Poisson-type equations:

$$\Delta \theta_m = -\rho_m,$$

$$\Delta \theta_e = -\rho_e.$$
When we let \( \chi = \theta_m + \theta_e, \psi = \theta_m - \theta_e, \omega_1 = -\rho_m - \rho_e, \) and \( \omega_2 = -\rho_m + \rho_e, \) we get

\[
\begin{align*}
\Delta \chi &= \omega_1, \\
\Delta \psi &= \omega_2.
\end{align*}
\]

(3)

It is required that \( \rho_m \geq |\rho_e|, \) therefore \( \omega_1 \leq 0 \) and \( \omega_2 \leq 0. \) Assuming suitable asymptotic decay of \( \omega_1 \) and \( \omega_2 \) and boundary conditions which require \( \chi \) and \( \psi \) to approach some constant at infinity, solutions to this equation are once again existent and unique.

Now, in general relativity, a similar equation relating the metric to mass and charge distributions may be derived by guessing the metric

\[ g = (\chi \psi)^2 \delta \]

under the constraints

\[
\begin{align*}
R(g) &= 16\pi \rho_m + 2\|\vec{E}\|^2, \\
\text{div} \vec{E} &= 4\pi \rho_e,
\end{align*}
\]

(4)

and a third equation relating \( \rho_m \) to \( \Delta \chi \) and \( \Delta \psi. \) These constraints, and a definition of \( \omega_1 \) and \( \omega_2 \) as metric independent forms which are related to \( -\rho_m - \rho_e \) and \( -\rho_m + \rho_e, \) respectively, lead to the equations

\[
\begin{align*}
\Delta \chi \cdot \psi &= \omega_1, \\
\Delta \psi \cdot \chi &= \omega_2
\end{align*}
\]

where \( \omega_1 \leq 0 \) and \( \omega_2 \leq 0. \) These equations are analogous to both (2) and (3) for relativity in the absence of electromagnetism, and electromagnetism in the absence of relativity, respectively. In analogy/extension of similar results for (1), (2) and (3) this paper proves the existence, uniqueness, and smoothness of solutions to the following problem:

Given smooth, compactly supported, non-positive functions \( \omega_1 \) and \( \omega_2 \) with their domain in \( \mathbb{R}^3, \) find \( \chi \) and \( \psi \) for which

\[
\begin{align*}
\chi, \psi &> 0 \\
\chi, \psi &\to 1 \text{ as } \|x\| \to \infty \\
\Delta \chi \cdot \psi &= \omega_1, \\
\Delta \psi \cdot \chi &= \omega_2.
\end{align*}
\]

(5)

The techniques used in the existence proof were inspired by a proof of the method of sub and super solutions from the lecture notes of Professor Iva Stavrov. As these have not been published, we have included a reference to a similar proof by Kazdan and Warner [2]. The proof of existence constitutes the bulk of the paper, but we believe the uniqueness proof is more original and of more value to the reader. To find it, we imagined two sets of solutions \((\chi_1, \psi_1)\) and \((\chi_2, \psi_2)\) satisfying equations (6) and (7), then simplified the problem by letting
\( \omega_1 = \omega_2, \psi_2 = \chi_1 \) and \( \psi_1 = \chi_2 \). These equalities reduce the two equations to one. We changed the form of this equation, and found a new, linear PDE solved by both \( \chi_1 \) and \( \chi_2 \) by fixing some variables and letting others vary. Since the equation is linear, linearly combining the the distinct solutions \( \chi_1 \) and \( \chi_2 \) produces a one parameter family of functions satisfying the equation and the boundary conditions imposed on \( \chi_1 \) and \( \chi_2 \). Some of these functions will have positive minimums and will therefore be unable to solve the equation under consideration. This contradiction disproves the assumption of two distinct solutions to the original problem. The actual proof is, of course, different, since in general it is not possible to assume \( \omega_1 = \omega_2 \) or \( \psi_2 = \chi_1 \) and \( \psi_1 = \chi_2 \), but this method inspired the proof in full generality.

In the first part of the existence section, Section 2, we build two sequences \((\chi_n, \psi_n)\) whose limits \((\chi_*, \psi_*)\) seem to satisfy our PDE system and which have properties allowing us to prove that their limits actually exist (Theorem 2.1). With this task accomplished, we prove that not only do the sequences converge pointwise, they also converge within the Sobolev spaces \(H^k(B[r])\). We break the proof of this result into the base case Theorem 2.2 and an induction argument that follows in the proof of Theorem 2.3. From this result it follows that the sequence limits are smooth and satisfy the PDE system. In the last step of the existence proof, we demonstrate that the limits \( \chi_* \) and \( \psi_* \) satisfy the boundary conditions of the problem.

Section 3 presents the uniqueness proof as a single theorem. We use proof by contradiction and a reformulation of the PDE system to show there is only one solution set solving our problem, not two.

Lastly, the appendix contains three technical lemmas concerning Cauchy sequences (used in Theorem 2.2), regularity (in Theorem 2.3), and the continuity of the infimum of a certain group of continuous functions (in Theorem 3.1).

## 2 Existence

To prove existence of solutions, we construct sequences \( \chi_n \) and \( \psi_n \) which solve PDEs approaching our PDE as \( n \to \infty \). Then we show that \( \chi_n \) and \( \psi_n \) are Cauchy in the Sobolev spaces \(H^k(B[r])\) for all positive integers \( k \) and all \( r > 0 \). Sequences \( \chi_n \) and \( \psi_n \) will therefore converge to smooth solutions to the problem, \( \chi \) and \( \psi \).

**Theorem 2.1.** Given smooth, non-positive, compactly supported functions \( \omega_1 \) and \( \omega_2 \), there exist sequences of functions \( \chi_n \) and \( \psi_n \) and functions \( \psi_+ \) and \( \chi_+ \) for which

\[
\begin{align*}
\text{(i) } & \psi_0 = 1, \\
\text{(ii) } & \Delta \chi_n \cdot \psi_n = \omega_1 \text{ for each } n, \\
\text{(iii) } & \Delta \psi_{n+1} \cdot \chi_n = \omega_1 \text{ for each } n, \\
\text{(iv) } & \psi_n \text{ and } \chi_n \text{ are smooth functions for each } n,
\end{align*}
\]
(v) the sequence $\psi_n$ is non-decreasing.

(vi) the sequence $\chi_n$ is non-increasing.

(vii) $\psi_+$ and $\chi_+$ approach 1 at infinity.

(viii) $1 \leq \psi_n \leq \psi_+$ and $1 \leq \chi_n \leq \chi_+$.

(ix) $\psi_+$ and $\chi_+$ are bounded above.

Proof. As $\omega_1$ and $\omega_2$ are continuous and compactly supported, they are also bounded. This allows us to define $\chi_+ = 1 - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega_1(y)}{\|y-x\|} dy$ and $\psi_+ = 1 - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega_2(y)}{\|y-x\|} dy$. The integrands are compactly supported and approach 0 for large $x$, thus $\chi_+$ and $\psi_+$ approach 1 near infinity. The functions $\chi_+$ and $\psi_+$ are differentiable by the Leibniz rule, therefore they are continuous. Their continuity and their boundary values imply that $\chi_+$ and $\psi_+$ are bounded. Also, $\psi_+ \geq 1$ and $\chi_+ \geq 1$, since $\omega_1$ and $\omega_2$ are non-positive. Note that $\Delta \chi_+ = \omega_1$ and $\Delta \psi_+ = \omega_2$. We show that for each $k$ there are partial sequences $\psi_0, \psi_1, \ldots, \psi_{k+1}$ and $\chi_0, \chi_1, \ldots, \chi_k$ satisfying (i)-(ix). This will demonstrate the existence of non-terminating sequences $\psi_n$ and $\chi_n$ satisfying equations (i)-(ix).

Base case: Let $\psi_0 = 1$. Let $\chi_0 = \chi_+$. Clearly $1 \leq \chi_0 \leq \chi_+$, $\chi_0$ approaches 1 at infinity, and $\chi_0$ is smooth. Since $\Delta \chi_0 = \omega_1$, (i) is satisfied for $n = 0$. Let $\psi_1$ be the smooth function approaching 1 at infinity for which $\Delta \psi_1 = \omega_2/\chi_0$. Such a function exists because $\omega_2/\chi_0$ is bounded and compactly supported. This satisfies (ii) for $n = 0$. It is also smooth, and $1 \leq \psi_1 \leq \psi_+$. Since

$$\Delta \psi_1 = \omega_2/\chi_0 \leq 0 = \Delta \psi_0$$

and

$$\Delta \psi_1 \geq \omega_2 = \Delta \psi_+$$

it follows from the weak maximum principle that

$$\psi_0 \leq \psi_1 \leq \psi_+.$$

Inductive step: Assume that there exist finite sequences of smooth functions $\psi_0, \psi_1, \ldots, \psi_k$ and $\chi_0, \chi_1, \ldots, \chi_{k-1}$ satisfying equations (i)-(ix). Because $\omega_1/\psi_{k+1}$ is bounded and compactly supported, we may define $\chi_k$ as the smooth function approaching 1 at infinity and satisfying $\Delta \chi_k = \omega_1/\psi_k$. This function satisfies (ii) for $n = k$. Furthermore

$$\Delta \chi_k = \omega_1/\psi_k \geq \omega_1/\psi_{k-1} = \Delta \chi_{k-1}$$

and

$$\Delta \chi_k = \omega_1/\psi_k \leq 0.$$

By the weak maximum principle

$$1 \leq \chi_k \leq \chi_{k-1} \leq \chi_+.$$
The function $\omega_2/\chi_k$ is bounded and compactly supported. This allows us to define $\psi_{k+1}$ as the smooth function approaching 1 at infinity and satisfying $\Delta \psi_{k+1} = \omega_2/\chi_k$. This satisfies (iii) for $n = k$. Furthermore

$$\Delta \psi_{k+1} = \omega_1/\chi_k \leq \omega_2/\chi_{k-1} = \Delta \psi_k$$

and

$$\Delta \psi_{k+1} = \omega_1/\chi_k \geq \omega_1 = \Delta \psi_+. $$

By the weak maximum principle

$$1 \leq \psi_k \leq \psi_{k+1} \leq \psi_+. $$

This completes our inductive proof. \qed

Since $\psi_n$ and $\chi_n$ are both bounded and monotonic sequences, they converge pointwise. In order to prove that their pointwise limits satisfy the PDE, we need convergence in $C^2(B[r])$ (with $B[r]$ the closed ball of radius $r$ centered about the origin) for any choice of $r$ to move a limit through the Laplacians in equations (ii) and (iii). By proving even stronger regularity results for $\psi_n$ and $\chi_n$ we will show that their limits, the functions solving the PDE, are in fact smooth. We start by proving the base case of the induction proof that $\psi_n$ and $\chi_n$ are Cauchy sequences in all $H^k$ and $C^k$ spaces over balls.

**Theorem 2.2.** Let $G_1(x, y) = \omega_1(x)/y$. Let $G_2(x, y) = \omega_2(x)/y$. For all positive $r$, the sequences $\psi_n$ and $\chi_n$ are Cauchy in $H^2(B[r])$ and $C^0(B[r])$, and the sequences $G_1(x, \psi_n(x))$ and $G_2(x, \chi_n)$ are Cauchy in $L^2(B[r])$.

**Proof.** Let $r > 0$. Recall that $\omega_1$ and $\omega_2$ are bounded. Since $\chi_n$ and $\psi_n$ are bounded below by 1, $\Delta \chi_n = \omega_1/\psi_n$ and $\Delta \psi_n = \omega_2/\chi_n$ are uniformly pointwise bounded, and therefore uniformly bounded in $L^2(B[r+1])$. By the Elliptic Regularity Theorem there exists a constant $C$ such that

$$\|\psi_n\|_{H^2(B[r+1])} \leq C\left(\|\Delta \psi_n\|_{L^2(B[r+2])} + \|\psi_n\|_{L^2(B[r+2])}\right).$$

It follows that the uniform boundedness of $\psi_n$ and $\Delta \psi_n$ in $L^2(B[r+2])$ implies the uniform boundedness of the sequence $\psi_n$ in $H^2(B[r+1])$. The Rellich Lemma asserts the existence of a subsequence $\psi_{n_k}$ which converges in $H^1(B[r+1])$ and, in particular, is Cauchy in $L^2(B[r+1])$. Since the sequence $\psi_n$ is monotonic and has a Cauchy subsequence in $L^2(B[r+1])$, $\psi_n$ itself is Cauchy in $L^2(B[r+1])$. In case this fact is not obvious, it is proven in Lemma 4.1.

Let $\psi_*$ be an upper bound on $\psi_+$ (and let $\chi_*$ be an upper bound on $\chi_+$). The partial derivatives of $G_1$ in $y$ are continuous. Because the set $B[r] \times [1, \psi_+]$ is compact we may define $C' = \max_{B[r] \times [1, \psi_+]} |\partial_y G_1(x, y)|$. The Mean Value Theorem then shows

$$|G_1(x, \psi_m(x)) - G_1(x, \psi_l(x))| \leq C'|\psi_m(x) - \psi_l(x)|.$$
Proof. We apply a bootstrap argument to show that the pointwise limit of equations (ii) and (iii) gives Theorem 2.3, $\psi$ Theorem 2.4. There exist smooth solutions to $G$ for all integers $k \geq 0$ and any non-negative integers $k$ and all $r > 0$. Let $r > 0$.

Base Case: By Theorem 2.2, $\psi_n$ and $\chi_n$ are Cauchy in $C^0(B[r])$ and $H^2(B[r])$, $G_1(x, \psi_n)$ and $G_2(x, \chi_n)$ are Cauchy in $H^0(B[r])$ for all positive $r$.

Inductive Step: Assume $\psi_n$ and $\chi_n$ are Cauchy in $C^k(B[r+1])$ and $H^{k+2}(B[r+1])$ for all $r > 0$. The sequence $\psi_n$ is bounded between the constants 1 and $\psi_{++}$, and the sequence $\chi_n$ is bounded between the constants 1 and $\chi_{++}$. The sequences are clearly also Cauchy in $H^{k+1}(B[r+1])$. From Lemma 4.2 in the appendix it follows that $G_1(x, \psi_n)$ and $G_2(x, \chi_n)$ are Cauchy in $H^{k+1}(B[r+1])$. Therefore $\Delta \chi_n = G_1(x, \psi_n)$ and $\Delta \psi_n = G_2(x, \chi_n)$ must be Cauchy in $H^{k+2}(B[r+1])$. By the Elliptic Regularity Theorem, there exists a $C'''$ for which

$$\|\psi_m - \psi_l\|_{H^{k+3}(B[r])} \leq C''' (\|\Delta \psi_m - \Delta \psi_l\|_{H^{k+1}(B[r+1])} + \|\psi_m - \psi_l\|_{L^2(B[r+1])}).$$

It follows that $\psi_n$ must be Cauchy in $H^{k+3}(B[r])$. The Sobolev Embedding Theorem implies $\psi_n$ and $\chi_n$ are Cauchy in $C^{k+1}(B[r])$. The principle of induction gives us, in particular, that $\psi_n$ and $\chi_n$ are Cauchy in $C^{k+2}(B[r])$ for all non-negative integers $k$.

Theorem 2.4. There exist smooth solutions to (5).

Proof. The following remarks hold for any positive $r$ and any non-negative integers $k$. By Theorem 2.3, $\psi_n$ and $\chi_n$ are Cauchy in $H^{k+2}(B[r])$. As the Sobolev spaces are complete, $\psi_n$ and $\chi_n$ converge in $H^{k+2}(B[r])$ for any $r$ and $k$. By the Sobolev Embedding Theorem, the limits $\psi_*$ and $\chi_*$ are elements of $C^k(B[r])$ and the sequences converge to these functions in $C^k(B[r])$. Because this applies for all positive integers $k$, $\psi_*$ and $\chi_*$ must be smooth. Taking the pointwise limit of equations (ii) and (iii) gives

$$\Delta \chi_* \cdot \psi_* = \omega_1$$
\[ \Delta \psi \cdot \chi = \omega_2 \]

Moving the limit through the product is possible since \( \psi_n, \chi_n \in C^0(B[r]) \) for all \( r \). Moving the limit through the laplacians is possible because convergence of \( \chi_n \) and \( \psi_n \) occurs in \( C^2(B[r]) \). Also, these equations work for all \( x \) because each \( x \) is a member of some closed ball centered around the origin. It is only left show that \( \psi \) and \( \chi \) satisfy the boundary condition. Since, for all \( n \), \( 1 \leq \psi_n \leq \psi_+ \) and \( 1 \leq \chi_n \leq \chi_+ \) it follows that \( 1 \leq \psi_+ \leq \psi_+ \) and \( 1 \leq \chi_+ \leq \chi_+ \). The functions \( 1, \psi_+ \), and \( \chi_+ \) all approach 1 at infinity, thus \( \psi \) and \( \chi \) must also approach 1 at infinity. \( \square \)

3 Uniqueness

We assume the existence of two distinct solutions to the problem, then modify them to produce a family of solutions to a related system of equations. We show that there must be members of this family with positive minimums, which yields a contradiction.

Theorem 3.1. Solutions to (5) are unique.

Proof. Assume that two distinct solutions to the problem exist. More explicitly, assume

\[ \Delta \chi \cdot \psi_1 = \omega_1 = \Delta \chi \cdot \psi_2 \] (6)

and

\[ \Delta \psi \cdot \chi_1 = \omega_2 = \Delta \psi \cdot \chi_2 \] (7)

where \( \chi_1, \psi_1 \) and \( \chi_2, \psi_2 \) satisfy the problem conditions and \( \chi_1 \neq \chi_2 \) or \( \psi_1 \neq \psi_2 \). It is easily seen that \( \chi_1 \equiv \chi_2 \) implies \( \psi_1 \equiv \psi_2 \) and vice-versa. Therefore \( \chi_1 \neq \chi_2 \) if and only if \( \psi_1 \neq \psi_2 \). From the assumptions that \( \chi_1 \) and \( \psi_2 \) are positive while \( \omega_1 \) and \( \omega_2 \) are non-positive, it follows that \( \Delta \psi_1 \leq 0 \) and \( \Delta \chi_2 \leq 0 \), which implies \( \psi_1 \geq 1 \) and \( \chi_2 \geq 1 \).

Subtracting the far right side of the above equations from the far left gives

\[ \Delta(\chi_1 - \chi_2)\psi_1 + \Delta\chi_2(\psi_1 - \psi_2) = 0 \]

\[ \Delta\psi_1(\chi_1 - \chi_2) + \Delta(\psi_1 - \psi_2)\chi_2 = 0 \]

We set \( v = \chi_1 - \chi_2 \) and \( w = \psi_1 - \psi_2 \) getting

\[ \Delta v \cdot \psi_1 + \Delta \chi_2 \cdot w = 0 \] (8)

\[ \Delta w \cdot \chi_2 + \Delta \psi_1 \cdot v = 0 \] (9)

It must also be true that

\[ \Delta v(\psi_1 + kw) + \Delta(\chi_2 - kv)w = 0 \] (10)

\[ \Delta w(\chi_2 - kv) + \Delta(\psi_1 + kw)v = 0 \] (11)
Therefore, \( \min(\inf_{x \in \mathbb{R}^3}(\chi_2 - kv), \inf_{x \in \mathbb{R}^3}(\psi_1 + kw)) \) are continuous in \( k \), by Lemma 4.3. Therefore, \( \min(\inf_{x \in \mathbb{R}^3}(\chi_2 - kv), \inf_{x \in \mathbb{R}^3}(\psi_1 + kw)) \) is also continuous in \( k \). Evaluated at \( k = 0 \), \( \min(\inf_{x \in \mathbb{R}^3}(\chi_2 - kv), \inf_{x \in \mathbb{R}^3}(\psi_1 + kw)) > 1/2 \). Since \( v \neq 0 \) there exists a \( k \in \mathbb{R} \) at which \( \min(\inf_{x \in \mathbb{R}^3}(\chi_2 - kv), \inf_{x \in \mathbb{R}^3}(\psi_1 + kw)) < 1/2 \). It follows from the Intermediate Value Theorem that there exists a \( k_* \) at which

\[
\min(\inf_{x \in \mathbb{R}^3}(\chi_2 - k_*v), \inf_{x \in \mathbb{R}^3}(\psi_1 + k_*w)) = 1/2.
\]

In particular, both \( \chi_2 - k_*v > 0 \) and \( \chi_2 + k_*w > 0 \), and there exists a point \( p \) at which \( \chi_2(p) - k_*v(p) < 1 \) or \( \psi_1(p) + k_*w(p) < 1 \).

We present the argument in the case where the first inequality holds. The other case may be treated similarly. At some point \( q \in \mathbb{R}^3 \), \( \Delta(\chi_2 - k_*v)(q) > 0 \), because otherwise the weak maximum principle implies that \( \chi_2 - k_*v \geq 1 \) everywhere. Because \( \Delta(\chi_2 - k_*v)(q) > 0 \) yet \( \Delta \chi_2 \leq 0 \), it follows that \( k_* \Delta v(q) < 0 \). Multiplying (10) by \( k_* \) gives

\[
(k_* \Delta v)(\psi_1 + k_*w) + \Delta(\chi_2 - k_*v)(k_*w) = 0
\]

Since \( k_* \Delta v(q) < 0 \), \( (\psi_1 + k_*w)(q) > 0 \), and \( \Delta(\chi_2 - k_*v)(q) > 0 \), it follows that \( k_*w(q) > 0 \). Multiplying (8) by \( k_* \) gives

\[
k_* \Delta v \cdot \psi_1 + \Delta \chi_2 \cdot k_*w = 0
\]

But plugging \( k_* \Delta v(q) < 0 \), \( \psi_1 > 0 \), \( \Delta \chi_2 \leq 0 \) and \( k_*w(q) > 0 \) into (9) shows \( 0 < 0 \), a contradiction.

The assumed statement, that \( \chi_1 \neq \chi_2 \) or \( \psi_1 \neq \psi_2 \), is false. Solutions to PDE problem (5) are unique. \( \square \)

4 Appendix

This section contains proofs of three technical lemmas referenced in the paper.

**Lemma 4.1.** If the sequence of functions \( f_n \) is monotonic (weakly), and has a subsequence \( f_{n_k} \) Cauchy in \( L^2(B[r]) \), then \( f_n \) is Cauchy in \( L^2(B[r]) \).

**Proof.** Assume for simplicity that \( f_n \) is non-decreasing. Let \( \epsilon > 0 \). Let \( K_1 \) be a number such that for any \( j, k \geq K_1 \), \( \| f_{n_j} - f_{n_k} \|_{L^2(B[r+1])} < \epsilon \). Then for \( l, m > n_{K_1} \) there is a \( K_2 \) for which \( n_{K_1} < l, m < n_{K_2} \). It follows from the monotonicity of \( f_n \) that \( f_{n_{K_1}} \leq f_l, f_m \leq f_{n_{K_2}} \). Therefore

\[
\| f_m - f_l \|_{L^2(B[r+1])} < \| f_{m_{K_2}} - f_{m_{K_1}} \|_{L^2(B[r+1])} < \epsilon.
\]

**Lemma 4.2.** If the sequence of functions \( f_n \) is bounded between constants \( a \) and \( b \), is Cauchy in \( C^k(B[r]) \) and \( H^{k+1}(B[r]) \) for some \( k > 0 \), and \( G(x, y) \) is a smooth function whose arguments are in \( \mathbb{R}^3 \) and \( \mathbb{R} \), respectively, then \( G(x, f_n(x)) \) is Cauchy in \( H^{k+1}(B[r]) \).
Proof. Let $\alpha$ be a multiset representing derivatives with respect to $x$ for which $|\alpha| \leq k+1$. It suffices to show that $D^{\alpha}G(x, f_n(x))$ is Cauchy in $L^2(B[r])$. By the product rule, the function $D^{\alpha}G(x, f_n(x))$ is a sum of terms of the form

$$D^X D^Y G|_{(x, f_n(x))} \cdot D^\beta f_n$$

where $X$ and $Y$ represent derivatives of the first and second components of $G$ respectively, $eta$ is a multi-index for which $\beta = (\beta_1, \beta_2, \ldots, \beta_{|Y|})$ and $D^{\beta_1} \psi_n = D^{\beta_1} \psi_n \cdot D^{\beta_2} \psi_n \cdot \ldots \cdot D^{\beta_{|Y|}} \psi_n$, $|X| + |Y| \leq k + 1$ and $\sum_{i=1}^{|Y|} |\beta_i| \leq k + 1$. Therefore it is enough to show that such terms are Cauchy in $L^2(B[r])$.

The term is of the form $D^X D^Y G|_{(x, f_n(x))} \cdot D^\beta f_n$. The smoothness of $G$ allows us to define $C^m = \max_{B[r] \times [a, b]} \frac{\partial}{\partial y} D^X D^Y G$. By the Mean Value Theorem

$$D^X D^Y G|_{(x, f_m(x))} - D^X D^Y G|_{(x, f_l(x))} \leq C^m(f_m(x) - f_l(x)).$$

Now $D^X D^Y G|_{(x, f_n(x))}$ is Cauchy in $C^0(B[r])$ since $f_n$ is Cauchy in $C^0(B[r])$.

Case 1: $|Y| \neq 1$. The term has either no derivatives of $f_n$ or multiple derivatives of $f_n$ in its product. Crucially, for each $\beta_i$, $|\beta_i| \leq k$ and $D^{\beta_i} f_n \in C^0(B[r])$ since $f_n \in C^k(B[r])$. For reasons mentioned above $D^X D^Y G|_{(x, f_n)}$ is also Cauchy in $C^0(B[r])$. It follows that $D^X D^Y G|_{(x, f_n(x))} \cdot D^{\beta_1} f_n \cdot \ldots \cdot D^{\beta_{|Y|}} f_n$ is Cauchy in $C^0(B[r])$, therefore Cauchy in $L^2(B[r])$.

Case 2: $|Y| = 1$. In this case it is more difficult to prove that the term is Cauchy, since it could include a partial derivative of $f_n$ of order $k + 1$. Our previous argument would fail because $f_n$ is not necessarily in $C^{k+1}(B[r])$.

$D^\beta f_n$ is Cauchy in $L^2(B[r])$ because $|\beta| \leq k + 1$ and $f_n$ is Cauchy in $H^{k+1}(B[r])$. Now, since

$$D^X D^Y G|_{(x, f_m(x))} \cdot D^\beta f_m - D^X D^Y G|_{(x, f_l(x))} \cdot D^\beta f_l = \left(D^X D^Y G|_{(x, f_m(x))} - D^X D^Y G|_{(x, f_l(x))}\right) \cdot D^\beta f_m + D^X D^Y G|_{(x, f_l(x))} \cdot (D^\beta f_m - D^\beta f_l),$$

the fact that $D^X D^Y G|_{(x, f_n(x))}$ is Cauchy (and thus bounded) in $C^0(B[r])$ and $D^\beta f_n$ is Cauchy (and thus bounded) in $L^2(B[r])$ implies $D^X D^Y G|_{(x, f_m(x))} \cdot D^\beta f_n$ is Cauchy in $L^2(B[r])$.

Lemma 4.3. Let $f$ be some function which is bounded below. Let $g$ be some bounded function. Then $\inf(f - kg)$ is a continuous function of $k$.

Proof.

$$\inf((k_2 - k_1)g) \leq \inf(f - k_1 g) - \inf(f - k_2 g) \leq \sup((k_2 - k_1)g)$$

As $g$ is bounded, the function $(k_2 - k_1)g$ can be made arbitrarily close to zero by choosing a sufficiently small $(k_2 - k_1)$. Thus $\inf(f - kg)$ is a continuous function of $k$. 

\end{proof}
References
