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CLASSIFICATION OF FOUR-COMPONENT ROTATIONALLY SYMMETRIC ROSE LINKS

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Abstract. A rose link is a disjoint union of a finite number of unknots. Each unknot is considered a component of the link. We study rotationally symmetric rose links, those that can be rotated in a way that does not change their appearance or true form. Brown used link invariants to classify 3-component rose links; we categorize 4-component rose links using the HOMFLY polynomial.

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1 Introduction

The study of knots began in the field of chemistry in the 1880s [1]. At the time, many chemists believed that there was a substance they called "ether" and that knots made up different forms of matter within it [1]. This belief led to William Thomson's Vortex-Axom Theory [4]. He hypothesized that all chemical elements had their own knot form, but later it was proven that ether did not exist [1]. While chemists had given up on knots, his research sparked interest among mathematicians at the time. Eventually the mathematicians' work on the subject compiled to form the branch of topology known as knot theory.

Figure 1: The unknot and the left-hand trefoil [1, 7].

A *knot* is a "closed curve in 3-dimensional space that does not intersect itself" [1]. In simpler terms, one can view a knot as a string with attached ends that cannot be cut. Figure 1 provides an example of two simple knots, the unknot and a trefoil. A knot is comprised of just one component, while a *link* is "a disjoint union of a finite number of knots, where each knot is called a component of the link." See Figure 2 for an example of a 4-component link [1].

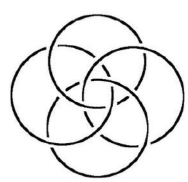

Figure 2: A 4-component rotationally symmetric rose link [1].

Both knots and links can be represented on paper, in 2 dimensions, by either diagrams or projections. A *link* or *knot diagram* depicts the crossings by a break in the underlying strand (such as in Figures 1 and 2), while a *knot* or *link projection* shows the link without denoting the overlapping and underlapping crossings (such as Figure 3, below) [1]. A major goal in knot theory is to determine when two knots (or links) are "equivalent," meaning intuitively that one knot or link diagram can be shifted/transformed to look like the other without untying the knot (or link). In order to prove two knots are not equivalent, knot invariants can be used [1]. An example of a knot invariant is crossing number, or the least number of times the knot crosses itself out of any of its arrangements. For instance, in Figure 1, the unknot has a crossing number of zero and the trefoil has a crossing number of three. Therefore, the unknot and trefoil are not equivalent. To keep track of all discovered knots (or links), Charles Livingston, Morwen Thistlethwaite, Jim Hoste and Jeff Weeks have tabulated knots up to a crossing number of sixteen [8, 9].

In this paper, we completely categorize a certain class of links, 4-component rotationally symmetric rose links, using the HOMFLY polynomial link invariant. A *rose link* is a specific type of link, recognized by its unknotted components [1]. A n-component rose link, as introduced by Amelia Brown in "Rotationally Symmetric Rose Links," is "a link comprised of n unknotted components that can be arranged to yield a n-component rose projection" [1]. The rose links that we will be studying are also rotationally symmetric, as shown in Figures 3 and 4. A rose link is rotationally symmetric "if [in the rose link diagram] the pattern of over and under crossings is the same for every component such that if the diagram is rotated $\frac{360}{n}$ degrees it looks the same" [1]. In Figure 3, we have 3-component and 4-component rotationally symmetric rose link projections. The 3-component rotationally symmetric rose link has $360/3 = 120$ degree rotational symmetry. That is, if we were to rotate the link diagram by 120 degrees, the diagram would appear the same. Similarly, the 4-component rotationally symmetric rose link exhibits 90 degree rotational symmetry.

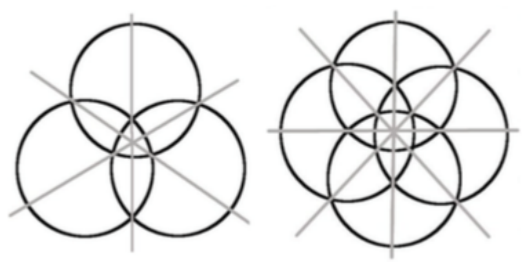

Figure 3: 3-component and 4-component rotationally symmetric rose link projections [1].

In Section 2, we introduce a naming convention for rotationally symmetric rose links and discuss the links we distinguish. In Section 3, we describe link invariants, Reidemeister moves, planar isotopy, and the cube octahedron method. In Section 4, we explain the HOMFLY polynomial and continue in Section 5 by using the HOMFLY polynomial to categorize 4-component rotationally symmetric rose links. We conclude with a discussion of future research possibilities in Section 6.

2 Rotationally Symmetric Rose Links

In this section we introduce a naming convention to distinguish rotationally symmetric rose links. Since the rose links we are studying are rotationally symmetric, each component has the same crossing arrangements [1]. This makes it relatively simple to name them. The crossing types are denoted by an "O" for over crossing, and a "U" for under crossing [1]. Then, after looking at just one component, one can write a pattern of under and over crossings by using these two letters. These patterns are arranged in alphabetical order and numbered from 1 to 2^{n-1}, where n is the number of components of the rotationally symmetric rose link [1]. Note that 2^{n-1} is the number of crossing arrangements for a link of n components. Lastly, this number is divided by two since the mirror images are included [1]. In our case, we are dealing with links of four components, 2^{4-1}, so there are eight links that we want to distinguish, including mirror images.

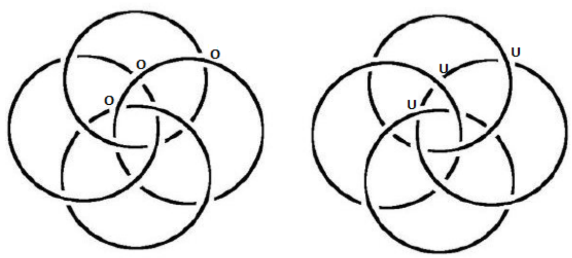

Figure 4: Labeling of under and over crossings for R_1^4 and R_1^{4*} [1].

Below, we have used this naming convention to name the eight 4-component rotationally symmetric rose links that appear in Figures 5 and 6. The first four are numbered according to alphabetical order, while the next four, their mirror images, are assigned the same number with the addition of an asterisk. Diagrams for the mirror images appear in Figure 6. We will determine which of these eight links are equivalent in Section 5, after discussing link equivalence and the HOMFLY polynomial in Sections 4 and 5.

R_1^4	OOO
R_2^4	OOU
R_3^4	OUO
R_4^4	OUU
$R_4^{4^*}$	UOO
$R_3^{4^*}$	UOU
$R_2^{4^*}$	UUO
$R_1^{4^*}$	UUU

Table 1: Crossing arrangements for the 4-component rotationally symmetric rose link diagrams.

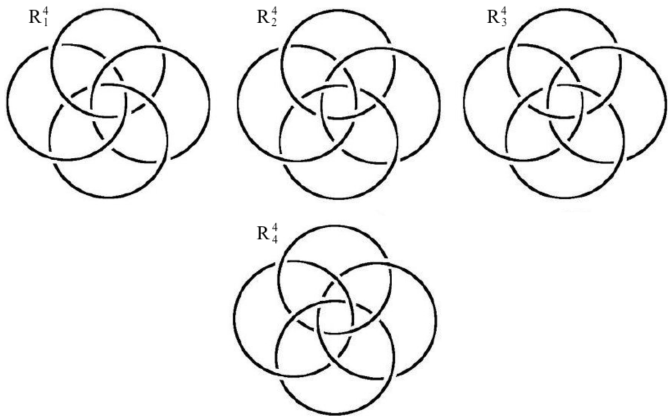

Figure 5: The link diagrams of the four 4-component rotationally symmetric rose links, R_1^4, R_2^4, R_3^4, and R_4^4 [1].

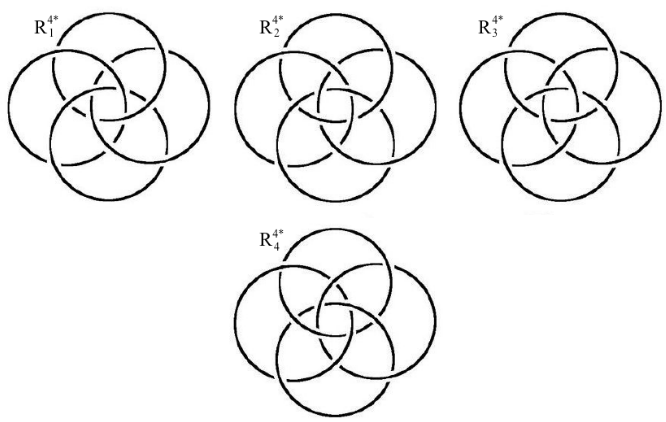

Figure 6: The link diagrams of the mirror images of the 4-component rotationally symmetric rose links, $R_1^{4^*}$, $R_2^{4^*}$, $R_3^{4^*}$, and $R_4^{4^*}$ [1].

3 Link Equivalence

One of the major goals of knot theory is to distinguish one link from another. In this section, we discuss three ways to show that links are equivalent: Reidemeister moves, planar isotopies, and the cube octahedron method. We employed these techniques to show the equivalence of certain rotationally symmetric rose links, as we mention in Section 5 below. We also discuss link invariants in more detail, leading to the development of the HOMFLY polynomial in the next section.

The three Reidemeister moves shown below are a way to change the appearance of the link without changing the link itself. A Type I Reidemeister move is illustrated in Figure 7, and consists of twisting or untwisting a component. A Type II Reidemeister move, shown in

Figure 8, is the overlapping or separation of two strands to remove or create two crossings. Lastly, a Type III Reidemeister move, or either move depicted in Figure 9, is the movement of a strand over a crossing of two strands.

A planar isotopy is the reshaping or stretching of a link. An example of a planar isotopy is shown in Figure 10.

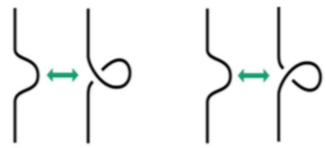

Figure 7: The two variations of the Reidemeister Type I Move [1].

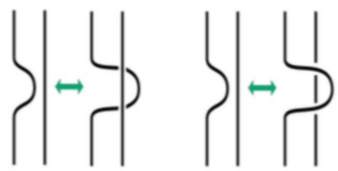

Figure 8: The two variations of the Reidemeister Type II Move [1].

Figure 9: The two variations of the Reidemeister Type III Move [1].

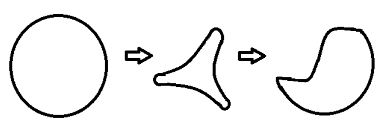

Figure 10: One example of a planar isotopy [1].

An interesting method for proving equivalence of two 4-component rotationally symmetric rose links is to view the link in three dimensions by using a cube octahedron. A cube octahedron is a three-dimensional object that has fourteen faces, twelve vertices and twenty-four edges. Of the fourteen faces, eight faces are triangles and the other six faces are squares [6]. Referring to the diagram of a 4-component rotationally symmetric rose link in Figure 11, notice that the link is also made up of eight triangles and six squares. The triangles are more easily recognized than the squares in Figure 11. Each time all four components meet in a rectangular region that region is a square face of the octahedron. For example, there is a square region in the middle, four square regions at 45°,135°, 225° and 315°, and the last square region is located at the edge of each component. After laying the link on an octahedron, such that its components line up with the edges, we can more clearly see the properties of the link. By flipping the 3-dimensional octahedron to change its position, with a new face on top, we can see the other link. Thus we can prove link equivalence without the explicit use of Reidemeister moves or planar isotopies.

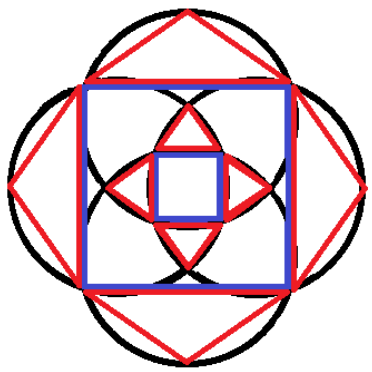

Figure 11: Three-sided and four-sided regions within a 4-component rotationally symmetric rose link.

A link invariant is a quantity or property that is the same for two links that are equivalent. Thus, if the invariant is different for two links, then one link is distinct from the other. However, if the links invariants are the same for two links, it does not neccessarily follow that the links are distinct. If two links do have the same value of a particular invariant, a stronger invariant, such as the HOMFLY polynomial, may distinguish them. As the links get more complicated, there is a developing need for stronger invariants. Some basic link invariants include the number of components, the linking number, and coloring [1]. Stronger link invariants include the Pairwise Linking Number Sum (PLNS), the Alexander, the Jones, and the HOMFLY polynomial [1]. The PLNS is a stronger invariant that builds on the linking number invariant [1]. The HOMFLY polynomial incorporates both the Alexander and Jones polynomials; we will use the HOMFLY polynomial to complete our investigation.

4 The HOMFLY Polynomial

The HOMFLY polynomial, a linking invariant developed in 1984 by Hoste, Ocneau, Millett, Freyd, Lickorish and Yetter, incorporated both the Jones and Alexander polynomials [1]. The HOMFLY polynomial received its unique name from the first letter of the last names of each of its developers. Przytycki and Traczyk are two other mathematicians who worked on the polynomial separately and simultaneously during its development. However, their work arrived too late. At times, Przytycki and Traczyk are credited for their research by adding PT to the end of HOMFLY. Brown used the HOMFLY polynomial in order to classify the 3-component rotationally symmetric rose links [1].

The HOMFLY polynomial is a two variable polynomial. The skein relation is an association of the three types of crossings used to build the HOMFLY polynomial. The diagram of the crossings used to build the skein relation are shown in Figure 11. We will explain the skein relation in further detail later.

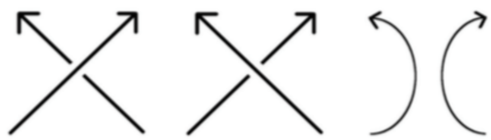

Figure 12: The crossings used to build the skein relation in the HOMFLY polynomial are, from left to right: L_+, L_-,and L_0 [1].

A L_+ crossing in Figure 12 occurs when the overlying strand is oriented upward and the underlying strand approaches from the right [1]. A L_- crossing occurs under the same condition with the underlying strand approaches from the left [1]. A L_0 crossing occurs when the two strands do not cross at all [1].

Let L be the oriented link. Then $P(L)(\ell, m)$ is the HOMFLY polynomial for the link. $P(L)$ is an isotopy invariant, or a polynomial that relies on the movement of the crossing in question. The HOMFLY Polynomial is defined below [3].

1. Let U be the unknot, then $P(U) = 1$.

2. The skein relationship for the HOMFLY polynomial is defined as listed below, where ℓ and m are variables. For 3 link diagrams that are identical except for one crossing (where each one has one of the crossings from Figure 10), label the lines L_+, L_-, and L_0 respectively. Then,

$$\ell P(L_+) + \ell^{-1} P(L_-) + mP(L_0) = 0$$

   Note that this a Laurent polynomial, or a polynomial that can contain negative exponents.

Depending on which crossing we are solving for, we apply the skein relation for the HOMFLY polynomial to relate this crossing to the other two, making diagrams of the link as we change its crossings. Then we solve for the other diagrams in the same fashion, sometimes involving the same process. Eventually, we reach links or knots for which we have already found the HOMFLY polynomial, such as an unknot, or a simpler link. After we have reached this point, we substitute these known polynomials back into the rows of skein relations and reduce the polynomial. We are then left with the HOMFLY polynomial.

For example, we will compute the HOMFLY polynomial of the 2-component unlink, or $P(U^2)$. To compute a HOMFLY polynomial for $P(U^2)$, we need to orient our links; note that we are solving for the L_0 crossing. We must then solve for the other two types of crossings, and then isolate the L_0 crossing in the HOMFLY skein relation. To do this, we introduce a crossing to both of the unknots, turning them into L_+ and L_- crossings.

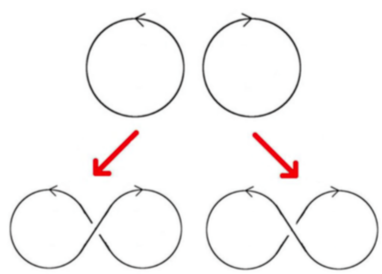

Figure 13: The 2-component unlink, U^2, resolved to a L_+ and L_- crossing, from left to right [1].

Now both of the crossings can be untwisted using Reidemeister Type I move, leaving two unknots. We know from the definition of the HOMFLY polynomial that $P(U) = 1$, so substituting this into the skein relation we get:

$$\ell P(L_+) + \ell^{-1} P(L_-) + mP(L_0) = 0$$

$$\ell(1) + \ell^{-1}(1) + mP(U^2) = 0$$

$$\begin{aligned} P(U^2) &= m^{-1}(-\ell - \ell^{-1}) \\ &= -\ell m^{-1} - \ell^{-1} m^{-1} \end{aligned}$$

In this example, the link could easily be solved by one skein relation. However, finding the HOMFLY polynomial for the 4-component rotationally symmetric rose links requires more than one skein relation. The solution for one crossing is then nested into another skein relation, and so on, until the HOMFLY polynomial is found. We will illustrate this principle by finding the HOMFLY polynomial for the Hopf link. The Hopf link is the top link in Figure 14.

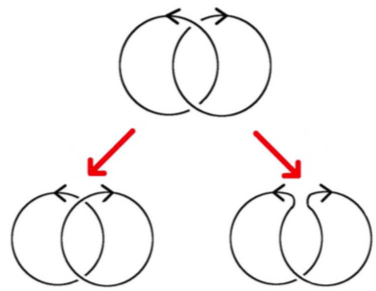

Figure 14: The Hopf Link resolved to a L_+ and L_0 crossing [1].

Here, we are solving for the L_- crossing, so we must solve for the other two crossings, L_+ and L_0, and isolate the L_- in the skein relation. We notice that solving for the L_+ crossing yields the 2-component unlink that we solved before, and that solving for the L_0 crossing yields an unknot. We know that $P(U^2) = m^{-1}(-\ell - \ell^{-1})$ and that $P(U) = 1$.

$$\ell P(L_+) + \ell^{-1} P(L_-) + mP(L_0) = 0$$
$$P(L_-) = \ell(-\ell P(L_+) - mP(L_0))$$

$$\begin{aligned} P(HopfLink) &= \ell(m^{-1}(-\ell - \ell^{-1}) - 1) \\ &= \ell(-\ell(-\ell m^{-1} - \ell^{-1} m^{-1}) - m) \\ &= -\ell m + \ell^3 m^{-1} + \ell m^{-1} \end{aligned}$$

Now, we have reached the HOMFLY polynomial for the Hopf Link by using the 2-component unlink that we solved for before.

In an effort to classify the eight 4-component rotationally symmetric rose links, we found the HOMFLY polynomial for each of the links. Several were calculated in the manner shown above in an effort to understand the HOMFLY polynomial better.

5 Results

In this section we distinguish 4-component rotationally symmetric rose links using HOMFLY polynomials. In all, there were eight different ways in which each link could be oriented, resulting in eight distinct HOMFLY polynomials for each rose link. For our comparison, we chose to orient each component of the four original links and their mirror images counter-clockwise. Unfortunately, computing the HOMFLY polynomial for every 4-component rose link may have lent itself to more human error. Instead, with the help of Dr. Thistlethwaite from the University of Tennessee in Knoxville, we used Ewing-Millett, a computer program, to calculate the HOMFLY polynomials for our eight links. HOMFLY polynomials for R_1^4 and R_3^4 were also calculated by hand to better understand the nested aspect of HOMFLY polynomials. Below are the HOMFLY polynomials for eight 4-component rotationally symmetric rose links:

$P(R_1^4) : -\ell^{15}m^{-3} - 3\ell^{13}m^{-3} - 3\ell^{11}m^{-3} - \ell^9m^{-3} - 6\ell^{13}m^{-1} - 12\ell^{11}m^{-1} + 6\ell^9m^{-1} - 4\ell^{13}m - 23\ell^{11}m - 19\ell^9m + \ell^{13}m^3 + 21\ell^{11}m^3 + 36\ell^9m^3 - 8\ell^{11}m^5 - 28\ell^9m^5 + \ell^{11}m^7 + 9\ell^9m^7 - \ell^9m^9$

$P(R_2^4) = P(R_4^{4^*}) : -\ell^7m^{-3} - 3\ell^5m^{-3} - 3\ell^3m^{-3} - \ell m^{-3} + 2\ell^5m^{-1} + 4\ell^3m^{-1} + 2\ell m^{-1} - \ell^3m - \ell m + \ell^7m^3 + 5\ell^5m^3 + 7\ell^3m^3 + 3\ell m^3 - 5\ell^5m^5 - 11\ell^3m^5 - 4\ell m^5 + \ell^5m^7 + 6\ell^3m^7 + \ell m^7 - \ell^3m^9$

$P(R_3^4) : -\ell^7m^{-3} - 3\ell^5m^{-3} - 3\ell^3m^{-3} - \ell^1m^{-3} + 2\ell^7m^{-1} + 8\ell^5m^{-1} + 10\ell^3m^{-1} + 4\ell^1m^{-1} - \ell^7m - 5\ell^5m - 6\ell^3m - 2\ell^1m + 2\ell^7m^3 + 3\ell^5m^3 - \ell^3m^3 - 2\ell^1m^3 - \ell^7m^5 - 5\ell^5m^5 - \ell^3m^5 - \ell^1m^5 + 2\ell^5m^7 + 3\ell^3m^7 + \ell^1m^7 - \ell^3m^9$

$P(R_4^4) = P(R_2^{4^*}) : -\ell^{-7}m^{-3} - 3\ell^{-5}m^{-3} - 3\ell^{-3}m^{-3} - \ell^{-1}m^{-3} + 2\ell^{-5}m^{-1} + 4\ell^{-3}m^{-1} + 2\ell^{-1}m^{-1} - \ell^{-3}m - \ell^{-1}m + \ell^{-7}m^3 + 5\ell^{-5}m^3 + 7\ell^{-3}m^3 + 3\ell^{-1}m^3 - 5\ell^{-5}m^5 - 11\ell^{-3}m^5 - 4\ell^{-1}m^5 + \ell^{-5}m^7 + 6\ell^{-3}m^7 + \ell^{-1}m^7 - \ell^{-3}m^9$

$P(R_1^{4^*}) : -\ell^{-15}m^{-3} - 3\ell^{-13}m^{-3} - 3\ell^{-11}m^{-3} - \ell^{-9}m^{-3} - 6\ell^{-13}m^{-1} - 12\ell^{-11}m^{-1} + 6\ell^{-9}m^{-1} - 4\ell^{-13}m - 23\ell^{-11}m - 19\ell^{-9}m + \ell^{-13}m^3 + 21\ell^{-11}m^3 + 36\ell^{-9}m^3 - 8\ell^{-11}m^5 - 28\ell^{-9}m^5 + \ell^{-11}m^7 + 9\ell^{-9}m^7 - \ell^{-9}m^9$

$P(R_3^{4^*}) : -\ell^{-7}m^{-3} - 3\ell^{-5}m^{-3} - 3\ell^{-3}m^{-3} - \ell^{-1}m^{-3} + 2\ell^{-7}m^{-1} + 8\ell^{-5}m^{-1} + 10\ell^{-3}m^{-1} + 4\ell^{-1}m^{-1} - \ell^{-7}m - 5\ell^{-5}m - 6\ell^{-3}m - 2\ell^{-1}m + 2\ell^{-7}m^3 + 3\ell^-5m^3 - \ell^-3m^3 - 2\ell^{-1}m^3 - \ell^{-7}m^5 - 5\ell^{-5}m^5 - \ell^{-3}m^5 - \ell^{-1}m^5 + 2\ell^{-5}m^7 + 3\ell^{-3}m^7 + \ell^{-1}m^7 - \ell^{-3}m^9$

In all, there are six different HOMFLY polynomials. This means that there are two sets of links that share the same HOMFLY polynomial, R_2^4, $R_4^{4^*}$ and R_4^4, $R_2^{4^*}$. However, this does not prove that there are exactly six equivalence classes for all 4-component rotationally symmetric rose links. Using both Reidemeister moves, planar isotopies, and the cube octahedron method, we found that R_2^4 is equivalent to $R_4^{4^*}$ and R_4^4 is equivalent to $R_2^{4^*}$. Thus, there are exactly six equivalence classes of 4-component rotationally symmetric rose links.

A rare quality of the HOMFLY polynomials is that the only difference between the HOMFLY polynomial of an original link and its mirror image is the variable ℓ. In the HOMFLY polynomial of a link's mirror image, all the ℓ's are negated. This means that ℓ^{-1} would be changed to ℓ and vice versa. All the other coefficients and exponents stay the same and the exponent associated with the variable m do not change. Our results in finding the HOMFLY polynomials for links and their mirror images prove this observation true. Notice the R_1^4 HOMFLY polynomial and the $R_1^{4^*}$ HOMFLY polynomial are the same throughout other than the negated variable ℓ.

6 Future Work

Thus far, research using HOMFLY polynomials has only been performed with 3 and 4-component rotationally symmetric rose links. This limited research seems to be mainly due to the lack of information available on links with crossing numbers greater than eleven. With the help of Dr. Thistlethwaite's computer program Ewing-Millett, it would be feasible to classify 5-component rotationally symmetric rose links using HOMFLY polynomials. Furthermore, there are other link invariants that can be used to classify rotationally symmetric rose links, such as the Seifert matrix. Just like rotationally symmetric rose links, there are many other types of links that could be classified with HOMFLY polynomials. One such link type is the Brunnian link. Brunnian links are a set of links that turn into unknots even if just one component is removed. These are yet to be categorized using HOMFLY polynomials.

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