Complex Symmetry of Truncated Composition Operators

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Abstract. We define a *truncated composition operator* on the spaces $\mathbb{P}_n$ of n-degree polynomials with complex coefficients. After doing so, we concern ourselves with the *complex symmetry* of such operators, that is, whether there is an orthonormal basis that gives them a symmetric matrix representation.

1 Introduction

Symmetric matrices, whose entries form a mirror image across the diagonal, are a familiar concept in the study of linear algebra and its applications. The concept of a complex symmetric operator is similar in nature and has been shown to have a wide variety of interesting applications within operator theory (Garcia, Putinar [2]). An operator is complex symmetric if it has a symmetric matrix representation with respect to some orthonormal basis. This is distinct from a self-adjoint matrix, in which entries across the diagonal are the complex conjugate of one another rather than identical.

In recent years, researchers have shown an interest in the connection between complex symmetry and composition operators (Narayan et. al. [4]). On any space of analytic functions, the composition operator with symbol C_φ can be defined by $C_\varphi f = f \circ \varphi$. Composition operators are most commonly studied on H^2, the Hilbert space of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ from the open unit disk $\mathbb{D}$ to $\mathbb{C}$ where $||f||^2 = \sum_{k=0}^{\infty} |a_k|^2 < \infty$ (square-summable Taylor series). To make such operators work in finite dimensions, we consider them on the polynomial subspace $\mathbb{P}_n$ of H^2: the space of all polynomials of degree n or less with complex coefficients. To make a composition operator with any given symbol map $\mathbb{P}_n$ back into $\mathbb{P}_n$, we view the operator as map from $\mathbb{P}_n$ to H^2, and then project the image back onto the polynomial space $\mathbb{P}_n$ by removing all terms of degree $n+1$ or higher (truncation).

We write $P_n C_\varphi$ for the operator that maps $\mathbb{P}_n$ to itself and call it a *truncated composition operator*. This is equivalent to considering the upper-left block of the matrix representation of C_φ on the Hardy space H^2 in its standard basis. In particular, the traditional dot product used in $\mathbb{R}^n$ aligns with the inner product that $\mathbb{P}_n$ inherits as a subspace of H^2, allowing us to use familiar tools, and to study complex symmetry in a finite-dimensional setting, yet with implications for study on infinite-dimensional function spaces. Although our work could technically be defined for a variety of symbols φ so that C_φ acts on $\mathbb{P}_n$, our ultimate goal is to inform further work on the Hardy space H^2. To that end, we will focus on symbols that are self-maps of $\mathbb{D}$.

For example, let $\varphi(z) = \frac{z}{2-z} = \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots$ and $f(z) = z^2$. Then $C_\varphi f$ will result in the following Taylor series.

$$(C_\varphi f)(z) = (f \circ \varphi)(z) = (\varphi^2)(z) = \frac{1}{4} z^2 + \frac{2}{8} z^3 + \frac{3}{16} z^4 + \frac{4}{32} z^5 + \frac{5}{64} z^6 + \cdots$$

This means that C_φ acts as a map from $\mathbb{P}_n$ to H^2; the operator P_n will truncate the higher-degree terms and define $P_n C_\varphi$ as a self-map of $\mathbb{P}_n$. Here, we have

$$(P_n C_\varphi f)(z) = (P_n \varphi^2)(z) = \frac{1}{4} z^2 + \frac{2}{8} z^3 + \cdots + \frac{n-1}{2^n} z^n.$$

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be some analytic function with Taylor series $\sum_{k=0}^{\infty} a_k z^k$. With respect to the canonical basis $\{1, z, z^2, z^3, \cdots, z^n\}$, $P_n C_\varphi$ can be written as an $(n+1) \times (n+1)$ matrix. The columns of this matrix correspond to $P_n C_\varphi$ acting on each element of the basis. Figure 1 shows $P_2 C_\varphi$ when $\varphi(z) = \frac{z}{2-z}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Figure 1: Matrix representation of $P_2 C_\varphi$

This matrix would also represent the upper left hand corner of the infinite matrix representation of C_φ with respect to the same canonical basis (now with infinitely many elements).

Next, we define a complex symmetric operator and give an example of a truncated composition operator exhibiting this property.

Definition. An operator T is said to be a **complex symmetric operator** (CSO) if there is an orthonormal basis with respect to which T has a symmetric matrix representation (which may contain complex entries).

As an example of a complex symmetric operator, we again consider $\varphi(z) = \frac{z}{2-z}$. With respect to the orthonormal basis

$$\left\{ 1, \frac{(1+\sqrt{2})e^{\frac{3i\pi}{8}}}{\sqrt{4+2\sqrt{2}}}z + \frac{e^{\frac{3i\pi}{8}}}{\sqrt{4+2\sqrt{2}}}z^2, \frac{(1-\sqrt{2})e^{\frac{-i\pi}{8}}}{\sqrt{4-2\sqrt{2}}}z + \frac{e^{\frac{-i\pi}{8}}}{\sqrt{4-2\sqrt{2}}}z^2 \right\},$$

$P_2 C_\varphi$ has the following symmetric matrix representation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8}(3+\sqrt{2}) & \frac{i}{8} \\ 0 & \frac{i}{8} & \frac{1}{8}(3-\sqrt{2}) \end{bmatrix}$$

Note that this matrix is symmetric, but not self-adjoint. That is, the complex entries are identical over the diagonal and not the complex conjugate of one another.

In the remainder of this paper, we will explore the question of when a truncated composition operator is complex symmetric. In Section 2, we describe our methodology and introduce the Strong Angle Test, which will be used throughout the remainder of the paper. We will then explore the relationship between the complex symmetry of C_φ and $P_2 C_\varphi$ for various symbols $\varphi(z)$. Finally, we will determine symbols $\varphi(z)$ for which $P_n C_\varphi$ is represented by a lower triangular matrix in Section 3, and for which $P_2 C_\varphi$ is represented by an upper triangular matrix in Section 4, with respect to the canonical basis. To conclude, we will present in Section 5 a few questions for further consideration in the future.

2 Methodology and Initial Findings

It is often difficult to find the correct orthonormal basis in order to determine if an operator is complex symmetric. The following theorem allows us to determine whether such a basis exists using just one matrix representation of our operator.

Theorem (The Strong Angle Test [1]). *Given an $n \times n$ matrix M, let $\mathbf{u}_1, \mathbf{u}_2...\mathbf{u}_n$ be the unit eigenvectors of M and let $\mathbf{v}_1, \mathbf{v}_2...\mathbf{v}_n$ be the corresponding unit eigenvectors of M^*, the conjugate transpose of M. Then M is a CSO if and only if*

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle \langle \mathbf{u}_j, \mathbf{u}_k \rangle \langle \mathbf{u}_k, \mathbf{u}_i \rangle = \overline{\langle \mathbf{v}_i, \mathbf{v}_j \rangle \langle \mathbf{v}_j, \mathbf{v}_k \rangle \langle \mathbf{v}_k, \mathbf{v}_i \rangle}$$

for all $1 \leq i \leq j \leq k \leq n$.

Example. Let $\varphi(z) = \frac{z}{2-z}$. With respect to the basis $\{1, z, z^2\}$, $P_2 C_\varphi$ has the following matrix representation and eigenvectors:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

From these eigenvectors, we have the following:

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle \langle \mathbf{u}_2, \mathbf{u}_3 \rangle \langle \mathbf{u}_3, \mathbf{u}_1 \rangle = 0 \times \frac{1}{\sqrt{2}} \times 0 = 0$$

$$\overline{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \langle \mathbf{v}_2, \mathbf{v}_3 \rangle \langle \mathbf{v}_3, \mathbf{v}_1 \rangle} = 0 \times -\frac{1}{\sqrt{2}} \times 0 = 0.$$

Thus, $P_2 C_\varphi$ is a complex symmetric operator by the Strong Angle Test.

We wish to examine whether the property of complex symmetry carries over from C_φ to $P_n C_\varphi$. We summarize the results from examining the case of $P_2 C_\varphi$ for 4 different definitions of φ. The table below displays the results of testing $P_2 C_\varphi$ using the Strong Angle Test alongside knowledge of C_φ from the work in Narayan et. al. [4]. The symbol ✓ indicates that the operator is a complex symmetric operator for the given φ and $\times$ indicates that it is not. Note that the complex symmetry of C_φ seems to have no bearing on the complex symmetry of $P_2 C_\varphi$, one way or the other.

Although we found some truncated composition operators that were complex symmetric, our main result was that for the symbols we considered, complex symmetry does not hold for any particular symbol φ as the dimension of the truncation was increased. In the next sections, we give our justification for this result, considering only symbols that induce upper- and lower-triangular matrix representations. This was inspired by our knowledge of the work of Narayan et. al. [4], which only considered symbols with this property. In every case, we consider symbols whose Taylor series have only real coefficients.

φ	P_2C_φ	C_φ
$\frac{1}{2}z + \frac{1}{2}$	$\times$	$\times$
$\frac{1}{3}z + \frac{1}{3}$	$\times$	$\checkmark$
$\frac{z}{2-z}$	$\checkmark$	$\times$
$\frac{z}{3-z}$	$\checkmark$	$\checkmark$

Table 1: Complex Symmetry of Operators given φ

3 The Lower Triangular Case

3.1 Examining P_2C_φ

For $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_0 = 0$, the matrix representation of P_2C_φ is lower triangular.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & a_2 & a_1{}^2 \end{bmatrix}$$

Figure 2: Matrix of P_2C_φ with $a_0 = 0$

Here, $f(z) \equiv 1$ is a unit eigenvector for both P_2C_φ and $(P_2C_\varphi)^*$, while all of the other unit eigenvectors for both operators do not have a constant term. Therefore, for all a_1 and a_2, $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$. So, P_2C_φ is always a complex symmetric operator by the Strong Angle Test. When will symbols of the same form produce complex symmetric P_nC_φ for larger values of n?

3.2 Examining $P_nC_\varphi, n > 2$

For further study, we consider P_nC_φ where $\varphi(z) = a_1 z + a_2 z^2$. For the following, we take advantage of the fact that for P_nC_φ, $\langle \mathbf{u}_1, \mathbf{u}_i \rangle = \langle \mathbf{v}_1, \mathbf{v}_i \rangle = 0$ for all $2 \le i \le n + 1$.
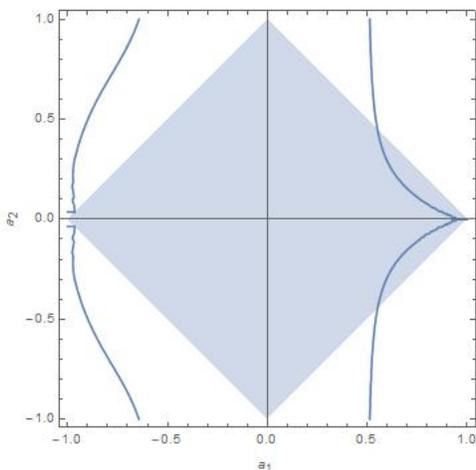
Figure 3: Values of a_1 and a_2 for which P_3C_φ is a CSO.

The graph in Figure 3 shows values of a_1 and a_2 for which P_3C_φ with $\varphi(z) = a_1 z + a_2 z^2$ is complex symmetric by the Strong Angle Test. The shaded region represents symbols that map the open unit interval to itself (those in which $|a_1| + |a_2| < 1$), which is our area of interest.
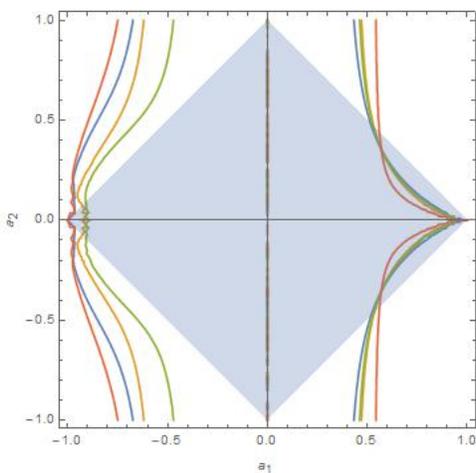

Figure 4: The Strong Angle Test on P_4C_φ

The four equations produced by applying the Strong Angle Test to P_4C_φ are graphed in Figure 4. There are at most two points satisfying all 4 equations when both coefficients are non-zero.
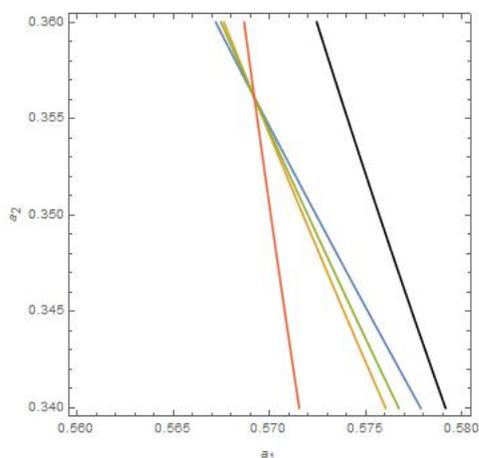
Figure 5: A closer view of Figure 4 along with Figure 3

In Figure 5, it appears that the four curves intersect near $a_1 = .569$ and $a_2 = .356$. The black line represents the curve plotted in Figure 3, which does not pass near this point. So, if there is a function $\varphi(z) = a_1 z + a_2 z^2$ for which $P_4 C_\varphi$ is complex symmetric, the previous truncation $P_3 C_\varphi$ was not.
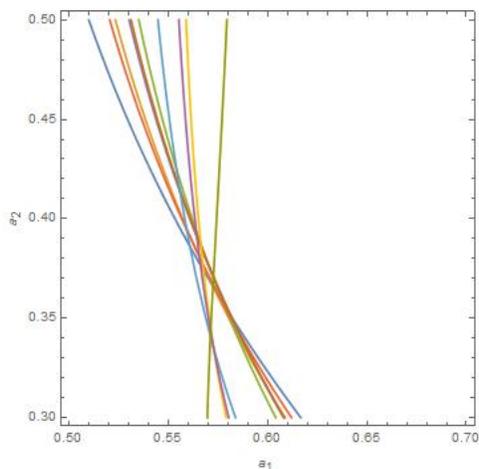

Figure 6: The Strong Angle Test on $P_5 C_\varphi$

By applying the same approach to $P_5 C_\varphi$, the resulting system of equations has no solution. This can be seen in Figure 6. Therefore, there is no $\varphi(z) = a_1 z + a_2 z^2$ for which $P_5 C_\varphi$ is complex symmetric.

In summary, it appears that there is no $\varphi(z) = a_1 z + a_1 z^2$ for which $P_n C_\varphi$ is a complex symmetric operator for all n. Moreover, we seem to lose a dimension in our solution set for each larger value of n. It seems that by forcing each truncation to be in terms of the same coefficients, we are limiting ourselves too far. In order to preserve complex symmetry from

one truncation to the next, perhaps φ needs to introduce new coefficients into the corresponding matrix.

Conjecture. If φ has a finite Taylor series, then there exists n such that P_nC_φ is not a complex symmetric operator.

4 Upper Triangular Case, P_2C_φ

For $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_2 = 0$, the matrix representation of P_2C_φ is upper triangular.

$$\begin{bmatrix} 1 & a_0 & a_0{}^2 \\ 0 & a_1 & 2a_0a_1 \\ 0 & 0 & a_1{}^2 \end{bmatrix}$$

Figure 7: Matrix of P_2C_φ with $a_2 = 0$

If $a_0 = 0$, the matrix above will be diagonal, and therefore symmetric. Other than this trivial solution, there are no real values of a_0 and a_1 for which P_2C_φ is complex symmetric by the Strong Angle Test.

5 Further Questions

We conclude with some questions for further research.

1. For $\varphi(z)=a_1z$, P_nC_φ is a CSO for all n because the corresponding matrix will always be diagonal. Besides this trivial case, are there any symbols φ for which P_nC_φ is a CSO for all n?

2. Composition operators on H^2 that are complex symmetric have yet to be completely classified, particularly in the difficult case when φ is not linear-fractional. Can truncated composition operators help solve this problem?

3. What other properties beyond complex symmetry, meaningful in both finite and infinite dimensions, can be studied for truncated composition operators?

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