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The Calculus of Proportional α-Derivatives

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Abstract. We introduce a new proportional α-derivative with parameter $\alpha \in [0, 1]$, explore its calculus properties, and give several examples of our results. We begin with an introduction to our proportional α-derivative and some of its basic calculus properties. We next investigate the system of α-lines which make up our curved yet Euclidean geometry, as well as address traditional calculus concepts such as Rolle's Theorem and the Mean Value Theorem in terms of our α-derivative. We also introduce a new α-integral to be paired with our α-derivative, which leads to proofs of the Fundamental Theorem of Calculus Parts I and II, as applied to our formulas. Finally, we provide instructions on how to locate α-maximum and α-minimum values as they are related to our type of Euclidean geometry, including an increasing and decreasing test, concavity test, and first and second α-derivative tests.

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1 Introduction

When thinking of derivatives of the function $f(t)$ in traditional calculus, it is common practice to take the first derivative, $\frac{df}{dt}$, the second derivative, $\frac{d^2f}{dt^2}$, the third derivative $\frac{d^3f}{dt^3}$, and so on. Here, we propose a type of proportional derivative $D^\alpha f(t)$, with $\alpha \in [0,1]$, to explore the behavior of a function $f(t)$ as it fluctuates between its identity and its first derivative.

We begin with some essential definitions pertaining to our topic of proportional α-derivatives.

Definition 1 (Conformable Differential Operator)**.** Let $\alpha \in [0,1]$. A differential operator D^α is conformable if and only if D^0 is the identity operator and D^1 is the classical differential operator. Specifically, D^α is conformable if and only if for a differentiable function $f(t)$,

$$D^0 f(t) = f(t) \quad \text{and} \quad D^1 f(t) = \frac{d}{dt} f(t) = f'(t).$$

Remark 2. In this paper we will investigate a specific type of derivative that is motivated by the idea of a proportional-derivative (PD) controller from control theory, such as that used by Piltan, Mehrara, Meigolinedjad, and Bayat [1, Equations (4) and (5)], for example. PD controllers are used to tune the measured signal, say in a robot manipulator, toward the desired input signal. In particular, if U is the controller output over time, then

$$U_\alpha = K_{V_\alpha} e' + K_{p_\alpha} e,$$

where

$$e(t) = \theta_{target}(t) - \theta_d(t)$$

is the error between the measured and desired signals, and the tuning parameters K_{V_α} and K_{p_α} stand for the derivative gain and the proportional gain, respectively.

We now define the main object of interest in our paper.

Definition 3 (A Type of Conformable Derivative)**.** Let $\alpha \in [0,1]$. The following differential operator D^α, defined via

$$D^\alpha f(t) = \alpha f'(t) + (1-\alpha) f(t) \tag{1}$$

is the α-derivative of $f(t)$, and is conformable provided the function $f(t)$ is differentiable at t and $f'(t) := \frac{d}{dt} f(t)$. Furthermore, the partial α-derivative with respect to t is given by

$$D_t^\alpha f(t,s) = \alpha \frac{\partial}{\partial t} f(t,s) + (1-\alpha) f(t,s).$$

This definition is a special case of a more theoretical definition given in Anderson and Ulness [2, Definition 1.3].

Remark 4. We note that another definition of a conformable derivative appears in Camrud [3].

Example 5. Let $\alpha \in [0,1]$. If we take the α-derivative of a function $f(t) = t^2$ at $\alpha = \frac{n}{8}$, $0 \leq n \leq 8$, $n \in \mathbb{Z}$, we have the following:

$$\begin{aligned}
\alpha &= 0: \quad D^0 f(t) = t^2 \\
\alpha &= \frac{1}{8}: \quad D^{\frac{1}{8}} f(t) = .25t + .875t^2 \\
\alpha &= \frac{2}{8}: \quad D^{\frac{2}{8}} f(t) = .5t + .75t^2 \\
\alpha &= \frac{3}{8}: \quad D^{\frac{3}{8}} f(t) = .75t + .625t^2 \\
\alpha &= \frac{4}{8}: \quad D^{\frac{4}{8}} f(t) = t + .5t^2 \\
\alpha &= \frac{5}{8}: \quad D^{\frac{5}{8}} f(t) = 1.25t + .375t^2 \\
\alpha &= \frac{6}{8}: \quad D^{\frac{6}{8}} f(t) = 1.5t + .25t^2 \\
\alpha &= \frac{7}{8}: \quad D^{\frac{7}{8}} f(t) = 1.75t + .125t^2 \\
\alpha &= 1: \quad D^1 f(t) = 2t.
\end{aligned}$$

Graphically, these look like

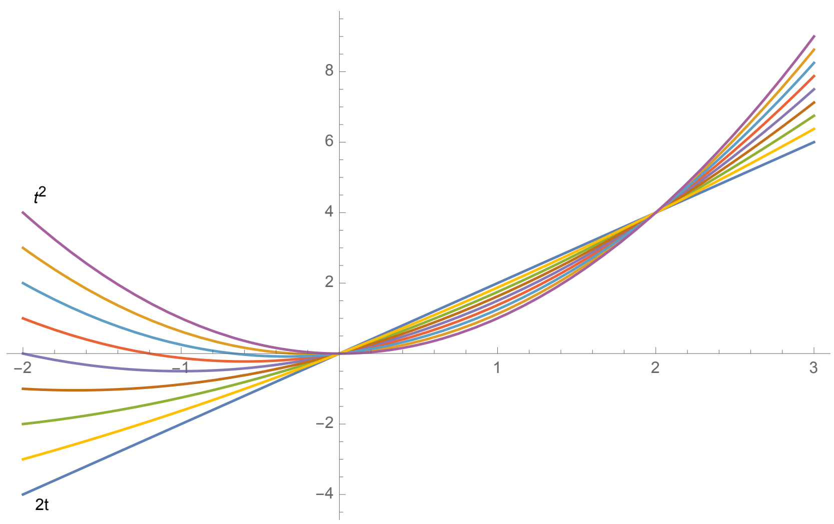


Figure 1: In this graph, we can see that as α increases from 0 to 1, the α-derivative of t^2 bends from t^2 toward $2t$.

Throughout the rest of the paper, we will be exploring the properties of this proportional α-derivative by investigating its implications with laws of traditional calculus, such as basic derivative and integration rules, as well as evaluating function behavior through local maximums and minimums, increasing and decreasing tests, and concavity tests. In particular, in Section 2 we cover basic α-derivative rules, what a "horizontal line" looks like according to our proportional derivative, the definitions of both an α-secant line and an α-tangent line, as well as proofs of Rolle's Theorem and the Mean Value Theorem, followed by several examples. In Section 3, we provide a definition for an α-integral to be paired with our α-derivative, a proof of the Fundamental Theorem of Calculus with respect to our α-integral and α-derivative, and additional examples. Finally, in Section 4, we state definitions of α-increasing and α-decreasing, a critical point, and α-maximums and α-minimums, as well as proving theorems relating to an increasing/decreasing test, first and second α-derivative tests, and a concavity test, ending with a final example and theorems using the α-derivative to locate critical points in certain functions.

2 Lines and the MVT of Proportional α-Derivatives

Using formula (1), in this section we cover basic α-derivative rules, what a "horizontal line" looks like according to our proportional derivative, the definitions of both an α-secant line and an α-tangent line, as well as proofs of Rolle's Theorem and the Mean Value Theorem, followed by several examples. We begin with the following basic α-derivative formulas, which are special cases of those found in Anderson and Ulness [2], but are included here for completeness:

Lemma 6 (Basic α-Derivatives). *Let the conformable differential operator D^α be given as in* (1)*, where $\alpha \in [0,1]$. Assume the functions f and g are differentiable as needed. Then*

(*i*) $D^\alpha[af+bg] = aD^\alpha[f] + bD^\alpha[g]$ *for all* $a, b \in \mathbb{R}$*;*

(*ii*) $D^\alpha c = c(1-\alpha)$ *for all constants* $c \in \mathbb{R}$*;*

(*iii*) $D^\alpha[fg] = fD^\alpha[g] + gD^\alpha[f] - (1-\alpha)[fg]$*;*

(*iv*) $D^\alpha[f/g] = \frac{gD^\alpha[f]-fD^\alpha[g]}{g^2} + (1-\alpha)\frac{f}{g}$.

Proof. The proofs of (*i*) and (*ii*) follow immediately from Definition 3. We prove (*iii*), the α-Product Rule. Let the conditions given in Lemma 6 be satisfied. By traditional calculus,

$$[fg]' = fg' + f'g.$$

Applying the α-derivative rather than the traditional derivative, we get

$$\begin{aligned} D^\alpha[fg] &= \alpha[fg]' + (1-\alpha)[fg] \\ &= \alpha[fg' + f'g] + (1-\alpha)[fg] \\ &= \alpha[fg'] + \alpha[f'g] + (1-\alpha)[fg] \\ &= f[\alpha g' + (1-\alpha)g] + [\alpha f' + (1-\alpha)f]g - (1-\alpha)[fg] \\ &= fD^\alpha[g] + gD^\alpha[f] - (1-\alpha)[fg]. \end{aligned}$$

This completes the proof of (*iii*). A similar process is used in proving (*iv*), the α-Quotient Rule. □

We now present a vital definition which establishes a type of exponential function for the α-derivative (1). Note that this is a special case of the exponential function introduced in Anderson and Ulness [2]. However, the results below that follow from the definition are new.

Definition 7 (Conformable Exponential Function)**.** Let $\alpha \in (0,1]$, the points $s, t \in \mathbb{R}$, and let the function $p : [s,t] \to \mathbb{R}$ be continuous. Then the conformable exponential function with respect to D^α in (1) is defined to be

$$e_p(t,s) := e^{\int_s^t \frac{p(\tau)-(1-\alpha)}{\alpha} d\tau}, \quad e_0(t,s) = e^{-\left(\frac{1-\alpha}{\alpha}\right)(t-s)}, \tag{2}$$

and, for a fixed value of s, satisfies

$$D_t^\alpha e_p(t,s) = p(t) e_p(t,s), \quad D_t^\alpha e_0(t,s) = 0. \tag{3}$$

The following useful properties of the exponential function are a direct result of (2).

Lemma 8 (Exponential Function Properties)**.** *Let p, q be continuous functions, and let $t, s, r \in \mathbb{R}$. For the conformable exponential function given in* (2)*, the following properties hold:*

(i) $e_p(t,t) \equiv 1$;

(ii) $e_p(t,s) e_p(s,r) = e_p(t,r)$;

(iii) $\frac{1}{e_p(t,s)} = e_p(s,t)$;

(iv) $e_{(1-\alpha)}(t,s) \equiv 1$;

(v) $e_{(\alpha-1)}(t,s) = e_0^2(t,s)$;

(vi) $e_p(t,s) e_q(t,s) = e_{(p+q-1+\alpha))}(t,s)$;

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{(p-q+1-\alpha)}(t,s)$.

Proof. We will prove (iii) and (vi): For (iii), note that

$$\begin{aligned} \frac{1}{e_p(t,s)} &= e^{-\int_s^t \frac{p(\tau)-(1-\alpha)}{\alpha} d\tau} \\ &= e^{\int_t^s \frac{p(\tau)-(1-\alpha)}{\alpha} d\tau} \\ &= e_p(s,t). \end{aligned}$$

For (vi),

$$\begin{aligned} e_p(t,s) e_q(t,s) &= e^{\int_s^t \frac{p(\tau)-(1-\alpha)}{\alpha} d\tau} e^{\int_s^t \frac{q(\tau)-(1-\alpha)}{\alpha} d\tau} \\ &= e^{\int_s^t \frac{p(\tau)-(1-\alpha)}{\alpha} d\tau + \int_s^t \frac{q(\tau)-(1-\alpha)}{\alpha} d\tau} \\ &= e^{\int_s^t \frac{[p(\tau)+q(\tau)-(1-\alpha)]-(1-\alpha)}{\alpha} d\tau} \\ &= e_{(p+q-1+\alpha))}(t,s). \end{aligned}$$

The others are proved similarly. $\square$

Definition 9. Following from formula (2), we define a "horizontal" α-line to be a function with slope $D^\alpha f(t) = 0$ and thus of the form

$$f(t) = ce_0(t, t_0),$$

and a general α-line to be a function of the form

$$f(t) = c_1 e_0(t, t_0) + c_2 e_0(t, t_0)t,$$

where t_0 is fixed.

Remark 10. Due to formula (3) and Definition 9, the geodesics of this operator, namely those curves with zero acceleration (see, for example, McCleary and O'Neill [4, 5]), include the horizontal α-lines and the general α-lines. Nevertheless, the calculus employed in this paper can be considered a Euclidean calculus. Recall that a Euclidean geometry is one in which the parallel postulate holds; namely, that given a line y_1 and a point p not on y_1, there exists exactly one line y_2 through p and parallel to y_1.
For example, consider the α-line $y_1 = e_0(t, 0) + e_0(t, 0)t$, and the point $(0, 0)$ not on y_1. It is considered an α-line because $D^\alpha[D^\alpha[y_1]] = 0$; this is comparable to traditional calculus in which $y'' = 0$ for $y = c_1 + c_2 t$, a traditional line. In order to prove that this geometry is Euclidean, we must (i) prove that there exists exactly one α-line through the origin parallel to y_1, and (ii) prove that those two α-lines never intersect.

(i) A general α-line is $y_2 = c_1 e_0(t, 0) + c_2 e_0(t, 0)t$. To go through the origin, we need $c_1 = 0$. Thus, $y_2 = c_2 e_0(t, 0)t$ is an α-line which goes through the point $(0, 0)$. In order for y_2 to be parallel to y_1, their slopes must be equivalent. Thus,

$$D^\alpha y_1 = D^\alpha y_2.$$

By the Product Rule (iii) in Lemma 6,

$$\begin{aligned} 0 + e_0(t,0)D^\alpha[t] + t(0) - (1-\alpha)e_0(t,0)t &= c_2[e_0(t,0)D^\alpha[t] + t(0)] - (1-\alpha)te_0(t,0) \\ 1 &= c_2. \end{aligned}$$

Thus, y_2 is parallel to y_1 when $c_2 = 1$.

(ii) If y_1 and y_2 were to intersect, then the following would have to be true at some t:

$$\begin{aligned} y_1 &= y_2 \\ e_0(t,0)(1+t) &= c_2 e_0(t,0)t \\ 1+t &= c_2 t \\ 1 &= (c_2 - 1)t, \quad \text{which is} \\ t &= \frac{1}{c_2 - 1}, \quad c_2 \neq 1. \end{aligned}$$

Thus, the α-lines y_1 and y_2 never intersect when $c_2 = 1$;

$$y_1 = e_0(t,0) + e_0(t,0)t \neq e_0(t,0)t = y_2 \quad \forall t.$$

Therefore, the α-lines are parallel and their geometry is Euclidean.

Theorem 11. *Two important geodesics are the α-secant line for a function f from a to b given by*

$$\sigma(t) := e_0(t,a)f(a) + h_1(t,a)\frac{e_0(t,b)f(b) - e_0(t,a)f(a)}{h_1(b,a)}, \quad h_1(t,a) := \frac{t-a}{\alpha}, \tag{4}$$

and the α-tangent line for a function f differentiable at a given by

$$\ell(t) := e_0(t,a)f(a) + h_1(t,a)e_0(t,a)D^\alpha f(a). \tag{5}$$

Proof. We offer an explanation for formulas (4) and (5). We use the equation of the general α-line in Definition 9, except we introduce some slight modifications, for later convenience:

$$\sigma(t) = c_1 e_0(t,a) + c_2 e_0(t,a)\left(\frac{t-a}{\alpha}\right).$$

These modifications do not change the α-linear quality of this equation, as can be verified by checking that $D^\alpha[D^\alpha[\sigma(t)]] = 0$.
We use the more general function $\sigma(t)$ to find the specific function of the α-secant line by solving for the c_1 and c_2 values. We know that a secant line $\sigma(t)$ of a function $f(t)$ intercepts the function at two points: $(a, f(a))$ and $(b, f(b))$. Plugging a and b into $\sigma(t)$ and solving for the constants c_1 and c_2, we have $\sigma(a) = f(a)$, which yields

$$c_1 = f(a),$$

and $\sigma(b) = f(b)$, which yields

$$c_2 = \frac{f(b) - f(a)e_0(b,a)}{e_0(b,a)\left(\frac{b-a}{\alpha}\right)}.$$

Substituting these values into $\sigma(t)$, we have

$$\sigma(t) = f(a)e_0(t,a) + \frac{f(b) - f(a)e_0(b,a)}{e_0(b,a)\left(\frac{b-a}{\alpha}\right)} e_0(t,a)\left(\frac{t-a}{\alpha}\right).$$

Simplifying the expression, we have

$$\begin{aligned}
\sigma(t) &= f(a)e_0(t,a) + \frac{f(b) - f(a)e_0(b,a)}{h_1(b,a)} e_0(t,a)e_0(a,b)h_1(t,a) \\
&= f(a)e_0(t,a) + \frac{f(b) - f(a)e_0(b,a)}{h_1(b,a)} e_0(t,b)h_1(t,a) \\
&= e_0(t,a)f(a) + h_1(t,a)\frac{e_0(t,b)f(b) - e_0(t,a)f(a)}{h_1(b,a)},
\end{aligned}$$

which is the definition of the α-secant line. This equation can be compared to the familiar equation $y = b + tm$, where b is the y-intercept, t is the variable, and m is the slope.

Similarly, we can find the equation of the α-tangent line

$$\ell(t) = c_1 e_0(t,a) + c_2 e_0(t,a)\left(\frac{t-a}{\alpha}\right)$$

to the point $(a, f(a))$ by plugging a into $\ell(t)$ and $D^\alpha \ell(t)$ and solving for c_1 and c_2:

$$\ell(a) = c_1 = f(a)$$

and

$$D^\alpha \ell(a) = c_2 = D^\alpha f(a).$$

Substituting these values into $\ell(t)$, we have

$$\ell(t) = f(a)e_0(t,a) + D^\alpha f(a) e_0(t,a)\left(\frac{t-a}{\alpha}\right),$$

which is equivalent to

$$\ell(t) = e_0(t,a)f(a) + h_1(t,a)e_0(t,a)D^\alpha f(a),$$

the definition of the α-tangent line. □

Remark 12. Note that if the α-secant line for a function f is a line with slope $= 0$ from $(a, f(a))$ to $(b, f(b))$, then

$$f(a) = e_0(a,b)f(b).$$

Theorem 13 (Rolle's Theorem)**.** *Let $\alpha \in (0,1]$. If a function f is continuous on $[a,b]$ and differentiable on (a,b), with*

$$f(a) = e_0(a,b)f(b),$$

then there exists at least one number $c \in (a,b)$ such that $D^\alpha f(c) = 0$.

Proof. Let f satisfy the hypotheses of the theorem, and set

$$g(t) := e_0(a,t)f(t) - f(a).$$

Then $g(a) = 0$, and $g(b) = 0$ by assumption. Now g is continuous on $[a,b]$ and differentiable on (a,b), so by classical Rolle's Theorem there exists $c \in (a,b)$ such that $g'(c) = 0$. Thus we have

$$\begin{aligned} 0 &= g'(c) = e_0(a,c)f'(c) + f(c)e_0(a,c)\frac{1-\alpha}{\alpha} \\ &= \frac{e_0(a,c)}{\alpha}\left(\alpha f'(c) + (1-\alpha)f(c)\right) = \frac{e_0(a,c)}{\alpha}D^\alpha f(c) \end{aligned}$$

by (1), which yields $D^\alpha f(c) = 0$ as $e_0(a,c)$ is never zero. □

Theorem 14 (Mean Value Theorem)**.** *Let $\alpha \in (0,1]$. If the function f is continuous on $[a,b]$ and differentiable on (a,b), then there exists at least one number $c \in (a,b)$ such that*

$$D^\alpha f(c) = \frac{e_0(c,b)f(b) - e_0(c,a)f(a)}{h_1(b,a)},$$

where $h_1(b,a) = \frac{b-a}{\alpha}$.

Proof. Let f satisfy the hypotheses of the theorem, and set

$$g(t) := f(t) - e_0(t,a)f(a) - \frac{h_1(t,a)}{h_1(b,a)}\left(e_0(t,b)f(b) - e_0(t,a)f(a)\right),$$

where $h_1(t,a) = \frac{t-a}{\alpha}$. Then g is continuous on $[a,b]$, differentiable on (a,b), and $g(a) = 0 = g(b)$. By Rolle's Theorem 13 above, there exists $c \in (a,b)$ such that $D^\alpha g(c) = 0$. Thus we have

$$0 \;=\; D^\alpha g(c) = D^\alpha f(c) - 0 - \left(\frac{e_0(c,b)f(b) - e_0(c,a)f(a)}{h_1(b,a)}\right),$$

which gives the result. □

Example 15. In this example, we illustrate the Mean Value Theorem. Let $\alpha \in (0,1)$. Given $f(t) = 1$, $a = 0$, and $b = 1$, find c such that

$$D^\alpha f(c) = \frac{e_0(c,b)f(b) - e_0(c,a)f(a)}{h_1(b,a)},$$

where $h_1(b,a) = \frac{b-a}{\alpha}$.
We begin by substituting the given values into the Mean Value Theorem equation:

$$D^\alpha[1] = \frac{e_0(c,1)(1) - e_0(c,0)(1)}{h_1(1,0)}.$$

Using the property listed in Lemma (8) (ii) and solving, we have

$$\begin{aligned}
\alpha(0) + (1-\alpha)(1) &= \frac{e_0(c,0)e_0(0,1) - e_0(c,0)}{\frac{1-0}{\alpha}} \\
1-\alpha &= \alpha e_0(c,0)[e_0(0,1) - 1], \quad \text{which is} \\
e_0(c,0) &= \frac{1-\alpha}{\alpha[e_0(0,1) - 1]}.
\end{aligned}$$

Recall Definition (2). Solving, we have

$$\begin{aligned}
e^{-\left(\frac{1-\alpha}{\alpha}\right)c} &= \frac{1-\alpha}{\alpha\left(e^{\frac{1-\alpha}{\alpha}}-1\right)} \\
-\left(\frac{1-\alpha}{\alpha}\right)c &= \ln\left[\frac{1-\alpha}{\alpha\left(e^{\frac{1-\alpha}{\alpha}}-1\right)}\right] \\
c &= -\left(\frac{\alpha}{1-\alpha}\right)\ln\left[\frac{1-\alpha}{\alpha\left(e^{\frac{1-\alpha}{\alpha}}-1\right)}\right].
\end{aligned}$$

Although we have solved for the value of c, we must check that $c \in (0,1)$. If $\frac{1-\alpha}{\alpha\left(e^{\frac{1-\alpha}{\alpha}}-1\right)} < 1$, we know that $\ln\left[\frac{1-\alpha}{\alpha\left(e^{\frac{1-\alpha}{\alpha}}-1\right)}\right]$ is negative and thus $c > 0$. To prove this, we start by stating that

$$0 < \frac{1}{2!}\left(\frac{1}{\alpha}-1\right)^2 + \frac{1}{3!}\left(\frac{1}{\alpha}-1\right)^3 + \cdots,$$

which is always true since $\alpha \in (0,1)$. With some minor algebraic adjustments, we see that the right hand side of

$$\frac{1}{\alpha} < 1 + \left(\frac{1}{\alpha}-1\right) + \frac{1}{2!}\left(\frac{1}{\alpha}-1\right)^2 + \frac{1}{3!}\left(\frac{1}{\alpha}-1\right)^3 + \cdots$$

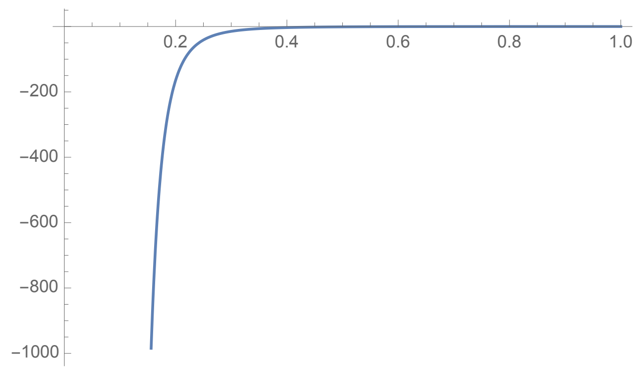
is the Maclaurin series for $e^{\frac{1-\alpha}{\alpha}}$ (see Example 19 for the traditional definition of a Maclaurin series). Substituting, we have

$$\begin{aligned}
\frac{1}{\alpha} &< e^{\frac{1-\alpha}{\alpha}} \\
1 &< \alpha e^{\frac{1-\alpha}{\alpha}} \\
1-\alpha &< \alpha e^{\frac{1-\alpha}{\alpha}} - \alpha = \alpha\left(e^{\frac{1-\alpha}{\alpha}}-1\right) \\
\frac{1-\alpha}{\alpha\left(e^{\frac{1-\alpha}{\alpha}}-1\right)} &< 1.
\end{aligned}$$

Thus, $c > 0 \ \forall \ \alpha \in (0,1)$.
Now we must check that $c < 1$. We start with the function

$$g(\alpha) := \left(\frac{2\alpha-1}{\alpha}\right)e^{\frac{1-\alpha}{\alpha}} - 1$$

and its graph



As seen from the graph of $g(\alpha)$ above, $g(\alpha) < 0 \; \forall \; \alpha \in (0,1)$. Thus

$$\begin{aligned}
\left(\frac{2\alpha-1}{\alpha}\right) e^{\frac{1-\alpha}{\alpha}} &< 1 \\
(2\alpha-1)e^{\frac{1-\alpha}{\alpha}} &< \alpha \\
2\alpha e^{\frac{1-\alpha}{\alpha}} - \alpha &< e^{\frac{1-\alpha}{\alpha}} \\
\alpha e^{\frac{1-\alpha}{\alpha}} - \alpha &< e^{\frac{1-\alpha}{\alpha}} - \alpha e^{\frac{1-\alpha}{\alpha}} \\
\frac{\alpha\left(e^{\frac{1-\alpha}{\alpha}} - 1\right)}{1-\alpha} &< e^{\frac{1-\alpha}{\alpha}} \\
\ln\left[\frac{\alpha\left(e^{\frac{1-\alpha}{\alpha}} - 1\right)}{1-\alpha}\right] &< \frac{1-\alpha}{\alpha} \\
c = \left(\frac{\alpha}{1-\alpha}\right)\ln\left[\frac{\alpha\left(e^{\frac{1-\alpha}{\alpha}} - 1\right)}{1-\alpha}\right] &< 1.
\end{aligned}$$

Hence, we have verified that $0 < c < 1 \; \forall \; \alpha \in (0,1)$.

3 α-Antiderivatives, the α-Integral, and the FTC

In this section, we provide a definition for an α-integral to be paired with our α-derivative, a proof of the Fundamental Theorem of Calculus with respect to our α-integral and α-derivative, and additional examples.

Definition 16. Let $\alpha \in [0,1]$. If a function F is differentiable and $D^{\alpha}F = f$, then F is called an α-antiderivative of f.

Definition 17. Let $\alpha \in (0,1]$. Define the α-integral $\int\!\!\!\!\alpha \; f dt = F$ provided that F is an α-antiderivative of f.

Theorem 18 (Indefinite α-Integral)**.** *Let $\alpha \in (0,1]$. If F is an α-antiderivative of f, then F satisfies this formula:*

$$F(t) = \int_{\alpha} f(t)dt = \frac{e_0(t,0)}{\alpha} \int f(t)e_0(0,t)dt + ce_0(t,0).$$

Proof. Suppose F is the α-antiderivative of f. Then

$$f = D^{\alpha} F$$

$$f = \alpha F' + (1-\alpha)F.$$

For $\alpha \in (0,1]$, solve for F:

$$F' + \frac{1-\alpha}{\alpha} F = \frac{1}{\alpha} f.$$

This is in the form of a first-order differential equation. To solve, we find the integrating factor $e^{\int \frac{1-\alpha}{\alpha} dt} = e^{\frac{1-\alpha}{\alpha} t}$. Distributing, we have

$$\begin{aligned}
e^{\frac{1-\alpha}{\alpha} t} F' + \left(\frac{1-\alpha}{\alpha}\right) e^{\frac{1-\alpha}{\alpha} t} F &= \frac{1}{\alpha} e^{\frac{1-\alpha}{\alpha} t} f \\
\left[e^{\frac{1-\alpha}{\alpha} t} F\right]' &= \frac{1}{\alpha} e^{\frac{1-\alpha}{\alpha} t} f \\
e^{\frac{1-\alpha}{\alpha} t} F &= \frac{1}{\alpha} \int e^{\frac{1-\alpha}{\alpha} t} f dt + c \\
F &= \frac{e^{-\frac{1-\alpha}{\alpha} t}}{\alpha} \int e^{\frac{1-\alpha}{\alpha} t} f dt + ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}.
\end{aligned}$$

This completes the proof. □

Example 19. In this example, we show how the α-antiderivative F is found of a specific function f, and how this α-antiderivative compares to traditional calculus as $\alpha \to 0^+$ and $\alpha \to 1^-$. Let $\alpha \in (0,1)$. If $f(t) = t$, then

$$\int_{\alpha} t dt = F(t) = \frac{e^{-\left(\frac{1-\alpha}{\alpha} t\right)}}{\alpha} \int te^{\frac{1-\alpha}{\alpha} t} dt + ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}.$$

Using integration by parts, we have

$$\begin{aligned}
F(t) &= \frac{e^{-\left(\frac{1-\alpha}{\alpha}\right)t}}{\alpha} \left[\frac{\alpha}{1-\alpha}\left(te^{\frac{1-\alpha}{\alpha} t} - \int e^{\frac{1-\alpha}{\alpha} t} dt\right)\right] + ce^{-\left(\frac{1-\alpha}{\alpha}\right)t} \\
&= \frac{e^{-\left(\frac{1-\alpha}{\alpha}\right)t}}{1-\alpha} \left[te^{\frac{1-\alpha}{\alpha} t} - \frac{\alpha}{1-\alpha} e^{\frac{1-\alpha}{\alpha} t}\right] + ce^{-\left(\frac{1-\alpha}{\alpha}\right)t} \\
&= \frac{1}{1-\alpha}\left[t - \frac{\alpha}{1-\alpha}\right] + ce^{-\left(\frac{1-\alpha}{\alpha}\right)t},
\end{aligned}$$

which is the general expression of $F(t)$. For convenience, we choose c so that $F(0) = C$. Thus

$$F(t) = \frac{1}{1-\alpha}\left[t - \frac{\alpha}{1-\alpha}\right] + \left(\frac{\alpha}{(1-\alpha)^2} + C\right)e^{-\left(\frac{1-\alpha}{\alpha}\right)t}.$$

We now test the equation as $\alpha \to 0^+$ and $\alpha \to 1^-$ to compare our results with those obtained through methods of traditional calculus. As we let $\alpha \to 0^+$, we see that

$$\lim_{\alpha\to 0^+} F(t) = \lim_{\alpha\to 0^+}\left[\frac{1}{1-\alpha}\left(t - \frac{\alpha}{1-\alpha}\right) + \left(\frac{\alpha}{(1-\alpha)^2} + C\right)e^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right].$$

This simplifies to

$$\lim_{\alpha\to 0^+} F(t) = t = f(t), \quad t \geq 0,$$

which is what we expect, since by setting $\alpha = 0$ we are essentially leaving the function $f(t)$ untouched. As we let $\alpha \to 1^-$,

$$\lim_{\alpha\to 1^-} F(t) = \lim_{\alpha\to 1^-}\left[\frac{1}{1-\alpha}\left(t - \frac{\alpha}{1-\alpha}\right) + \left(\frac{\alpha}{(1-\alpha)^2} + C\right)e^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right].$$

However, as can been seen from this limit, the solution does not exist due to dividing by zero, so we must take a different approach. According to the definition of a Maclaurin series,

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots.$$

If we factor an $\frac{\alpha}{(1-\alpha)^2}$ from our equation, we have

$$\begin{aligned} F(t) &= \frac{\alpha}{(1-\alpha)^2}\left[\frac{1-\alpha}{\alpha}\left(t - \frac{\alpha}{1-\alpha}\right) + e^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right] + Ce^{-\left(\frac{1-\alpha}{\alpha}\right)t} \\ &= \frac{\alpha}{(1-\alpha)^2}\left[\frac{1-\alpha}{\alpha}t - 1 + e^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right] + Ce^{-\left(\frac{1-\alpha}{\alpha}\right)t} \end{aligned}$$

When compared to the Maclaurin series, we see that if we set $x = -\left(\frac{1-\alpha}{\alpha}\right)t$, then

$$e^{-\left(\frac{1-\alpha}{\alpha}\right)t} - 1 - \left(-\left(\frac{1-\alpha}{\alpha}\right)t\right) = \frac{1}{2!}\left(-\left(\frac{1-\alpha}{\alpha}\right)t\right)^2 + \frac{1}{3!}\left(-\left(\frac{1-\alpha}{\alpha}\right)t\right)^3 + \cdots.$$

Substituting the right-hand side of this equation into our equation, we get

$$\begin{aligned} F(t) &= \frac{\alpha}{(1-\alpha)^2}\left[\frac{1}{2!}\left(-\left(\frac{1-\alpha}{\alpha}\right)t\right)^2 + \frac{1}{3!}\left(-\left(\frac{1-\alpha}{\alpha}\right)t\right)^3 + \cdots\right] + Ce^{-\left(\frac{1-\alpha}{\alpha}\right)t} \\ &= \frac{\alpha}{(1-\alpha)^2}\left(-\left(\frac{1-\alpha}{\alpha}\right)t\right)^2\left[\frac{1}{2!} + \frac{1}{3!}\left(-\left(\frac{1-\alpha}{\alpha}\right)t\right) + \cdots\right] + Ce^{-\left(\frac{1-\alpha}{\alpha}\right)t} \\ &= \frac{1}{\alpha}t^2\left[\frac{1}{2!} + \frac{1}{3!}\left(-\left(\frac{1-\alpha}{\alpha}\right)t\right) + \cdots\right] + Ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}. \end{aligned}$$

Now as we let $\alpha \to 1^-$, we have

$$\begin{aligned}\lim_{\alpha\to 1^-} F(t) &= \lim_{\alpha\to 1^-} \frac{1}{\alpha}t^2\left[\frac{1}{2!}+\frac{1}{3!}\left(-\left(\frac{1-\alpha}{\alpha}\right)t\right)+\cdots\right]+Ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}\\ &\overset{\text{DS}}{=} t^2\left[\frac{1}{2}+0+\cdots\right]+Ce^0=\frac{1}{2}t^2+C.\end{aligned}$$

This agrees with traditional calculus, which states that the antiderivative of $f(t)=t$ is $F(t)=\frac{1}{2}t^2+c$.

Example 20. We present another example of how the α-antiderivative F is found of a specific function f, and how this α-antiderivative compares to traditional calculus as $\alpha \to 0^+$ and $\alpha \to 1^-$. Let $\alpha \in (0,1)$. If $f(t)=\cos(t)$, then

$$F(t) = \int_\alpha \cos(t)dt = \frac{e^{-\frac{1-\alpha}{\alpha}t}}{\alpha}\int \cos(t)e^{\frac{1-\alpha}{\alpha}t}dt + ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}.$$

Applying integration by parts, we have

$$F(t)=\frac{e^{-\frac{1-\alpha}{\alpha}t}}{\alpha}\left[\frac{\alpha}{1-\alpha}\left(\cos(t)e^{\frac{1-\alpha}{\alpha}t}+\int \sin(t)e^{\frac{1-\alpha}{\alpha}t}dt\right)\right]+ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}.$$

Applying integration by parts a second time, we have

$$F(t)=\frac{e^{-\frac{1-\alpha}{\alpha}t}}{\alpha}\left[\frac{\alpha}{1-\alpha}\left(\cos(t)e^{\frac{1-\alpha}{\alpha}t}+\frac{\alpha}{1-\alpha}\left\{\sin(t)e^{\frac{1-\alpha}{\alpha}t}-\int \cos(t)e^{\frac{1-\alpha}{\alpha}t}dt\right\}\right)\right]+ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}.$$

Setting the equation equal to the original equation $F(t)$ and isolating the integral, we have

$$\int \cos(t)e^{\frac{1-\alpha}{\alpha}t}dt=\frac{\alpha(1-\alpha)}{(1-\alpha)^2+\alpha^2}e^{\frac{1-\alpha}{\alpha}t}\left[\cos(t)+\frac{\alpha}{1-\alpha}\sin(t)\right].$$

Substituting this back into the original equation, we see that

$$\begin{aligned}F(t) &= \frac{e^{-\frac{1-\alpha}{\alpha}t}}{\alpha}\left[\frac{\alpha(1-\alpha)}{(1-\alpha)^2+\alpha^2}e^{\frac{1-\alpha}{\alpha}t}\left(\cos(t)+\frac{\alpha}{1-\alpha}\sin(t)\right)\right]+ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}\\ &= \frac{1-\alpha}{(1-\alpha)^2+\alpha^2}\left[\cos(t)+\frac{\alpha}{1-\alpha}\sin(t)\right]+ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}.\end{aligned}$$

We now test the equation as $\alpha \to 0^+$ and $\alpha \to 1^-$ to compare our results with those obtained through methods of traditional calculus. As we let $\alpha \to 0^+$, we see that

$$\lim_{\alpha\to 0^+} F(t) = \lim_{\alpha\to 0^+}\left[\frac{1-\alpha}{(1-\alpha)^2+\alpha^2}\left[\cos(t)+\frac{\alpha}{1-\alpha}\sin(t)\right]+ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right].$$

This simplifies to

$$\lim_{\alpha\to 0^+} F(t) = \cos(t) = f(t), \quad t \geq 0.$$

As we let $\alpha \to 1^-$, we have

$$\lim_{\alpha\to 1^-} F(t) = \lim_{\alpha\to 1^-} \left[\frac{1-\alpha}{(1-\alpha)^2+\alpha^2}\left[\cos(t) + \frac{\alpha}{1-\alpha}\sin(t)\right] + ce^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right],$$

which, when expanded, simplifies to

$$\int\!\!\!\!\!\alpha\; f(t)dt = \lim_{\alpha\to 1^-} F(t) = \sin(t) + c = \int f(t), \qquad \alpha \to 1^-.$$

Thus the α-integral agrees with traditional calculus.

Definition 21. Let $\alpha \in (0,1]$. Then, drawn from Theorem 18, the definite α-integral from a to b of an integrable function f is

$$\int\!\!\!\!\!\alpha_{\,a}^{\,b} f(t)dt = \frac{e_0(b,a)}{\alpha}\int_a^b f(t)e_0(a,t)dt = \frac{1}{\alpha}\int_a^b f(t)e_0(b,t)dt.$$

Theorem 22 (Fundamental Theorem of Calculus)**.** *Let $\alpha \in (0,1]$, and let the exponential function e_0 be given as in equation* (2)*.*

(FTC I) If f is integrable, then

$$D^\alpha\left[\int\!\!\!\!\!\alpha_{\,a}^{\,t} f(s)ds\right] = f(t). \tag{6}$$

(FTC II) If $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and f' is integrable on $[a,b]$, then

$$\int\!\!\!\!\!\alpha_{\,a}^{\,b} \left[D^\alpha[f(t)]\right]dt = f(b) - f(a)e_0(b,a).$$

Proof. (FTC II) We first prove the Fundamental Theorem of Integral Calculus, FTC II. Let P be any partition of $[a,b]$, $P = \{t_0, t_1, \cdots, t_n\}$, and recall that $h_1(t,a) = \frac{t-a}{\alpha}$. By the Mean Value Theorem 14 applied to f on $[t_{i-1}, t_i]$, there exist $c_i \in (t_{i-1}, t_i)$ such that

$$D^\alpha f(c_i) = \frac{e_0(c_i,t_i)f(t_i) - e_0(c_i,t_{i-1})f(t_{i-1})}{h_1(t_i,t_{i-1})},$$

or equivalently

$$D^\alpha[f(c_i)]e_0(b,c_i)h_1(t_i,t_{i-1}) = e_0(b,t_i)f(t_i) - e_0(b,t_{i-1})f(t_{i-1}).$$

After forming the Riemann-Stieltjes sum

$$S(P,f,\mu) = \sum_{i=1}^n D^\alpha[f(c_i)]e_0(b,c_i)\left(\mu(t_i) - \mu(t_{i-1})\right), \quad \mu(t) := \frac{t-a}{\alpha} = h_1(t,a),$$

we see that

$$S(P,f,\mu) = \sum_{i=1}^{n} e_0(b,t_i)f(t_i) - e_0(b,t_{i-1})f(t_{i-1}) = f(b) - e_0(b,a)f(a).$$

Since P was arbitrary,

$$\begin{aligned}
\int_a^b D^\alpha[f(t)]e_0(b,t)d\mu(t) &= \int_a^b D^\alpha[f(t)]e_0(b,t)\mu'(t)dt \\
&= \frac{1}{\alpha}\int_a^b D^\alpha[f(t)]e_0(b,t)dt \\
&= \alpha\int_a^b D^\alpha[f(t)]dt \\
&= f(b) - e_0(b,a)f(a).
\end{aligned}$$

This completes the proof of FTC II. □

Proof. (FTC I) We next prove the Fundamental Theorem of Differential Calculus, FTC I. Let the conditions given above be satisfied. By FTC II,

$$\frac{1}{\alpha}\int_a^t D^\alpha[g(s)]e_0(t,s)ds = g(t) - g(a)e_0(t,a).$$

By taking the α-derivative of both sides, we have

$$D^\alpha\left[\frac{1}{\alpha}\int_a^t D^\alpha[g(s)]e_0(t,s)ds\right] = D^\alpha[g(t)] - 0.$$

Setting the general equation $D^\alpha g(x) := f(x)$, x being an arbitrary variable, we get

$$D^\alpha\left[\frac{1}{\alpha}\int_a^t f(s)e_0(t,s)ds\right] = f(t), \quad \text{so}$$

$$f(t) = \alpha\left[\frac{1}{\alpha}\int_a^t f(s)e_0(t,s)ds\right]' + (1-\alpha)\left[\frac{1}{\alpha}\int_a^t f(s)e_0(t,s)ds\right].$$

Using Leibniz Rule to solve, we have

$$f(t) = f(t)e_0(t,t) + \int_a^t f(s)\frac{d}{dt}\left[e^{-\left(\frac{1-\alpha}{\alpha}\right)(t-s)}\right]ds + \left(\frac{1-\alpha}{\alpha}\right)\int_a^t f(s)e_0(t,s)ds$$

$$f(t) = f(t) + \int_a^t f(s)e_0(t,s)\left(-\frac{1-\alpha}{\alpha}\right)ds + \left(\frac{1-\alpha}{\alpha}\right)\int_a^t f(s)e_0(t,s)ds$$

$$f(t) = f(t).$$

This completes the proof of FTC I. □

Example 23. Let $\alpha \in (0,1]$, $[a,b] \subset \mathbb{R}$, and $r > 0$. In this example, we evaluate

$$\int_a^b \left(\frac{t-a}{\alpha}\right)^{r-1} \left[r + (1-\alpha)\left(\frac{t-a}{\alpha}\right)\right] d_\alpha t$$

using the Fundamental Theorem of Integral Calculus, Theorem 22. Note that

$$\begin{aligned}
\left(\frac{t-a}{\alpha}\right)^{r-1} \left[r + (1-\alpha)\left(\frac{t-a}{\alpha}\right)\right] &= r\left(\frac{t-a}{\alpha}\right)^{r-1} + (1-\alpha)\left(\frac{t-a}{\alpha}\right)^{r} \\
&= \alpha r\left(\frac{t-a}{\alpha}\right)^{r-1}\frac{1}{\alpha} + (1-\alpha)\left(\frac{t-a}{\alpha}\right)^{r} \\
&= D^\alpha\left[\left(\frac{t-a}{\alpha}\right)^{r}\right].
\end{aligned}$$

Thus by Theorem 22,

$$\begin{aligned}
\int_a^b \left(\frac{t-a}{\alpha}\right)^{r-1} \left[r + (1-\alpha)\left(\frac{t-a}{\alpha}\right)\right] d_\alpha t &= \int_a^b D^\alpha\left[\left(\frac{t-a}{\alpha}\right)^{r}\right] d_\alpha t \\
&= \left(\frac{b-a}{\alpha}\right)^{r}.
\end{aligned}$$

This concludes the example. □

4 Finding α-Maximums and α-Minimums

In this section, we state definitions of α-increasing and α-decreasing, a critical point, and α-maximums and α-minimums, as well as proving theorems relating to an increasing/decreasing test, first and second α-derivative tests, and a concavity test, ending with a final example and theorems using the α-derivative to locate critical points in certain functions. Many of these are extensions of ideas from traditional calculus. See, for example, Stewart [6].

Definition 24. Let $\alpha \in (0,1]$. A function f is α-increasing on an interval $\mathcal{I}$ if

$$f(t_1) \le e_0(t_1,t_2)f(t_2), \quad \text{whenever} \quad t_1 < t_2, \quad t_1,t_2 \in \mathcal{I},$$

and is strictly α-increasing if

$$f(t_1) < e_0(t_1,t_2)f(t_2), \quad \text{whenever} \quad t_1 < t_2, \quad t_1,t_2 \in \mathcal{I}.$$

A function f is α-decreasing on an interval $\mathcal{I}$ if

$$e_0(t_2,t_1)f(t_1) \ge f(t_2), \quad \text{whenever} \quad t_1 < t_2, \quad t_1,t_2 \in \mathcal{I},$$

and is strictly α-decreasing if

$$e_0(t_2,t_1)f(t_1) > f(t_2), \quad \text{whenever} \quad t_1 < t_2, \quad t_1,t_2 \in \mathcal{I}.$$

Theorem 25 (Increasing/Decreasing Test)**.** *Letting $\alpha \in (0,1]$, suppose that $D^\alpha f(t)$ exists on some interval $\mathcal{I}$.*

(*i*) *If $D^\alpha f(t) > 0$ for all $t \in \mathcal{I}$, then $f(t)$ is strictly α-increasing on $\mathcal{I}$.*

(*ii*) *If $D^\alpha f(t) < 0$ for all $t \in \mathcal{I}$, then $f(t)$ is strictly α-decreasing on $\mathcal{I}$.*

Proof. (*i*) Let $t_1, t_2 \in \mathcal{I}$ with $t_2 > t_1$. Since $D^\alpha f(t) > 0$ for all $t \in \mathcal{I}$, f is differentiable on $[t_1, t_2]$, and the Mean Value Theorem 14 yields a number $c \in (t_1, t_2)$ such that

$$\frac{e_0(c,t_2)f(t_2) - e_0(c,t_1)f(t_1)}{h_1(t_2,t_1)} = D^\alpha f(c) > 0.$$

Thus $e_0(c,t_2)f(t_2) > e_0(c,t_1)f(t_1)$, or equivalently $e_0(t_1,t_2)f(t_2) > f(t_1)$ after multiplying both sides by $e_0(t_1,c) > 0$ and using Lemma 8. Referring to Definiton 24, this proves that f is strictly α-increasing on $\mathcal{I}$. The proof of (*ii*) is similar and hence omitted. □

Definition 26 (Critical Point)**.** Let $\alpha \in [0,1]$. A function f has a critical point at t_0 if $D^\alpha f(t_0) = 0$ or $D^\alpha f(t_0)$ does not exist.

Definition 27. Let $\alpha \in (0,1]$. A function f has

(*i*) a local α-maximum at t_0 if $f(t_0) \geq e_0(t_0,t)f(t)$;

(*ii*) a local α-minimum at t_0 if $f(t_0) \leq e_0(t_0,t)f(t)$

for all t near t_0.

Theorem 28 (First α-Derivative Test)**.** *Letting $\alpha \in (0,1]$, suppose that t_0 is a critical point of a continuous function f.*

(*i*) *If $D^\alpha f$ changes from positive to negative at t_0, then f has a local α-maximum at t_0.*

(*ii*) *If $D^\alpha f$ changes from negative to positive at t_0, then f has a local α-minimum at t_0.*

(*iii*) *If $D^\alpha f$ does not change sign at t_0, then f has neither an α-maximum nor α-minimum at t_0.*

Proof. This follows directly from the statement of Theorem 25. □

Theorem 29 (Concavity Test)**.** *Letting $\alpha \in (0,1]$, suppose that $D^\alpha D^\alpha f(t)$ exists on some interval $\mathcal{I}$.*

(*i*) *If $D^\alpha D^\alpha f(t) > 0 \; \forall \, t \in \mathcal{I}$, then the graph of f is concave upward on $\mathcal{I}$.*

(*ii*) *If $D^\alpha D^\alpha f(t) < 0 \; \forall \, t \in \mathcal{I}$, then the graph of f is concave downward on $\mathcal{I}$.*

Here concave upward means the function f lies above all of its α-tangents (see equation 2.4) on $\mathcal{I}$, and concave downward means the function f lies below all of its α-tangents on $\mathcal{I}$.

Proof. (i) Let $a, t \in \mathcal{I}$. We will show that the function f lies above the α-tangent line (see equation (5)) through the point $(a, f(a))$. First, let $t > a$. By the Mean Value Theorem 14 applied to f on $[a, t]$, there exists a number $c \in (a, t)$ such that

$$D^\alpha f(c) = \frac{e_0(c,t)f(t) - e_0(c,a)f(a)}{h_1(t,a)};$$

solving this for $f(t)$ we have

$$f(t) = e_0(t,a)f(a) + h_1(t,a)e_0(t,c)D^\alpha f(c).$$

Since $D^\alpha D^\alpha f(t) > 0$ for all $t \in \mathcal{I}$, we know that $D^\alpha f$ is strictly α-increasing on $\mathcal{I}$, and thus

$$e_0(a,t)D^\alpha f(t) > D^\alpha f(a), \quad t \in \mathcal{I}, \quad t > a,$$

so that in particular we have

$$e_0(t,c)D^\alpha f(c) > e_0(t,a)D^\alpha f(a).$$

Consequently,

$$f(t) \;=\; e_0(t,a)f(a) + h_1(t,a)e_0(t,c)D^\alpha f(c) > e_0(t,a)f(a) + h_1(t,a)e_0(t,a)D^\alpha f(a) = \ell(t),$$

where ℓ is the α-tangent line to f at a given in equation (5). If $t < a$, then

$$e_0(t,c)D^\alpha f(c) < e_0(t,a)D^\alpha f(a),$$

but multiplication by the negative number $h_1(t,a)$ reverses the inequality, so the proof is still valid. The proof of (ii) is similar and thus omitted. □

Theorem 30 (Second α-Derivative Test)**.** *Letting $\alpha \in (0, 1]$, suppose $D^\alpha D^\alpha f$ is continuous near t_0.*

(i) *If $D^\alpha f(t_0) = 0$ and $D^\alpha D^\alpha f(t_0) > 0$, then f has a local α-minimum at t_0.*

(ii) *If $D^\alpha f(t_0) = 0$ and $D^\alpha D^\alpha f(t_0) < 0$, then f has a local α-maximum at t_0.*

Proof. This follows directly from the Concavity Test and the definition of a Critical Point. □

Example 31. In this example, we classify a critical point and explain its features in terms of our α-geometry.

If $f(t) = t$ and $\alpha \in [0, 1)$, there exists a critical point t_0 where $D^\alpha f(t) = 0$ or $D^\alpha f(t)$ does not exist, as stated in Definition 26. Therefore, $f(t)$ has a critical point where

$$D^\alpha f(t) = \alpha + (1 - \alpha)t = 0,$$

namely at

$$t_0 = -\frac{\alpha}{1-\alpha}.$$

Taking the second α-derivative, we see that

$$\begin{aligned} D^\alpha D^\alpha f(t) &= \alpha(1-\alpha) + (1-\alpha)(\alpha + (1-\alpha)t) \\ &= (1-\alpha)(2\alpha + (1-\alpha)t). \end{aligned}$$

Plugging in $t_0 = -\frac{\alpha}{1-\alpha}$, we have

$$\begin{aligned} D^\alpha D^\alpha f\left(-\frac{\alpha}{1-\alpha}\right) &= (1-\alpha)\left(2\alpha + (1-\alpha)\left(-\frac{\alpha}{1-\alpha}\right)\right) \\ &= \alpha(1-\alpha) > 0. \end{aligned}$$

This states that $f(t)$ is concave upward, and therefore the critical point t_0 is an α-minimum, according to Theorems 29 and 30. Note that as $\alpha \to 1^-$, $t_0 \to -\infty$, which makes sense in traditional calculus because the "minimum" of the line $f(t) = t$ is $-\infty$.

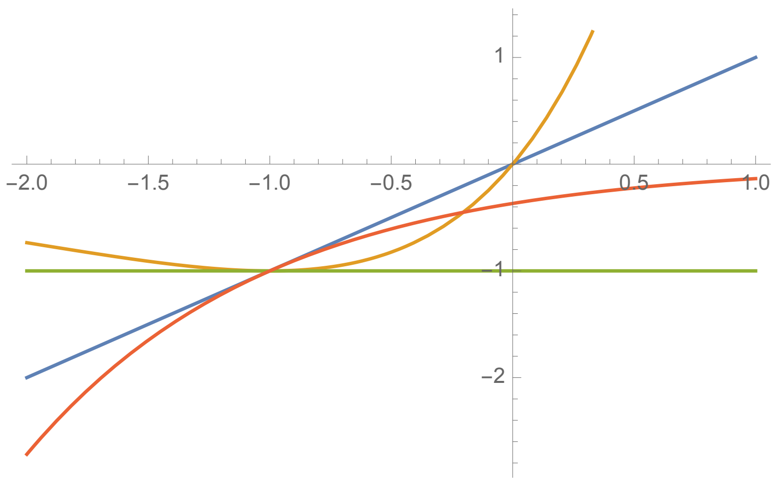


Figure 2: The functions are graphed setting $\alpha = 1/2$. The blue function is the original function, $f(t) = t$. The red function is the α-tangent line (5) to $f(t)$ at the point $t_0 = -\frac{\alpha}{1-\alpha} = -1$, which is the local minimum. We can better visualize this relationship through the orange and green functions, which act as an interpretation of the red and blue functions, respectively, to traditional calculus. Taken from Definition 27, the formula for the orange function is $e_0(t_0, t)f(t) = e_0(-1, t)t$, and the green function is $f(t_0) = -1$. As we would expect from traditional calculus, the orange function $t\left(e^{\left(\frac{(1-\alpha)}{\alpha}\right)(1+t)}\right)$ lies above the green function, which is a horizontal line tangent to the curve at $t_0 = -1$ and is thus a minimum.

Theorem 32 (α-Minimum and α-Maximum Values of $\sin kt$)**.** *Let $\alpha \in (0, 1]$, and assume k*

is any real, positive number and n is an integer. The function $\sin kt$ has an α-maximum of

$$\frac{k\alpha}{\sqrt{(k^2+1)\alpha^2-2\alpha+1}} \quad at \quad -\frac{1}{k}\arctan\left(\frac{k\alpha}{1-\alpha}\right)+\frac{n\pi}{k}, \quad n \text{ odd,}$$

and an α-minimum of

$$\frac{-k\alpha}{\sqrt{(k^2+1)\alpha^2-2\alpha+1}} \quad at \quad -\frac{1}{k}\arctan\left(\frac{k\alpha}{1-\alpha}\right)+\frac{n\pi}{k}, \quad n \text{ even.}$$

Proof. Let $\alpha \in (0,1)$. Setting $D^\alpha[\sin kt]=0$, we have

$$D^\alpha[\sin kt] = k\alpha\cos kt + (1-\alpha)\sin kt = 0.$$

Solving for t, we have

$$(1-\alpha)\sin kt = -k\alpha\cos kt$$

$$\frac{\sin kt}{\cos kt} = -\frac{k\alpha}{1-\alpha}$$

$$\tan kt = -\frac{k\alpha}{1-\alpha}.$$

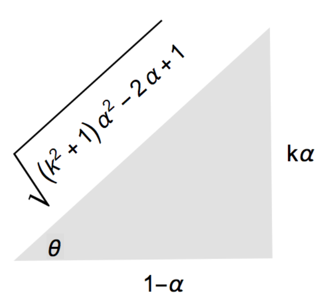
Recalling the periodic nature of $\tan kt$, we can solve for $t = t_n$:

$$t_n = -\frac{1}{k}\arctan\left(\frac{k\alpha}{1-\alpha}\right)+\frac{n\pi}{k},$$

in particular,

$$t_0 = -\frac{1}{k}\arctan\left(\frac{k\alpha}{1-\alpha}\right) = -\frac{1}{k}\theta.$$

Before solving for the α-maximum and α-minimum values, it is important to note that $\theta = \arctan\left(\frac{k\alpha}{1-\alpha}\right)$; this is visually represented in the right triangle



.

Solving for the function values at the critical points t_n, we insert $t_n = t_0 + \frac{n\pi}{k}$ into $\sin kt$:

$$\begin{aligned}
\sin kt_n &= \sin kt_0 \cos\left[k\left(\frac{n\pi}{k}\right)\right] + \sin\left[k\left(\frac{n\pi}{k}\right)\right]\cos kt_0 \\
&= (-1)^n \sin kt_0 + (0)\cos kt_0 \\
&= (-1)^n \sin\left[k\left(-\frac{1}{k}\theta\right)\right] \\
&= (-1)^{n+1}\sin\theta \\
&= (-1)^{n+1}\frac{k\alpha}{\sqrt{(k^2+1)\alpha^2 - 2\alpha + 1}}.
\end{aligned}$$

It is apparent that the quality of the critical points $(t_n, \sin kt_n)$ depend on the value of n. We conduct the Second α-Derivative Test listed in Theorem 30 to determine which points are α-minimums and which are α-maximums:

$$D^\alpha[D^\alpha[\sin kt]] = (-1)^n \frac{k\alpha(k^2\alpha^2 + (1-\alpha)^2)}{\sqrt{(k^2+1)\alpha^2 - 2\alpha + 1}}$$

$$\begin{aligned}
n \text{ odd} : D^\alpha[D^\alpha[\sin kt]] &< 0; \quad \text{maximum} \\
n \text{ even} : D^\alpha[D^\alpha[\sin kt]] &> 0; \quad \text{minimum.}
\end{aligned}$$

Thus, the theorem holds for $\alpha \in (0,1)$. Note that as $\alpha \to 1^-$, $\sin kt$ has an α-maximum of

$$\lim_{\alpha \to 1^-} \frac{k\alpha}{\sqrt{(k^2+1)\alpha^2 - 2\alpha + 1}} = \frac{k(1)}{\sqrt{(k^2+1)(1)^2 - 2(1) + 1}} = \frac{k}{\sqrt{k^2+1-1}} = \frac{k}{k} = 1$$

at

$$\lim_{\alpha \to 1^-} -\frac{1}{k}\arctan\left(\frac{k\alpha}{1-\alpha}\right) + \frac{n\pi}{k} = \frac{-\pi}{2k} + \frac{n\pi}{k}, \quad n \text{ odd.}$$

Similarily, as $\alpha \to 1^-$, $\sin kt$ has a minimum of -1 at $\frac{-\pi}{2k} + \frac{n\pi}{k}$, n even, which agrees with traditional calculus. □

Theorem 33 (α-Minimum and α-Maximum Values of $\cos kt$)**.** *Let $\alpha \in (0,1)$, and assume k is any real, positive number and n is an integer. The function $\cos kt$ has an α-maximum of*

$$\frac{k\alpha}{\sqrt{(k^2+1)\alpha^2 - 2\alpha + 1}} \quad \textit{at} \quad -\frac{1}{k}\arctan\left(\frac{k\alpha}{1-\alpha}\right) + \frac{n\pi}{k}, \quad n \textit{ even,}$$

and an α-minimum of

$$\frac{-k\alpha}{\sqrt{(k^2+1)\alpha^2 - 2\alpha + 1}} \quad \textit{at} \quad -\frac{1}{k}\arctan\left(\frac{k\alpha}{1-\alpha}\right) + \frac{n\pi}{k}, \quad n \textit{ odd.}$$

The proof of Theorem 33 is similar to that of $\sin kt$, and is thus omitted.

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