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An Algebraic Characterization of Highly Connected $2n$-Manifolds

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Abstract. All surfaces, up to homeomorphism, can be formed by gluing the edges of a polygon. This process is generalized into the idea of a $(n, 2n)$-cell complex: forming a space by attaching a $(2n-1)$-sphere into a wedge sum of n-spheres. In this paper, we classify oriented, $(n-1)$-connected, compact and closed $2n$-manifolds up to homotopy by treating them as $(n, 2n)$-cell complexes. To simplify the calculation, we create a basis called the Hilton basis for the homotopy class of the attaching map of the $(n, 2n)$-cell complex. At the end, we show that two attaching maps give the same, up to homotopy, manifold if and only if their homotopy classes, when written in a Hilton basis, differ only by a change-of-basis matrix that is in the image of a certain map Φ, which we define explicitly in the paper.

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1 Introduction

By gluing together the edges of a square with the right combination and orientation, we can form a 2-dimensional sphere, torus, Klein bottle, or real projective plane. In fact, all surfaces, up to homeomorphism, can be obtained by gluing the edges of a polygon, which is equivalent to attaching the boundary of a 2-ball to a wedge sum of 1-spheres. This attaching map can be viewed as an element of the fundamental group of the wedge sum. Furthermore, it is possible for two different elements of this fundamental group to give the same manifold. The attaching map depends only on its homotopy type, which is an element of $\pi_1(\bigvee^r S^1)$, the homotopy classes of (pointed) continuous maps from the S^1 to $(\bigvee^r S^1)$. Furthermore, by combinatorial group theory methods in topology, two different elements of this fundamental group could give attaching maps that construct the same surface, up to homeomorphism.

This is the essential idea of a $(n, 2n)$-cell complex: forming a space by attaching the boundary of a $2n$-ball, which is a $(2n-1)$-sphere, to a wedge sum of n-spheres. What we know is that any $2n$-manifold can be constructed as a $(n, 2n)$-cell complex provided that the manifold is oriented, $(n-1)$-connected, compact, and closed. We call such manifolds here *highly connected* $2n$*-manifolds*.

Our goal is to classify highly connected $2n$-manifolds up to homotopy by treating them as $(n, 2n)$-cell complexes. For an $(n, 2n)$-cell complex, the process of attaching is described by an attaching map $f : S^{2n-1} \to \bigvee^r S^n$. We focus our attention on the homotopy class of the attaching map (which is an element of $\pi_{2n-1}(\bigvee^r S^n)$), because it alone determines the homotopy type of the $(n, 2n)$-cell complex. To simplify the calculation, we will choose a basis for $\pi_{2n-1}(\bigvee^r S^n)$, called the Hilton basis, which is formed from a basis for $\pi_n(\bigvee^r S^n)$. The question we ask is that if we have two elements of $\pi_{2n-1}(\bigvee^r S^n)$ written in the same Hilton basis, when do they give the same (up to homotopy) $(n, 2n)$-cell complex? Now, pick a Hilton basis for $\pi_{2n-1}(\bigvee^r S^n)$ and an element from $\pi_{2n-1}(\bigvee^r S^n)$ whose representation in the Hilton basis is an integer vector $\hat{a}$. Moreover, pick a change-of-basis matrix M in $\pi_n(\bigvee^r S^n)$. We construct a certain map $\Phi_{n,r}$ that sends M to a change-of-basis matrix $\Phi_{n,r}(M)$ for $\pi_{2n-1}(\bigvee^r S^n)$. Our main theorem (Theorem 5.1.7) then shows that $\hat{a}^T \Phi_{n,r}(M)$ gives the same $(n, 2n)$-cell complex as $\hat{a}$, up to homotopy. Our method for the proof involves properties of primary operations on the homotopy group. Collectively, this is called a Π-algebra.

In Section 2, we lay the background for Π-algebras and give their definition. In Section 3 we demonstrate properties of Π-algebras, especially how their three operations (addition, bracket, composition) interact with each other. Section 4 introduces $(n, 2n)$-cell complexes, which are used to construct oriented, $(n-1)$-connected, compact, closed $2n$-manifolds. The main theorem of this paper (Theorem 5.1.7) comes in Section 5. In this section, we give the definition of $\Phi_{n,r}$ and show that it preserves the multiplication operation as well as invertibility. The application of these two properties in $(n, 2n)$-cell complexes using Hilton basis leads directly to the result stated in the main theorem. Next, we give examples of the main theorem on 4-manifolds in Section 6 and present in Section 7 a further problem to investigate. We also include an appendix where we present in detail some of the long proofs (Appendix A) and long examples (Appendix B) used in the paper , as well as a reference sheet for the values of the important constants in the n^{th} homotopy group of the $(2n-1)$-sphere (Appendix C).

2 Definition of Π-Algebra

We define the Π-algebra by following Blanc [2] and Dwyer and Kan [3]. A Π-algebra is a mix of two algebraic structures: a graded Lie algebra structure and a semi-module structure over a semi-ring defined from the homotopy groups of spheres. We begin by describing the Lie algebra side of things.

2.1 Lie Algebras and Hall Basis

Definition 2.1.1 (Lie Algebra)**.** Recall that a Lie algebra $(L, [\ ,\ ])$ is an abelian group together with an operation

$$[\ ,\ ] : L \times L \to L$$

with properties

1. For $x, x', y \in L$,
$$[x + x', y] = [x, y] + [x', y]$$
2. For $x \in L$,
$$[x, x] = 0$$
3. (Jacobi identity) For $x, y, z \in L$
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

Exercise 2.1.2. Show that $[y, x] = -[x, y]$ for $x, y \in L$.

Let $\mathcal{L}_k$ be the free Lie algebra generated by $x_1, x_2, \ldots, x_k$. This is the set of linear combinations of all possible monomials of brackets of the $x_1, x_2, \ldots, x_k$ with only relations determined just by those generated by the basic relations defining a Lie algebra. The **weight** of a monomial b is the number of x's in b. For example, the weight of $b = [x_1, [[[[x_1, x_2], x_3], x_2], x_1]]$ is 6. Note that it follows from this definition that if m, m' are two monomials then

$$\text{weight}([m, m']) = \text{weight}(m) + \text{weight}(m')$$

We now describe a basis for $\mathcal{L}_k$.

Definition 2.1.3 (Hall Basis)**.** Let $HB = (w_1, w_2, w_3, \ldots)$ be an ordered set consisting of a sequence of bracketed monomials $w_i = w_i(x_1, \ldots, x_k)$. This order must be a linear ordering, chosen so that the following conditions hold:

(i.) For $i < j$, $x_i < x_j$.

(ii.) If m, m' are monomials with weight$(m) <$ weight(m') then $m < m'$.

Given such an order, the set HB is determined by the further requirement:

(iii.) Each $[w_i, w_j] \in HB$ if and only if:

- (a) $w_i, w_j \in HB$ with $i < j$, and
- (b) either $w_j = x_t$ for some t, or $w_j = [w_l, w_r]$ for some $w_l, w_r \in HB$ with $l \leq i$.

Example 2.1.4. For $k = 3$, we have, sorted by weight:

1. $x_1,\ x_2,\ x_3$
2. $[x_1, x_2],\ [x_1, x_3],\ [x_2, x_3]$
3. $[x_1, [x_1, x_2]],\ [x_1, [x_1, x_3]],\ [x_2, [x_1, x_2]],\ [x_2, [x_1, x_3]]$
   $[x_2, [x_2, x_3]],\ [x_3, [x_1, x_2]],\ [x_3, [x_1, x_3]],\ [x_3, [x_2, x_3]]$

Under the graded lexicographic ordering (that is, sorting the elements by the weight first, and then sort the elements with the same weight by the lexicographic ordering), these elements represent the w_i for $1 \leq i \leq 14$ in the order they are presented.

Remark 2.1.5.

1. $[x_1, [x_2, x_3]] = [x_2, [x_1, x_3]] - [x_3, [x_1, x_2]]$ by the Jacobi identity.
2. Let $\mathcal{A}_k$ be the Lie algebra generated by $y_1, y_2, \ldots, y_k$. There is a map $\mathcal{L}_k \to \mathcal{A}_k$ defined by sending a basic product $w(x_1, \ldots, x_k)$ to the element $w(y_1, \ldots, y_k)$. For example, the basic product b above is sent to $[y_2, [[[y_1, y_3], y_2], y_2, y_1]]$. The images of all basic products of $\mathcal{L}_k$ in $\mathcal{A}_k$ form an additive basis of $\mathcal{A}_k$.

For a monomial $m \in HB$, we define its **serial number** $s(m)$ to be

$$s(m) = \#\{m' \mid m' \leq m\}$$

In particular, $s(x_i) = i$ and $s(w_j) = j$ in general. In addition, we define inductively the **rank** of m as follows:

a. For each i, $r(x_i) = 0$.

b. Assume for monomials m of weight $< n$, $r(m)$ is defined.

c. If $m = [m_1, m_2]$ has weight n with $m_1 < m_2$ and $r(m_2) \leq s(m_1)$, define $r(m) = s(m_1)$.

Remark 2.1.6. If B is a Lie algebra and $x_1, \ldots, x_k \in B$ generate B as a Lie algebra, then $\{w_j(x_1, \ldots, x_k)\}_{j=1}^{\infty}$ is an ordered additive spanning set of B. This can be seen as follows: the Hall basis is an ordered additive basis for the Lie algebra $\mathcal{L}_k$ freely generated on $y_1, \ldots, y_k$. The natural map of Lie algebras $\mathcal{L}_k \to B$ that sends $y_i \mapsto x_i$ for $1 \leq i \leq k$ is clearly onto. The result follows.

Exercise 2.1.7.

1. Determine the Hall basis elements of weights 4 and 5 for the generators x_1, x_2, and x_3.

2. Determine the rank and serial numbers for all the Hall basis elements of weights through 5.

3. For the following, determine if it belongs to the Hall basis and, if not, express it as a combination of Hall basis elements.

   (a) $[[x_1, x_3], [x_1, x_2]]$

   (b) $[x_2, [x_3, [x_1, x_2]]]$

   (c) $[[x_2, x_3], [[x_1, x_2], x_2]]]$

2.2 Whitehead Algebras

Before we define Whitehead algebra, we need to recall what a graded group is.

Definition 2.2.1 (Graded Group)**.** A **graded group** is a sequence $X = \{X_i\}_{i=1}^{\infty}$ with X_i a group for each $i \geq 1$. We will say that such an X is **quasi-abelian** if X_i is abelian for $i \geq 2$ and call it **abelian** if, additionally, X_1 is abelian. A quasi-abelian X is **simply connected** (and hence abelian) provided $X_1 \cong *$. Furthermore, if X is graded and $x \in X_j$ then we call j the **degree** of x and write $|x| = j$.

Now a Whitehead algebra is defined as follows.

Definition 2.2.2 (Whitehead Algebra)**.** A **Whitehead algebra** is a pair $(X, [\ ,\ ])$ consisting of a quasi-abelian graded group X together with a map

$$[\ ,\ ] : X_{i+1} \times X_{j+1} \to X_{i+j+1}$$

for each pair $i, j \geq 0$, satisfying the following properties:

1. Assume $i, j, k \geq 1$:

   (a) For $x, x' \in X_{i+1}$ and $y \in X_{j+1}$,

   $$[x + x', y] = [x, y] + [x', y]$$

   in X_{i+j+1}.

(b) For $x \in X_{i+1}$ and $y \in X_{j+1}$,

$$[y, x] = (-1)^{(i+1)(j+1)}[x, y]$$

in X_{i+j+1}

(c) (Jacobi identity) For $x \in X_{i+1}$, $y \in X_{j+1}$, and $z \in X_{k+1}$

$$(-1)^{(i+1)(k+1)}[[x, y], z] + (-1)^{(i+1)(j+1)}[[y, z], x] + (-1)^{(j+1)(k+1)}[[z, x], y] = 0$$

in $X_{i+j+k+1}$.

2. If $\xi, \lambda \in X_1$ and $x \in X_{i+1}$ then

(a) $[\xi, \lambda] = \xi\lambda\xi^{-1}\lambda^{-1}$ in X_1;

(b) there is an associated $x^\xi \in X_{i+1}$ so that $x^1 = x$ and $(x^\xi)^\lambda = x^{\xi\lambda}$ in X_{i+1} and

$$[x, \xi] = x^\xi - x$$

in X_{i+1}.

Exercise 2.2.3.

1. Show that if $|x|$ is odd then $2[x, x] = 0$.

2. Show that if $|x|$ is even then $3[[x, x], x] = 0$.

3. Show that if B is a Whitehead algebra and $x_1, \ldots, x_k \in B_{p+1}$ then $w_j(x_1, \ldots, x_k) \in B_{pq+1}$ where $q = \text{weight}(w_j)$.

Moreover, a particular type of Whitehead algebras that we are interested in is the so-called homotopy Whitehead algebra.

Definition 2.2.4 (Homotopy Whitehead Algebra)**.** For a connected topological space Y, let $\pi_*Y = \{\pi_i Y\}_{i=1}^{\infty}$ be the quasi-abelian graded group consisting of the homotopy groups of Y. In particular, $\pi_k Y = [S^k, Y]$, that is, the homotopy classes of (pointed) continuous maps $S^k \to Y$ from the k-sphere S^k. This has the structure of a Whitehead algebra which we will call the **homotopy Whitehead algebra associated to** Y.

To see this, let $x \in \pi_{i+1}Y$ and $y \in \pi_{j+1}Y$. We may view x as represented by $f : (e^{i+1}, S^i) \to (Y, *)$ and y is represented by $g : (e^{j+1}, S^j) \to (Y, *)$. Then

$$S^{i+j+1} \approx \partial(e^{i+1} \times e^{j+1}) = (S^i \times e^{j+1}) \cup (e^{i+1} \times S^j),$$

so define $h : S^{i+j+1} \to Y$ by setting $h = f\pi_1$ on $e^{i+1} \times S^j$ and $h = g\pi_2$ on $S^i \times e^{j+1}$ and set $[x, y] = [h] \in \pi_{i+j+1}Y$. Note that $h = *$ on $(S^i \times e^{j+1}) \cap (e^{i+1} \times S^j) = S^i \times S^j$.

We let $\mathcal{S} := \{\mathcal{S}_{p,q}\}_{p,q\geq 1}$ be the **sphere algebra**, where $\mathcal{S}_{p,q} := \pi_q(S^p)$. We note that $\mathcal{S}_n := \mathcal{S}_{n,*} := \{\mathcal{S}_{n,p}\}_{p\geq 1}$ is the homotopy Whitehead algebra $\pi_*(S^n)$, which we refer to as the n-**sphere algebra**.

We note that for $p, q \geq 1$ there is an operation

$$\circ : \pi_p(Y) \times \mathcal{S}_{p,q} \to \pi_q(Y)$$

defined as follows: for $\alpha \in \mathcal{S}_{p,q}$ represented by $f : S^q \to S^p$ and $y \in \pi_p(Y)$ represented by $g : S^p \to Y$ then the composite

$$S^q \xrightarrow{f} S^p \xrightarrow{g} Y$$

represents the homotopy class $y\circ\alpha \in \pi_q(Y)$. We will refer to this operation as **composition**.

Remark 2.2.5. There is a homomorphism $E : \mathcal{S}_{p,q} \to \mathcal{S}_{p+1,q+1}$, for any $p, q \geq 1$, called the **suspension homomorphism**. Topologically, this is the map

$$[S^q, S^p] \to [S^{q+1}, S^{p+1}]$$

defined by sending $f \mapsto \Sigma f$ where Σ denotes the reduced suspension. Moreover, E satisfies, for $\alpha \in \mathcal{S}_{p+1,q+1}$ and $\beta \in \mathcal{S}_{p+1,r+1}$,

$$E[\alpha, \beta] = 0$$

in $\mathcal{S}_{q+r+2,p+2}$. See Whitehead [6] p. 485.

2.3 Pseudo-$\mathcal{S}$-Modules

So far we have finished introducing the Lie algebra side of the definition of Π-algebras. We now move on to the other side, that is, a semi-module structure over a semi-ring defined from the homotopy groups of sphere.

Definition 2.3.1 (Pseudo-$\mathcal{S}$-Modules)**.** A pair $(X, \circ)$ is a **pseudo-$\mathcal{S}$-module** provided X is a quasi-abelian graded group and we have composition

$$\circ : X_p \times \mathcal{S}_{p,q} \to X_q \qquad p, q \geq 1$$

satisfying:

1. If $\iota_p \in \mathcal{S}_{p,p}$ represents the identity map id : $S^p \to S^p$. then, for $x \in X_p$, $x \circ \iota_p = x$.
2. for $x \in X_p$, $\alpha \in \mathcal{S}_{q,r}$, and $\beta \in \mathcal{S}_{p,q}$

   $$x \circ (\beta \circ \alpha) = (x \circ \beta) \circ \alpha$$

   in X_r
3. for $x \in X_p$ and $\alpha, \alpha' \in \mathcal{S}_{p,q}$

   $$x \circ (\alpha + \alpha') = x \circ \alpha + x \circ \alpha'$$

   in X_q.

Remark 2.3.2.

1. The definition of pseudo-module encodes no specific information pertaining to left distribution under the composition operation. An $\mathcal{S}$**-module** is a pseudo-$\mathcal{S}$-module X such that for $x, x' \in X_p$ and $\alpha \in \mathcal{S}_{q,p}$

$$(x + x') \circ \alpha = x \circ \alpha + x' \circ \alpha$$

   We will see below that a Π-algebra is a pseudo-$\mathcal{S}$-module in which right $\mathcal{S}$-composition satisfies a formula of Hilton.

2. $X = \mathcal{S}_m$ is a pseudo-$\mathcal{S}$-module since composition gives

$$\circ : \mathcal{S}_{m,p} \times \mathcal{S}_{p,q} \to \mathcal{S}_{m,q}$$

   Furthermore, if $\alpha \in \mathcal{S}_{m,p}$ and $\beta \in \mathcal{S}_{p,q}$ then

$$E(\alpha \circ \beta) = E\alpha \circ E\beta \in \mathcal{S}_{m+1,q+1}$$

2.4 Π-algebras

Before we give the definition of Π-algebra, we need to introduce an important type of homomorphisms called the Hilton invariants. (See Whitehead [6] §X.8 for definition and properties.)

Proposition 2.4.1. *Let $j \geq 0$ and $q = q(j+3) = weight(w_{j+3})$. Then there exist unique homomorphisms*

$$h_j : \mathcal{S}_{p+1,n+1} \to \mathcal{S}_{pq+1,n+1},$$

called the **Hopf-Hilton invariants,** *such that for any topological space Y, any $x, y \in \pi_{p+1}Y$, and $\alpha \in \mathcal{S}_{p+1,n+1}$, Hilton's formula holds:*

$$(x + y) \circ \alpha = x \circ \alpha + y \circ \alpha + \sum_{j \geq 0} w_{j+3}(x, y) \circ h_j(\alpha)$$

Moreover, we give a name to the Hopf-Hilton invariant in the special case of $j = 0$, since the the main results of this paper are concerned with this case.

Definition 2.4.2 (Hopf invariant)**.** Consider the special case of the Hopf-Hilton invariants when $j = 0$. Then $w_3(x, y) = [x, y]$ has weight 2 so $q = 2$ and

$$H = h_0 : \mathcal{S}_{p+1,n+1} \to \mathcal{S}_{2p+1,n+1}$$

is called the **Hopf invariant**. In particular, when $n = 2p$ then $H(\alpha) = H_0(\alpha)\iota_{2p+1}$ where $H_0(\alpha) \in \mathbb{Z}$.

Finally, we have everything needed to define Π-algebra.

Definition 2.4.3 (Π-Algebra)**.** A **Π-algebra** is a triple $(X, [\,,\,], \circ)$ such that $(X, [\,,\,])$ is a Whitehead algebra, $(X, \circ)$ is a pseudo-$\mathcal{S}$-module and the two structures interact according to the two formulas:

1. for $x \in X_p$, $\alpha \in \mathcal{S}_{p,\,q+1}$, and $\beta \in \mathcal{S}_{p,\,r+1}$

$$x \circ [\alpha, \beta] = [x \circ \alpha, x \circ \beta]$$

   in X_{q+r+1}

2. Hilton's formula: For any $x, y \in X_{p+1}$, and $\alpha \in \mathcal{S}_{p+1,\,n+1}$

$$(x + y) \circ \alpha = x \circ \alpha + y \circ \alpha + \sum_{j \geq 0} w_{j+3}(x, y) \circ h_j(\alpha)$$

   in X_{n+1}

3. Barcus-Barratt formula: For $x \in X_{q+1}$, $y \in X_{p+1}$, and $\alpha \in \mathcal{S}_{q+1,\,r+1}$

$$[x \circ \alpha, y] = [x, y] \circ (E^p\alpha) + \sum_{n=0}^{\infty} [x^{n+2}, y] \circ (E^p h_{j_n}(\alpha))$$

   in X_{p+r+1}, where

$$[x^k, y] = [x, [x, [x, \ldots [x, y] \ldots]]]$$

   and $j_n = s([x^{n+1}, y])$.

There is an important type of Π-algebras that is associated with the homotopy groups of a space.

Lemma 2.4.4. *For a space Y, the homotopy groups $\pi_* Y$ have the structure of a Π-algebra, called the* **homotopy Π-algebra associated to** Y.

Proof. That the majority of the Π-algebra properties hold for $\pi_* Y$ can be found primarily demonstrated in chapters X and XI of Whitehead [6]. The proof of the Barratt-Barcus formula can be found in Baues [1] §II.3.

For (1), let $x \in \pi_p(X)$ be represented by $f : S^p \to X$ in its homotopy class. Then $f_* : \pi_*(S^p) \to \pi_*(Y)$ is a map of Whitehead algebras. Hence for $\alpha, \beta \in \pi_*(S^p)$

$$x \circ [\alpha, \beta] = f_*([\alpha, \beta]) = [f_*(\alpha), f_*(\beta)] = [x \circ \alpha, x \circ \beta]$$

in $\pi_*(Y)$.

□

Just as other abstract algebraic structures, Π-algebras have homomorphisms, defined as follows.

Definition 2.4.5 (Homomorphism of Π-Algebras)**.** A homomorphism $f : X \to Y$ of graded groups X and Y is a sequence $f_j : X_j \to Y_j$ of group homomorphisms for $j \geq 1$. If X and Y are, additionally, Whitehead algebras then f is a **homomorphism of Whitehead algebras** provided if $x \in X_{i+1}$ and $y \in X_{j+1}$ then

$$f_{i+j+1}([x, y]) = [f_{i+1}(x), f_{j+1}(y)]$$

Finally, if X and Y are, additionally, Π-algebras then f is a **homomorphism of Π-algebras** provided if, for $x \in X_p$ and $\alpha \in \mathcal{S}_{p,q}$, we have

$$f_q(x \circ \alpha) = f_p(x) \circ \alpha$$

Now, let

$$\mathrm{Hom}_\Pi(X, Y) = \{f : X \to Y \mid f \text{ a } \Pi\text{-algebra homomorphism}\}$$

called the **set of Π-algebra homomorphisms from X to Y.**

Regarding the properties of Π-algebras, we have the following two important theorems.

Theorem 2.4.6 (Π-algebra Yoneda Lemma)**.** *For X a Π-algebra then the map*

$$\lambda : Hom_\Pi(\mathcal{S}_p, X) \to X_p$$

given by $\lambda(f) = f(\iota_p)$ is one-to-one and onto. Moreover, the map λ has as its inverse

$$\theta : X_p \to Hom_\Pi(\mathcal{S}_p, X)$$

given by $\theta(x)(\alpha) = x \circ \alpha$ (in particular, $\theta(x)(\iota_p) = x$).

Proof. First, for $x \in X_p$, observe that $\theta(x)([\alpha, \beta]) = x \circ [\alpha, \beta] = [x \circ \alpha, x \circ \beta] = [\theta(x)(\alpha), \theta(x)(\beta)]$. Also, $\theta(x)(\alpha \circ \beta) = x \circ (\alpha \circ \beta) = (x \circ \alpha) \circ \beta = \theta(x)(\alpha) \circ \beta$, for $\alpha \in \mathcal{S}_{p,q}$ and $\beta \in \mathcal{S}_{q,r}$. Thus, $\theta(x) : \mathcal{S}_p \to X$ is a homomorphism of Π-algebras.

Now, we have, for $x \in X_p$,

$$(\lambda \circ \theta)(x) = \lambda(\theta(x)) = \theta(x)(\iota_p) = x.$$

Also, for $f \in \mathrm{Hom}_\Pi(\mathcal{S}_p, X)$,

$$((\theta \circ \lambda)(f))(\alpha) = \theta(\lambda(f))(\alpha) = \theta(f(\iota_p))(\alpha) = f(\iota_p) \circ \alpha = f(\iota_p \circ \alpha) = f(\alpha),$$

that is, $(\theta \circ \lambda)(f) = f$.

We conclude that θ is a two sided inverse of λ. □

Theorem 2.4.7. *Let $f : X \to Y$ be a continuous map of topological spaces and let $f_* : \pi_* X \to \pi_* Y$ be the induced map of graded groups.*

1. *(Whitehead) If f_* is an isomorphism of graded groups, then f is homotopy equivalence, that is, there is a continuous map $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.*

2. *f_* is a homomorphism of Π-algebras.*

3 Properties of Π-Algebras

We now prove some basic properties of the Π-algebra, mostly regarding the interaction between the addition, bracket, and composition operations.

Lemma 3.0.1. *For a Whitehead algebra X and an integer $n \geq 2$, let $x_k \in X_n$, $a_k, b_k \in \mathbb{Z}$ for $1 \leq k \leq m$, where r is a positive integer. Then*

$$\left[\sum_{k_1=1}^{r} a_{k_1} x_{k_1}, \sum_{k_2=1}^{r} b_{k_2} x_{k_2}\right] = \sum_{1\leq k_1<k_2\leq r} (a_{k_1}b_{k_2} + (-1)^n a_{k_2}b_{k_1})[x_{k_1}, x_{k_2}] + \sum_{k=1}^{r} a_k b_k [x_k, x_k]$$

Proof. By the axioms of the bracket operator of Whitehead algebras,

$$\left[\sum_{k_1=1}^{r} a_{k_1} x_{k_1}, \sum_{k_2=1}^{r} b_{k_2} x_{k_2}\right] = \sum_{1\leq k_1,k_2\leq m} a_{k_1}b_{k_2}[x_{k_1}, x_{k_2}],$$

which, after breaking into cases by the relation between the two indexes, is equal to

$$\sum_{1\leq k_1<k_2\leq m} a_{k_1}b_{k_2}[x_{k_1}, x_{k_2}] + \sum_{1\leq k_1>k_2\leq m} a_{k_1}b_{k_2}[x_{k_1}, x_{k_2}] + \sum_{k=1}^{r} a_k b_k [x_k, x_k].$$

But

$$\sum_{1\leq k_1>k_2\leq m} a_{k_1}b_{k_2}[x_{k_1}, x_{k_2}] = \sum_{1\leq k_1<k_2\leq m} a_{k_2}b_{k_1}[x_{k_2}, x_{k_1}] = \sum_{1\leq k_1<k_2\leq m} a_{k_1}b_{k_2}(-1)^n[x_{k_1}, x_{k_2}],$$

This gives us the desired result. □

Lemma 3.0.2. *In the homotopy* Π*-algebra associated to space X, suppose $n \geq 2$, and let $x, y \in \pi_n X$ and $\alpha \in \mathcal{S}_{n,n}$. Then*

$$(x + y) \circ \alpha = x \circ \alpha + y \circ \alpha.$$

Proof. In this case of the Hilton formula, $w_3(x, y) = [x, y]$ has weight 2, so $q_0 = 2$, where $h_j(\alpha) \in \mathcal{S}_{nq_j+1,n+1}$. Then $q_j \geq 2$ for all $j \geq 0$, by the axioms of Hall basis. But $\mathcal{S}_{nq_j+1,n+1} \cong \{0\}$ for all $q_j \geq 2$ (since any sphere S^{m+1} is m-connected), so

$$w_{j+3}(x, y) \circ h_j(\alpha) = 0$$

for all $j \geq 0$. □

Lemma 3.0.3. *In the homotopy* Π*-algebra associated to space X, suppose $n \geq 2$, and let $x, y \in \pi_n X$ and $\alpha \in \mathcal{S}_{n,2n-1}$. Then*

$$(x + y) \circ \alpha = x \circ \alpha + y \circ \alpha + H_0(\alpha)[x, y].$$

Proof. In this case of Hilton's formula, the terms of the sum vanish for all $j \geq 1$, by the same argument as in the proof for Lemma 3.0.2. As for $j = 0$, we have

$$w_3(x,y) \circ h_0(\alpha) = [x,y] \circ (H_0(\alpha)\iota_{2n-1}) = H_0(\alpha)[x,y].$$

This completes the proof. □

Lemma 3.0.4. *In the homotopy* Π*-algebra associated to space* X*, suppose* $n \geq 2$*, and let* $\alpha \in \mathcal{S}_{n,2n-1}$ *and* $x_k \in \pi_n X$ *for all* $1 \leq k \leq r$*, where* r *is a positive integer. Then*

$$\left(\sum_{k=1}^{r} x_k\right) \circ \alpha = \sum_{k=1}^{r} x_k \circ \alpha + \sum_{1 \leq k_1 < k_2 \leq r} H_0(\alpha)[x_{k_1}, x_{k_2}]$$

Proof. We prove this by induction on r. For the base case we have

$$\sum_{1 \leq k \leq 1} x_k \circ \alpha + \sum_{1 \leq k_1 < k_2 \leq 1} H_0(\alpha)[x_{k_1}, x_{k_2}] = x_1 \circ \alpha + 0 = \left(\sum_{1 \leq k \leq 1} x_k\right) \circ \alpha.$$

For the inductive step, we have

$$\left(\sum_{1 \leq k \leq m+1} x_k\right) \circ \alpha = \left(\left(\sum_{1 \leq k \leq m} x_k\right) + x_{m+1}\right) \circ \alpha.$$

By Lemma 3.0.3, this is equal to

$$\left(\sum_{1 \leq k \leq m} x_k\right) \circ \alpha + x_{m+1} \circ \alpha + H_0(\alpha)\left[\sum_{1 \leq k \leq m} x_k, x_{m+1}\right].$$

By the inductive hypothesis,

$$\left(\sum_{1 \leq k \leq m} x_k\right) \circ \alpha = \sum_{1 \leq k \leq m} x_k \circ \alpha + \sum_{1 \leq k_1 < k_2 \leq m} H_0(\alpha)[x_{k_1}, x_{k_2}],$$

and we also have

$$H_0(\alpha)\left[\sum_{1 \leq k \leq m} x_k, x_{m+1}\right] = H_0(\alpha) \sum_{1 \leq k \leq m} [x_k, x_{m+1}] = \sum_{1 \leq k \leq m} H_0(\alpha)[x_k, x_{m+1}]$$

by Lemma 3.0.2. Thus

$$\begin{aligned}\left(\sum_{1 \leq k \leq m+1} x_k\right) \circ \alpha &= \sum_{1 \leq k \leq m} x_k \circ \alpha + \sum_{1 \leq k_1 < k_2 \leq m} H_0(\alpha)[x_{k_1}, x_{k_2}] + x_{m+1} + \sum_{1 \leq k \leq m} H_0(\alpha)[x_k, x_{m+1}] \\ &= \sum_{1 \leq k \leq m+1} x_k \circ \alpha + \sum_{1 \leq k_1 < k_2 \leq m+1} H_0(\alpha)[x_{k_1}, x_{k_2}]\end{aligned}$$

This completes the proof. □

Lemma 3.0.5. *In the homotopy* Π*-algebra associated to space* X*, suppose* $n \geq 2$*, and let* $\alpha \in \mathcal{S}_{n,2n-1}$*,* $x \in \pi_n X$*, and* $a \in \mathbb{Z}$*. Then*

$$(ax) \circ \alpha = a(x \circ \alpha) + \binom{a}{2} H_0(\alpha)[x, x],$$

where $\binom{a}{2}$ *is defined as* $\frac{1}{2}a(a-1)$ *for all* $a \in \mathbb{Z}$*.*

Proof. This follows from Lemma 3.0.4 if $a > 0$. Now suppose $a < 0$. Since

$$0 = 0 \circ \alpha = (ax + (-ax)) \circ \alpha = (ax) \circ \alpha + (-ax) \circ \alpha + H_0(\alpha)[ax, -ax],$$

by Lemma 3.0.3, we have

$$\begin{aligned}(ax) \circ \alpha =& -(-ax) \circ \alpha - H_0(\alpha)[ax, -ax] \\ =& -\big(-a(x \circ \alpha) + \frac{1}{2}(-a)(-a-1)H_0(\alpha)[x, x]\big) + a^2 H_0(\alpha)[x, x] \\ =& a(x \circ \alpha) + \frac{1}{2}a(a-1)H_0(\alpha)[x, x].\end{aligned}$$

This completes the proof. (The case when $a = 0$ is trivial.) □

Lemma 3.0.6. *In the homotopy* Π*-algebra associated to space* X*, suppose* $n \geq 2$*, and let* $\alpha \in \mathcal{S}_{n,2n-1}$ *and* $x_k \in \pi_n X$*,* $a_k \in \mathbb{Z}$ *for all* $1 \leq k \leq r$*, where* r *is a positive integer. Then*

$$\left(\sum_{k=1}^{r} a_k x_k\right) \circ \alpha = \sum_{1 \leq k_1 < k_2 \leq r} a_{k_1} a_{k_2} H_0(\alpha)[x_{k_1}, x_{k_2}] + \sum_{k=1}^{r} \binom{a_k}{2} H_0(\alpha)[x_k, x_k] + a_k(x_k \circ \alpha)$$

Proof. We prove this by induction on r. For the base case we have

$$(a_1 x) \circ \alpha = \binom{a_1}{2} H_0(\alpha)[x_1, x_1] + a_1(x_1 \circ \alpha)$$

by Lemma 3.0.5. For the inductive step, we have

$$\left(\sum_{k=1}^{r+1} a_k x_k\right) \circ \alpha = \left(\left(\sum_{k=1}^{r} a_k x_k\right) + a_{r+1} x_{r+1}\right) \circ \alpha,$$

which by Lemma 3.0.3 is equal to

$$\left(\sum_{k=1}^{r} a_k x_k\right) \circ \alpha + (a_{r+1} x_{r+1}) \circ \alpha + H_0(\alpha)\left[\sum_{k=1}^{r} a_k x_k, a_{r+1} x_{r+1}\right]$$

Moreover, we have

$$\left(\sum_{k=1}^{r} a_k x_k\right) \circ \alpha = \sum_{k=1}^{r} a_k(x_k \circ \alpha) + \sum_{k=1}^{r} \binom{a_k}{2} H_0(\alpha)[x_k, x_k] + \sum_{1 \leq k_1 < k_2 \leq r} a_{k_1} a_{k_2} H_0(\alpha)[x_{k_1}, x_{k_2}]$$

by the inductive hypothesis,

$$(a_{r+1}x_{r+1}) \circ \alpha = a_{r+1}(x_{r+1} \circ \alpha) + \binom{a_{r+1}}{2} H_0(\alpha)[x_{r+1}, x_{r+1}]$$

by Lemma 3.0.5, and

$$H_0(\alpha)\left[\sum_{k=1}^{r} a_k x_k, a_{r+1}x_{r+1}\right] = \sum_{k=1}^{r} a_k a_{r+1} H_0(\alpha)[x_k, x_{r+1}]$$

by Lemma 3.0.2. After adding these three expressions up and rearranging the terms, we get

$$\begin{aligned}
\left(\sum_{k=1}^{r+1} a_k x_k\right) \circ \alpha = & \sum_{1\le k_1<k_2\le r} a_{k_1}a_{k_2}H_0(\alpha)[x_{k_1}, x_{k_2}] + \sum_{k=1}^{r} a_k a_{r+1} H_0(\alpha)[x_k, x_{r+1}] \\
& + \left(\sum_{k=1}^{r} \binom{a_k}{2} H_0(\alpha)[x_k, x_k]\right) + \binom{a_{r+1}}{2} H_0(\alpha)[x_{r+1}, x_{r+1}] \\
& + \left(\sum_{k=1}^{r} a_k(x_k \circ \alpha)\right) + a_{r+1}(x_{r+1} \circ \alpha) \\
= & \sum_{1\le k_1<k_2\le r+1} a_{k_1}a_{k_2}H_0(\alpha)[x_{k_1}, x_{k_2}] + \sum_{k=1}^{r+1} \binom{a_k}{2} H_0(\alpha)[x_k, x_k] + a_k(x_k \circ \alpha)
\end{aligned}$$

□

4 $(n, 2n)$-Cell Complexes

As mentioned in the introduction, in this paper we focus on classifying oriented, $(n-1)$-connected, compact, closed $2n$-manifolds. One way to construct spaces that behave like such manifolds is as follows: Let e^{2n} be the $2n$-ball which is bounded by the $(2n-1)$-sphere S^{2n-1} under the inclusion to the boundary $\iota : S^{2n-1} \to e^{2n}$. Let $f : S^{2n-1} \to \bigvee^r S^n$ be a continuous map and define the $(n, 2n)$**-cell complex induced by** f to be the pair (X, r, f) where

$$X = e^{2n}\left(\cup_f \left(\bigvee S^n\right)\right) = e^{2n} \cup \left(\bigvee S^n\right)/\sim$$

where the equivalence relation $\sim$ is defined as follows: $w \sim x$ for $w \in e^{2n}$ and $x \in \bigvee S^n$ provided $x = f(b)$ and $w = \iota(b)$ for some $b \in S^{2n-1}$. We will denote by q_f the induced map

$$q_f : \bigvee S^n \to e^{2n} \cup_f \left(\bigvee S^n\right) = X$$

Here f is called the **attaching map** of (X, r, f) and the process of forming X is called **cell attachment**.

Remark 4.0.1. The attaching map $f : S^{2n-1} \to \bigvee_{i=1}^{r} S^n$ corresponds to an element in the $(2n-1)^{\text{th}}$ group of the homotopy Π-algebra associated to $\bigvee^r S^n$.

We will need to know when two attaching maps give homotopic $(n, 2n)$-cell complexes. For this, we define the following two concepts.

Definition 4.0.2. A $(n, 2n)$**-cell map** $(A, B) : (X, r, f) \to (Y, r, g)$ consists of two continuous maps $B : X \to Y$ and $A : \bigvee S^n \to \bigvee S^n$ satisfying:

1. $B \circ q_f \simeq q_g \circ A$.
2. $A \circ f \simeq g$

In particular, the diagram

$$\begin{array}{ccccc} S^{2n-1} & \xrightarrow{f} & \bigvee S^n & \xrightarrow{q_f} & X \\ {\scriptstyle \mathrm{id}_{S^{2n-1}}}\downarrow & & {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle B} \\ S^{2n-1} & \xrightarrow{g} & \bigvee S^n & \xrightarrow[q_g]{} & Y \end{array}$$

commutes up to homotopy.

Definition 4.0.3. A $(n, 2n)$**-cell equivalence** is a $(n, 2n)$-cell map (A, B) with A and B homotopy equivalences.

Remark 4.0.4. If $A : \bigvee S^n \to \bigvee S^n$ is a homotopy equivalence, then A induces a Π-algebra homomorphism from $\pi_*(\bigvee S^n)$ to itself.

5 Main Theorem

In this section, we present and prove our main theorem, Theorem 5.1.7.

5.1 Definitions and the Statement of Theorem

We start with the necessary definitions and a statement of our main theorem. First, we define a basis for $\pi_{2n-1}(\bigvee^r S^n)$ that is generated from a basis in $\pi_n(\bigvee^r S^n)$.

Definition 5.1.1 (Hilton Basis)**.** Let $\hat{x} = (x_k)_{k=1}^{r}$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$. Suppose $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle \alpha_k \rangle$. Then we define the Hilton basis $\varphi_{n,r}(\hat{x})$ of $\pi_{2n-1}(\bigvee^r S^n)$ as

$$\varphi_{n,r}(\hat{x}) = ([x_{i_1}, x_{i_2}]_{1 \le i_1 < i_2 \le r} \cup (x_{i_1} \circ \alpha_{i_3})_{1 \le i_1 \le r,\, 1 \le i_3 \le m}).$$

Notice that the basis elements in a Hilton basis have different orders, so in order to convert one Hilton basis into another, we cannot use the normal integer matrices. For this reason, we define a new kind of change-of-basis matrices that are applicable to Hilton basis.

Definition 5.1.2 ($\mathcal{M}_{n,r}$)**.** Let $n \geq 2$, $\hat{x} = (x_k)_{k=1}^r$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$. We define $\mathcal{M}_{n,r}$ as the collection of $\binom{r}{2} + rm$ by $\binom{r}{2} + rm$ matrices such that the entries in the j^{th} column of the matrix are elements in $\mathbb{Z}_{u_j}$, where u_j is the order of the j^{th} basis element of $\varphi_{n,r}(\hat{x})$.

Moreover, we need a new matrix multiplication for composing two change-of-basis matrices.

Definition 5.1.3 ($*$)**.** Define matrix multiplication $*$ on $\mathcal{M}_{n,r}$ the same as the regular matrix multiplication except that the entries in each column is reduced with the appropriate modulus, that is, the order of the basis element that the column corresponds to.

The operation of $*$ on $\mathcal{M}_{n,r}$ is illustrated in Example A.0.1 in Appendix A. We also know the following properties of $*$ and $\mathcal{M}_{n,r}$.

Remark 5.1.4. $*$ is associative on $\mathcal{M}_{n,r}$.

Remark 5.1.5. The identity matrix I on $\text{Mat}(\binom{r}{2} + rm, \mathbb{Z})$ is also the identity matrix on $\mathcal{M}_{n,r}$.

Finally, we define a matrix map called $\Phi_{n,r}$, which is central to our main theorem.

Definition 5.1.6 ($\Phi_{n,r}$)**.** Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^m \mathbb{Z}_{d_k}\langle \alpha_k \rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^m c_k \alpha_k$. Moreover, let $\hat{x} = (x_k)_{k=1}^r$ be a basis of $\pi_n(\bigvee^r S^n)$ and decide an ordering for $\varphi_{n,r}(\hat{x})$. Define $\Phi_{n,r} : \text{Mat}(r, \mathbb{Z}) \to \mathcal{M}_{n,r}$ such that for a $M \in \text{Mat}(r, \mathbb{Z})$, we have $\Phi_{n,r}(M)_{ij}$, the entry in the i^{th} row and j^{th} column of $\Phi_{n,r}(M)$, equal to

$$\Phi_{n,r}(M)_{ij} = \begin{cases} M_{i_1 j_1} M_{i_2 j_2} + (-1)^n M_{i_1 j_2} M_{i_2 j_1} & \text{if } \varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}],\ \varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}] \\ c_{j_3} M_{i_1 j_1} M_{i_2 j_1} \bmod d_{j_3} & \text{if } \varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}],\ \varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{j_3} \\ t_{i_3} M_{i_1 j_1} M_{i_1 j_2} & \text{if } \varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{i_3},\ \varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}] \\ t_{i_3} c_{j_3} \binom{M_{i_1 j_1}}{2} + \theta_{i_3 = j_3} M_{i_1 j_1} \bmod d_{j_3} & \text{if } \varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{i_3},\ \varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{j_3} \end{cases}$$

where $\theta : \{False, True\} \to \{0, 1\}$ is defined as

$$\theta_P = \begin{cases} 0, & \text{if } P \text{ is False} \\ 1, & \text{if } P \text{ is True} \end{cases}.$$

and $\binom{a}{2}$ is defined as $\frac{1}{2}a(a-1)$ for all $a \in \mathbb{Z}$.

Now we are ready to state our main theorem.

Theorem 5.1.7. *Let $n \geq 2$, $r \geq 1$, and $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^m \mathbb{Z}_{d_k}\langle \alpha_k \rangle$. Moreover, let $\hat{y}$ be a Hilton basis of $\pi_{2n-1}(\bigvee^r S^n)$, f, g be continuous maps from S^{2n-1} to $\bigvee^r S^n$ and $[f] = \hat{a}^T \hat{y}$ for some coefficient vector $\hat{a}$. Then for $(n, 2n)$-cell complexes (X, r, f) and (Y, r, g), $(X, r, f) \simeq (Y, r, g)$ if and only if there exists a $L \in \text{Im}\,\Phi_{n,r} \cap \text{GL}(\binom{r}{2} + rm, \mathbb{Z})$ such that $[g] = \hat{a}^T L \hat{y}$.*

We will present the proof for the theorem at the end of this section, after establishing a series of lemmas.

5.2 Properties of Hilton Basis

First, we need to show that the Hilton basis for $\pi_{2n-1}(\bigvee^r S^n)$ is indeed a basis for it, as in the following.

Lemma 5.2.1. *Let $n \geq 2$, $r \geq 1$, and $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^m \mathbb{Z}_{d_k}\langle \alpha_k \rangle$. If $\hat{x} = (x_k)_{k=1}^r$ is a basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$, then*

$$\pi_{2p-1}\left(\bigvee^m S^p\right) \cong \left(\bigoplus_{1\leq k_1<k_2\leq m} \mathbb{Z}\langle [x_{k_1}, x_{k_2}]\rangle\right) \bigoplus \left(\bigoplus_{k_1=1}^r \bigoplus_{k_3=1}^m \mathbb{Z}_{d_{k_3}}\langle x_{k_1} \circ \alpha_{k_3}\rangle\right)$$

Proof. This is a consequence of the Hilton-Milnor Theorem. See Whitehead [6] for more details. □

Next, we show how the bracket and composition operations in Π-algebra transform elements in $\pi_n(\bigvee^r S^n)$ and $\pi_{2n-1}(\bigvee^r S^n)$ when they are written in the Hilton basis.

Lemma 5.2.2. *Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^m \mathbb{Z}_{d_k}\langle \alpha_k \rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^m c_k \alpha_k$. Moreover, let $(x_k)_{k=1}^r$ be a basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$, and $(a_k)_{k=1}^r, (b_k)_{k=1}^r \in \mathbb{Z}^r$ be some coefficient vectors. Then*

$$\begin{aligned}\left[\sum_{k_1=1}^r a_{k_1}x_{k_1}, \sum_{k_2=1}^r b_{k_2}x_{k_2}\right] &= \sum_{1\leq j_1<j_2\leq r} (a_{j_1}b_{j_2} + (-1)^n a_{j_2}b_{j_1})[x_{j_1}, x_{j_2}] \\ &\quad + \sum_{j_1=1}^r \sum_{j_3=1}^m (c_{n,j_3}a_{j_1}b_{j_1} \bmod d_{n,j_3})(x_{j_1} \circ \alpha_{j_3}).\end{aligned}$$

Proof. This follows immediately from Lemma 3.0.1. □

Lemma 5.2.3. *Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^m \mathbb{Z}_{d_k}\langle \alpha_k \rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^m c_k \alpha_k$. Moreover, let $(x_k)_{k=1}^r$ be a basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$, $(a_k)_{k=1}^r \in \mathbb{Z}^r$ a coefficient vector, and α_l a generator of $\mathcal{S}_{n,2n-1}$. Then*

$$\begin{aligned}\left(\sum_{k=1}^r a_k x_k\right) \circ \alpha_l &= \sum_{1\leq j_1<j_2\leq r} t_l a_{j_1} a_{j_2} [x_{j_1}, x_{j_2}] \\ &\quad + \sum_{j_1=1}^r \sum_{j_3=1}^m \left(\left(t_l c_{n,j_3} \binom{a_{j_1}}{2} + \theta_{j_3=l} a_{j_1}\right) \bmod d_{n,j_3}\right)(x_{j_1} \circ \alpha_{j_3}),\end{aligned}$$

where θ is defined the same as in Definition 5.1.6.

Proof. This follows immediately from Lemma 3.0.6. □

These two lemmas give us the following relationship between $\varphi_{n,r}$ and $\Phi_{n,r}$. In fact, this relationship is exact motivation for defining $\Phi_{n,r}$ as it is in Definition 5.1.6.

Lemma 5.2.4. *Let $\hat{x}$ be a basis of $\pi_n(\bigvee^r S^n)$ and $M \in \mathrm{Mat}(r, \mathbb{Z})$. Then*

$$\varphi_{n,r}(M\hat{x}) = \Phi_{n,r}(M)\varphi_{n,r}(\hat{x})$$

Proof. This follows directly from Lemma 5.2.2 and Lemma 5.2.3. □

5.3 The Operator-Preserving Property of $\Phi_{n,r}$

Our map $\Phi_{n,r}$ turns out to have many interesting properties. An important one is that $\Phi_{n,r}$ preserves matrix multiplication.

Proposition 5.3.1. *Let $n \geq 2, r \geq 1$, and $M, M' \in \text{Mat}(r, \mathbb{Z})$. Then*

$$\Phi_{n,r}(MM') = \Phi_{n,r}(M) * \Phi_{n,r}(M')$$

Before we prove this proposition, we need the following lemmas. First, we show some properties about the operations in Π-algebras.

Lemma 5.3.2. *If $n \geq 2$ is even, then $H_0([\iota_n, \iota_n]) = 2$.*

Proof. Let X be a Π-algebra and $x, y \in X_n$. By Lemma 3.0.3, we have

$$\begin{aligned}(x+y) \circ [\iota_n, \iota_n] =& x \circ [\iota_n, \iota_n] + y \circ [\iota_n, \iota_n] + H_0([\iota_n, \iota_n])[x, y] \\ =& [x, x] + [y, y] + H_0([\iota_n, \iota_n])[x, y]\end{aligned}$$

On the other hand,

$$(x+y) \circ [\iota_n, \iota_n] = [x+y, x+y] = [x, x] + [y, y] + [x, y] + [y, x]$$

so

$$H_0([\iota_n, \iota_n])[x, y] = [x, y] + [y, x] = 2[x, y] \Rightarrow (H_0([\iota_n, \iota_n]) - 2)[x, y] = 0$$

Since $[x, y] \in \mathcal{S}_{2n-1,2n-1} \cong \mathbb{Z}$ is torsion-free, $H_0([\iota_n, \iota_n]) = 2$. □

Lemma 5.3.3. *If $n \geq 2$ is odd and $\alpha \in \mathcal{S}_{n,2n-1}$, then $H_0(\alpha) = 0$.*

Proof. Let X be a Π-algebra and $x, y \in X_n$. By Lemma 3.0.3, we have

$$x \circ \alpha + y \circ \alpha + H_0(\alpha)[x, y] = (x+y) \circ \alpha = (y+x) \circ \alpha = x \circ \alpha + y \circ \alpha + H_0(\alpha)[y, x]$$

so $H_0(\alpha)([x, y] - [y, x]) = 0$. Since n is odd, $[x, y] = -[y, x]$, so $2H_0(\alpha)[x, y] = 0$, which implies that $2H_0(\alpha) = 0$. Since $H(\alpha) \in \mathcal{S}_{2n-1,2n-1} \cong \mathbb{Z}$ is torsion-free, $H_0(\alpha) = 0$. □

The two lemmas above give us some information about the constants d_k, t_k, and c_k defined in the definition of $\Phi_{n,r}$.

Lemma 5.3.4. *Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle \alpha_k \rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^{m} c_k \alpha_k$. If n is even, then there exists a $1 \leq k' \leq m$ such that $d_{k'} = 0$, $t_{k'} c_{k'} = 2$, and for all $1 \leq k \leq m$ with $k \neq k'$, $d_k > 0$, $t_k = 0$.*

Proof. As shown in Hess [4], no more than one of the given generators of $\mathcal{S}_{n,2n-1}$ can be of infinite order, so $d_k > 0$ for all $1 \leq k \leq m$ except for some $k = k'$. Then α_k is of finite order for all $k \neq k'$. Since $H_0 : \mathcal{S}_{n,2n-1} \to \mathbb{Z}$ is a homomorphism, $H_0(\alpha_k)$ is of finite order for all $k \neq k'$. But the only element in $\mathbb{Z}$ that has a finite order is 0, so $t_k = H_0(\alpha_k) = 0$ for all $k \neq k'$. Moreover, by Lemma 5.3.2,

$$H_0\left(\sum_{k=1}^{m} c_k\alpha_k\right) = H_0([\iota_n, \iota_n]) = 2 \Rightarrow \sum_{k=1}^{m} c_k H_0(\alpha_k) = 2 \Rightarrow \sum_{k=1}^{m} t_k c_k = 2 \Rightarrow t_{k'}c_{k'} = 2$$

Then $d_{k'} \neq 0$, for otherwise $t_{k'} = 0$ by the argument in the beginning of the proof. □

Lemma 5.3.5. *Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle\alpha_k\rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^{m} c_k\alpha_k$. If n is odd, then $t_k = 0$ and $2c_k \bmod d_k = 0$ for all $1 \leq k \leq m$.*

Proof. By Lemma 5.3.3, $t_k = H_0(\alpha_k) = 0$ for all $1 \leq k \leq m$. Moreover, when n is odd,

$$[\iota_n, \iota_n] = -[\iota_n, \iota_n] \Rightarrow 2[\iota_n, \iota_n] = 0 \Rightarrow \sum_{k=1}^{m} 2c_k\alpha_k = 0$$

Since $(\alpha_k)_{k=1}^{m}$ is linearly independent, $2c_k = 0$ in $\mathbb{Z}_{d_k}$ for all $1 \leq k \leq m$. This completes the proof. □

Moreover, in order to show that $\Phi_{n,r}$ preserves matrix multiplication, we need to show that $\Phi_{n,r}$ maps an identity matrix to the identity matrix in the corresponding space.

Lemma 5.3.6. *Let I be the identity matrix in* $\mathrm{Mat}(r, \mathbb{Z})$*. Then $\Phi_{n,r}(I)$ is the identity matrix in $\mathcal{M}_{n,r}$.*

Proof. Let I be the identity matrix, that is, $I_{kl} = 1$ if $k = l$ and $I_{kl} = 0$ if $k \neq l$. We show that $\Phi_{n,r}(I)_{ij} = 1$ if $i = j$ and $\Phi_{n,r}(I)_{ij} = 0$ if $i \neq j$. Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle\alpha_k\rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^{m} c_k\alpha_k$. Moreover, let $\hat{x} = (x_k)_{k=1}^{r}$ be a basis of $\pi_n(\bigvee^r S^n)$ and decide an ordering for $\varphi_{n,r}(\hat{x})$.
Suppose $i = j$, so either $\varphi_{n,r}(\hat{x})_i = \varphi_{n,r}(\hat{x})_j = [x_{i_1}, x_{i_2}]$, or $\varphi_{n,r}(\hat{x})_i = \varphi_{n,r}(\hat{x})_j = x_{i_1} \circ \alpha_{i_3}$. In the former case,

$$\Phi_{n,r}(I)_{ij} = \Phi_{n,r}(I)_{ii} = I_{i_1i_1}I_{i_2i_2} + (-1)^n I_{i_1i_2}I_{i_2i_1} = 1 \cdot 1 + (-1)^n 0 \cdot 0 = 1$$

($I_{i_1i_2} = I_{i_2i_1} = 0$ because $i_1 < i_2$ by the definition of $\varphi_{n,r}$). In the latter case, Then

$$\Phi_{n,r}(I)_{ij} = \Phi_{n,r}(I)_i i = t_{i_3}c_{i_3}\binom{I_{i_1i_1}}{2} + \theta_{i_3=i_3}I_{i_3i_3} = t_{i_3}c_{i_3} \cdot 0 + 1 \cdot 1 = 1.$$

Now suppose $i \neq j$. We have four cases.

- Case 1: $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$, $\varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}]$.
  In this case $i_1 \neq j_1$ or $i_2 \neq j_2$, so $I_{i_1 j_1} = 0$ or $I_{i_2 j_2} = 0$. Moreover, $I_{i_1 j_2} I_{i_2 j_1} \neq 1$, for otherwise $i_1 < i_2 = j_1 < j_2 = i_1$, which is a contradiction. Thus $I_{i_1 j_2} I_{i_2 j_1} = 0$, so $\Phi_{n,r}(I)_{ij} = I_{i_1 j_1} I_{i_2 j_2} + (-1)^n I_{i_1 j_2} I_{i_2 j_1} = 0$.

- Case 2: $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$, $\varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{j_3}$.
  In thise case $I_{i_1 j_1} I_{i_2 j_1} \neq 1$, for otherwise $i_1 = j_1 = i_2 > i_1$, which is a contradiction. Thus $I_{i_1 j_1} I_{i_2 j_1} = 0$, so $\Phi_{n,r}(I)_{ij} = c_{j_3} I_{i_1 j_1} I_{i_2 j_1} \bmod d_{j_3} = 0$.

- Case 3: $\varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{i_3}$, $\varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}]$.
  In this case $\Phi_{n,r}(I)_{ij} = t_{i_3} I_{i_1 j_1} I_{i_1 j_2} = 0$ by an argument symmetric to that in Case 2.

- Case 4: $\varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{j_3}$, $\varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{j_3}$. In this case $\binom{I_{i_1 j_1}}{2} = 0$. Moreover, $\theta_{i_3 = j_3} M_{i_1 j_1} = 0$, for otherwise $i_3 = j_3$ and $i_1 = j_1$, which implies that $i = j$, contradicting our assumption that $i \neq j$, so $\Phi_{n,r}(I)_{ij} = t_{i_3} c_{j_3} \binom{I_{i_1 j_1}}{2} + \theta_{i_3 = j_3} I_{i_1 j_1} \bmod d_{j_3} = 0$.

Therefore, $\Phi_{n,r}(I)$ is the identity matrix. □

Now we have everything we need to show that $\Phi_{n,r}$ preserves matrix multiplication. The body of the proof consists of a lengthy case-by-case calculation. In order keep the flow of the paper, we defer it to the appendix. See Appendix B for the complete proof.

5.4 The Invertibility of $\Phi_{n,r}(M)$

Another important property of $\Phi_{n,r}$ is that it preserves the invertibility of matrices in both ways.

Proposition 5.4.1. *Let $n \geq 2$, $r \geq 1$, and $M \in \mathrm{Mat}(r, \mathbb{Z})$. Then $M \in GL(r, \mathbb{Z})$ if and only if $\Phi_{n,r}(M)$ has a two-side inverse under $*$.*

Again, we establish some lemmas first before we prove this proposition.

In the matrix $\Phi_{n,r}(M)$ for any $M \in \mathrm{Mat}(r, \mathbb{Z})$, we pay attention to the block intersected by the rows and columns that correspond to the basis elements of infinite order, that is, basis elements of the form $[x_{i_1}, x_{i_2}]$ (and one more basis element when n is even, as shown in Lemma 5.3.4). This block plays an important role in determining the invertibility of $\Phi_{n,r}(M)$, and we give it a name in the following definition.

Definition 5.4.2 ($\Psi_{n,r}$)**.** Let $n \geq 2$ and $r \geq 1$ and $\hat{x} = (x_k)_{k=1}^r$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$. Define function $\Psi_{n,r} : \mathcal{M}_{n,r} \to \mathrm{Mat}(\binom{r}{2} + r\big((n+1) \bmod 2\big), \mathbb{Z})$ as the following. For an arbitrary matrix $N \in \mathcal{M}_{n,r}$, let $\Psi_{n,r}(N) \in \mathrm{Mat}(\binom{r}{2} + r, \mathbb{Z})$ be the matrix contained in N that is obtained by taking the entries in the rows and the columns of N (without changing the order) that correspond to the basis elements in

1. $([x_{i_1}, x_{i_2}])_{1 \leq i_1 < i_2 \leq r} \cup (x_{i_1} \circ \alpha_{k'})_{i_1=1}^r$,
   if n is even, where $\alpha_{k'}$ is the only generator of $\mathcal{S}_{n,2n-1}$ that has an infinite order as shown in Lemma 5.3.4, and

2. $([x_{i_1}, x_{i_2}])_{1\le i_1<i_2\le r}$, if n is odd.

How $\Psi_{n,r}$ selects entries in a matrix in $\mathcal{M}_{n,r}$ is illustrated in Example A.0.2 and Example A.0.3.

Notice that all the entries in $\Phi_{n,r}(M)$ that are in the same columns as but different rows from $\Psi_{n,r}(\Phi_{n,r}(M))$ are equal to zero. This fact not only simplifies the calculation of $\Psi_{n,r}$ for a product of matrices but also links the invertibility of $\Phi_{n,r}(M)$ to $\Psi_{n,r}(\Phi_{n,r}(M)$, as shown in the following lemma.

Lemma 5.4.3. *Let $n \ge 2$, $r \ge 1$, and $M, M' \in \mathrm{Mat}(r, \mathbb{Z})$. Then*

$$\Psi_{n,r}(\Phi_{n,r}(M) * \Phi_{n,r}(M')) = \Psi_{n,r}(\Phi_{n,r}(M))\Psi_{n,r}(\Phi_{n,r}(M'))$$

Proof. Consider $\Psi_{n,r}(\Phi_{n,r}(M))$ as a block in $\Phi_{n,r}(M)$. By Lemmas 5.3.4 and 5.3.5, the entries in the same columns but different rows as $\Psi_{n,r}(\Phi_{n,r}(M))$ are all zeros. This gives us the desired result. □

We further notice that for any matrix $N \in \mathcal{M}_{n,r}$, $\Psi_{n,r}(N)$ is an integer matrix, since $\Psi_{n,r}$ selects only the entries that are of infinite order. This allows us to calculate the determinant of $\Psi_{n,r}(N)$ to determine its invertibility. But before we do that, we must fix an order for the elements that correspond to the rows and columns of $\Psi_{n,r}$, so that the determinant of $\Psi_{n,r}(N)$ is well-defined for any $N \in \mathcal{M}_{n,r}$.

Definition 5.4.4 (Lexicographical ordering)**.** Let $n \ge 2$, $r \ge 1$, $\hat{x} = (x_k)_{k=1}^r$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$. We define a lexicographical ordering for the elements of $\varphi_{n,r}(\hat{x})$ that correspond to the rows and columns of $\Psi_{n,r}$ (see the definition of $\Psi_{n,r}$).
If n is even, index $[x_{i_1}, x_{i_2}]$ with (i_1, i_2) and index $x_{i_1} \circ \alpha_{k'}$ with (i_1, i_1). Then use the regular lexicographical ordering for $([x_{i_1}, x_{i_2}])_{1\le i_1<i_2\le r} \cup (x_{i_1} \circ \alpha_{k'})_{i_1=1}^r$, that is, $(i_1, i_2) \le (j_1, j_2)$ if and only if $i_1 < j_1$, or $i_1 = j_1$ but $i_2 \le j_2$.
If n is odd, then still index $[x_{i_1}, x_{i_2}]$ with (i_1, i_2) and use the regular lexicographical ordering.

How the elements are arranged in the lexicographical ordering is illustrated in Example A.0.4 and *Example* A.0.5.

With a fixed ordering defined for the columns and rows of $\Psi_{n,r}$, we now prove a series of lemmas that will give the relationship between the determinant of a matrix in $\mathrm{Mat}(r, \mathbb{Z})$ and that of its image under $\Psi_{n,r} \circ \Phi_{n,r}$. We first show that triangular and permuation matrices are preserved under $\Psi_{n,r} \circ \Phi_{n,r}$, respectively.

Lemma 5.4.5. *Let $n \ge 2$ and $r \ge 1$, and we choose an ordering for $\varphi_{n,r}$ such that the basis elements that the columns and rows of $\Psi_{n,r}$ refer to are in the lexicographical ordering. If $T \in \mathrm{Mat}(r, \mathbb{Z})$ is a lower (upper) triangular matrix, then $\Psi_{n,r}(\Phi_{n,r}(T))$ is a lower (upper) triangular matrix.*

Proof. Let $T \in \mathrm{Mat}(r, \mathbb{Z})$ be a lower triangular matrix, that is, $T_{kl} = 0$ for $1 \le k < l \le r$. Suppose $1 \le i < j \le \binom{r}{2} + r\big((n+1) \bmod 2\big)$ in the lexicographical ordering. We show that

$\Psi_{n,r}(\Phi_{n,r}(T))_{ij} = 0$. Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle\alpha_k\rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^{m} c_k\alpha_k$. Moreover, let $\alpha_{k'}$ be the only generator of $\mathcal{S}_{n,2n-1}$ that has an infinite order, as shown by Lemma 5.3.4. Furthermore, let $\hat{x} = (x_k)_{k=1}^{r}$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$.
We first suppose n is even, so we have the following four cases.

- Case 1: $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$, $\varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}]$.
  In this case, $i < j$ in the lexicographical ordering implies $i_1 < j_1$ or ($i_1 = j_1$ and $i_2 < j_2$), so $T_{i_1j_1}T_{i_2j_2} = 0$. By the definition of $\varphi_{n,r}$, we have $i_1 < i_2$ and $j_1 < j_2$, so $i_1 < i_2 < j_2$ or $i_1 < j_1 < j_2$. Then $T_{i_1j_2} = 0$. Thus $\Phi_{n,r}(T)_{ij} = T_{i_1j_1}T_{i_2j_2} + (-1)^n T_{i_1j_2}T_{i_2j_1} = 0$.

- Case 2: $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$, $\varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{k'}$.
  In this case, $i < j$ in the lexicographical ordering implies $i_1 < j_1$ or ($i_1 = j_1$ and $i_2 < j_1$), so $T_{i_1j_1}T_{i_2j_1} = 0$. Thus $\Phi_{n,r}(T)_{ij} = c_{j_3}T_{i_1j_1}T_{i_2j_1} \bmod d_{j_3} = 0$.

- Case 3: $\varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{k'}$, $\varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}]$.
  In this case, $i < j$ in the lexicographical ordering implies $i_1 < j_1$ or ($i_1 = j_1$ and $i_1 < j_2$), so $T_{i_1j_1}T_{i_1j_2} = 0$. Thus $\Phi_{n,r}(T)_{ij} = t_{i_3}T_{i_1j_1}T_{i_1j_2} = 0$.

- Case 4: $\varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{k'}$, $\varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{k'}$.
  In this case, $i < j$ in the lexicographical ordering implies $i_1 < j_1$, so $T_{i_1j_1} = 0$. Thus $\Phi_{n,r}(T)_{ij} = t_{i_3}c_{j_3}\binom{T_{i_1j_1}}{2} + \theta_{i_3=j_3}T_{i_1j_1} \bmod d_{j_3} = 0$.

Therefore, $\Phi_{n,r}(T)_{ij} = 0$ in all the four cases when n is even. Observe that the argument in Case 1 also shows that $\Phi_{n,r}(T)_{ij} = 0$ when n is odd, that is, when the basis elements in $\varphi_{n,r}(\hat{x})$ that the rows and columns of $\Psi_{n,r}$ refer to only include $([x_{i_1}, x_{i_2}])_{1\leq i_1<i_2\leq r}$. Thus $\Phi_{n,r}(T)$ is an lower triangular matrix no matter n is even or odd. The same argument applies to the case when T is an upper triangular matrix. □

Lemma 5.4.6. *Let $n \geq 2$ and $r \geq 1$. If $P \in \mathrm{Mat}(r, \mathbb{Z})$ is a permutation matrix, then $\Psi_{n,r}(\Phi_{n,r}(P))$ is a signed permutation matrix.*

Proof. Suppose $P \in \mathrm{Mat}(r, \mathbb{Z})$ is a permutation matrix, so in every row and every column there is exactly one entry equal to one, and the rest of the entries equal to zero. We use the function ω to describe this relation, so in the k^{th} row of P, the entry in the $\omega(k)^{\text{th}}$ column is the only entry that is equal to one. Let $1 \leq i \leq \binom{r}{2} + r\big((n+1) \bmod 2\big)$. We show that there exactly one entry equal to ± 1 in the i^{th} row of $\Psi_{n,r}(\Phi_{n,r}(P))$.
We first suppose that n is even. Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle\alpha_k\rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^{m} c_k\alpha_k$. Moreover, let $\alpha_{k'}$ be the only generator of $\mathcal{S}_{n,2n-1}$ that has an infinite order, as shown by Lemma 5.3.4. Furthermore, let $\hat{x} = (x_k)_{k=1}^{r}$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$. We have the following four cases.

- Case 1: $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$, $\varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}]$.
  Let $k_1 = \min(\omega(i_1), \omega(i_2))$, $k_2 = \max(\omega(i_1), \omega(i_2))$, and column k be the column of $\Psi_{n,r}(\Phi_{n,r}(P))$ that corresponds to the basis element $[x_{k_1}, x_{k_2}]$. If $\omega(i_1) < \omega(i_2)$, then

$P_{i_1k_1}P_{i_2k_2} = 1$ and $P_{i_1k_2}P_{i_2k_1} = 0$. Otherwise, $\omega(i_1) > \omega(i_2)$, then $P_{i_1k_1}P_{i_1k_2} = 0$ and $P_{i_1k_2}P_{i_2k_1} = 1$. In both cases, $\Psi_{n,r}(\Phi_{n,r}(P))_{ik} = P_{i_1k_1}P_{i_1k_2} + (-1)^n P_{i_1k_2}P_{i_2k_1} = \pm 1$. Moreover, among all the entries in this case, $\Psi_{n,r}(\Phi_{n,r}(P))_{ik}$ this is the only entry that is not equal to zero, because in row i_1 and row i_2 of P, only column k_1 and column k_2 have nonzero entries.

- Case 2: $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$, $\varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{k'}$.
  All the entries in this case are equal to zero. We have $P_{i_1j_1}P_{i_2j_1} = 0$, since in column j_1 of P at most one entry is nonzero. Thus $\Phi_{n,r}(P)_{ij} = c_{j_3}P_{i_1j_1}P_{i_2j_1} \bmod d_{j_3} = 0$.

- Case 3: $\varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{k'}$, $\varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}]$.
  All the entries in this case are equal to zero, by an argument symmetric to that in Case 2.

- Case 4: $\varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{k'}$, $\varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{k'}$.
  Since $P_{i_1j_1} = 0$ or ± 1, $\binom{P_{i_1j_1}}{2} = 0$. Then the entries in this case is just a copy of row i_1 of P, which has one entry equal to one and all the rest are equal to zero.

Thus no matter $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$ or $\varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{k'}$, row i of $\Psi_{n,r}(\Phi_{n,r}(P))$ has exactly one entry equal to ± 1 and all the rest are equal to zero, so $\Psi_{n,r}(\Phi_{n,r}(P))$ is a signed permutation matrix when n is even. When n is odd, we apply the argument in Case 1 to show the same result. Therefore, $\Psi_{n,r}(\Phi_{n,r}(P))$ is a signed permutation matrix no matter n is even or odd. □

Next, we show how the determinant is modified for triangular and permutation matrices under $\Psi_{n,r} \circ \Phi_{n,r}$, respectively.

Lemma 5.4.7. *If $T \in \mathrm{Mat}(r, \mathbb{Z})$ is a triangular matrix, then*

$$\det \Psi_{n,r}(\Phi_{n,r}(T)) = (\det T)^{r+(-1)^n}.$$

Proof. Suppose $T \in \mathrm{Mat}(r, \mathbb{Z})$ is a lower triangular matrix. Then $\Psi_{n,r}(\Phi_{n,r}(T))$ is a lower triangular matrix by Lemma 5.4.5, and thus the determinant of $\Psi_{n,r}(\Phi_{n,r}(T))$ is equal to the product of its diagonal entries. We show that this product is equal to $(\det T)^{r+(-1)^n}$.
We first suppose that n is even. Let $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^m \mathbb{Z}_{d_k}\langle \alpha_k \rangle$, and for $1 \le k \le m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^m c_k \alpha_k$. Moreover, let $\alpha_{k'}$ be the only generator of $\mathcal{S}_{n,2n-1}$ that has an infinite order. Then

$$\det \Psi_{n,r}(\Phi_{n,r}(T)) = \prod_{1 \le i_1 < i_2 \le r} (T_{i_1i_1}T_{i_2i_2} + T_{i_1i_2}T_{i_2i_1}) \prod_{i_1=1}^{r} \left(t_{k'}c_{k'}\binom{T_{i_1j_1}}{2} + T_{i_1j_1}\right).$$

Since $i_1 < i_2$ and T is a lower triangular matrix, $T_{i_1i_2} = 0$. Moreover, $t_{k'}c_{k'} = 2$ by Lemma 5.3.4, so $t_{k'}c_{k'}\binom{T_{i_1j_1}}{2} + T_{i_1j_1} = (T_{i_1j_1})^2$. Then

$$\det \Psi_{n,r}(\Phi_{n,r}(T)) = \prod_{1 \le i_1 < i_2 \le r} T_{i_1i_1}T_{i_2i_2} \prod_{i_1=1}^{r} (T_{i_1j_1})^2 = \prod_{i_1=1}^{r} (T_{i_1i_1})^{r+1} = (\det T)^{r+1}.$$

Now suppose n is odd. Then

$$\det \Psi_{n,r}(\Phi_{n,r}(T)) = \prod_{1\leq i_1<i_2\leq r} (T_{i_1i_1}T_{i_2i_2} + T_{i_1i_2}T_{i_2i_1}).$$

By the same argument as in the case when n is even,

$$\det \Psi_{n,r}(\Phi_{n,r}(T)) = \prod_{1\leq i_1<i_2\leq r} T_{i_1i_1}T_{i_2i_2} = \prod_{i_1=1}^{r}(T_{i_1i_1})^{r-1} = (\det T)^{r-1}.$$

Thus $\det \Psi_{n,r}(\Phi_{n,r}(T)) = (\det T)^{r+(-1)^n}$.
A symmetric argument shows that the equality also holds when T is an upper triangular matrix. □

Lemma 5.4.8. *Let $n \geq 2$ and ≥ 1. If $E \in \mathrm{Mat}(r, \mathbb{Z})$ is a row-switching elementary matrix, then*

$$\det(\Psi_{n,r}(\Phi_{n,r}E)) = (\det E)^{r-1}.$$

Proof. Let $E \in \mathrm{Mat}(r, \mathbb{Z})$ be a row-switching elementary matrix, that is, E differs from the identity matrix by switching two rows, say row k_1 and row k_2. Then $\Psi_{n,r}(\Phi_{n,r}(E))$ is a signed permutation matrix by Lemma 5.4.6, so $\det \Psi_{n,r}(\Phi_{n,r}(E)) = \pm 1$. We need to consider all the effects on the sign of $\det \Psi_{n,r}(\Phi_{n,r}(E))$ caused by switching row k_1 and row k_2 in I_r.

We first suppose that n is odd, so $\Psi_{n,r}(\Phi_{n,r}(E))_{ij} = E_{i_1j_1}E_{i_1j_2} - E_{i_1j_2}E_{i_2j_1}$. Let I_r be the identity matrix of dimension r, so $\Psi_{n,r}(\Phi_{n,r}(E)) = I_{\binom{r}{2}}$, the identity matrix of dimension $\binom{r}{2}$, by Lemma 5.3.6. By switching row k_1 and row k_2 of I_r, the row of $\Psi_{n,r}(\Phi_{n,r}(E))$ that corresponds $[x_{i_1}, x_{k_1}]$ is switched with the row that corresponds to $[x_{i_1}, x_{k_2}]$ for all $1 \leq i_1 \leq k_1$, and the row of $\Psi_{n,r}(\Phi_{n,r}(E))$ that corresponds $[x_{k_1}, x_{i_2}]$ is switched with the row that corresponds to $[x_{k_2}, x_{i_2}]$ for all $k_2 \leq i_2 \leq r$. There are totally $r - (k_2 - k_1 + 1)$ mutually exclusive, pairwise row switching of this kind.

Moreover, for all $1 \leq i_1 \leq k_1$, switching row k_1 and row k_2 in I_r not only switches the row of $\Psi_{n,r}(\Phi_{n,r}(E))$ that corresponds to $[x_{k_1}, x_{i_1}]$ with the row that corresponds to $[x_{i_1}, x_{k_2}]$, but also changes both of the two nonzero entries in the two rows from 1 to -1. Then the changes that take place in the rows of this kind is $k_2 - k_1 - 1$ mutually exclusive, pairwise row switchings, and $2(k_2 - k_1 - 1)$ changes of signs of nonzero entries.

Finally, by switching row k_1 with row k_2 in I_r, the nonzero entry in the row of $\Psi_{n,r}(\Phi_{n,r}(E))$ that corresponds to $[x_{k_1}, x_{k_2}]$ is changed from 1 to -1.

In sum, switching row k_1 with row k_2 in I_r causes $r - (k_2 - k_1 + 1) + k_2 - k_1 - 1 = r - 2$ mutually exclusive, pairwise row switchings and $2(k_2 - k_1 - 1) + 1$ changes of signs of nonzero entries to take place in $\Psi_{n,r}(\Phi_{n,r}(I))$. Since in a permutation matrix a single pairwise row switching cause changes the sign of the determinant of the matrix from ± 1 to ∓ 1, and the change of sign of a nonzero entry of the permutation matrix has the same effect on its determinant, we conclude that

$$\det \Psi_{n,r}(\Phi_{n,r}(E)) = (-1)^{r-2+2(k_2-k_1-1)+1} = (-1)^{r-1} = (\det E)^{r-1}$$

when n is odd.

Now suppose n is even. Let $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle\alpha_k\rangle$, and for $1 \le k \le m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^{m} c_k\alpha_k$. Also, let $\alpha_{k'}$ be the only generator of $\mathcal{S}_{n,2n-1}$ that has an infinite order, as shown in Lemma 5.3.4. For the rows of $\Psi_{n,r}(\Phi_{n,r}(I_r))$ that corresponds to the basis elements in $([x_{i_1}, x_{i_2}])_{1\le i_1<i_2\le r}$, switching row k_1 and row k_2 in I_r has the same effect as described in the argument for the case when n is odd, only that there is no change of signs, so $r-2$ mutually exclusive, pairwise row switchings take place among these rows. Moreover, among the rows that corresponds to the basis elements in $(x_{i_1} \circ \alpha_{k'})_{i_1=1}^{r}$, the only change that takes place is the switching of the row that corresponds to $x_{k_1} \circ \alpha_{k'}$ and the row that corresponds to $x_{k_2} \circ \alpha_{k'}$.

In sum, switching row k_1 with row k_2 in I_r causes $r-1$ mutually exclusive, pairwise row switchings and no change of signs in $\Psi_{n,r}(\Phi_{n,r}(I))$. Thus $\det \Psi_{n,r}(\Phi_{n,r}(E)) = (\det E)^{r-1}$ no matter n is even or odd. □

Lemma 5.4.9. *Let $n \ge 2$ and $r \ge 1$. If $P \in \mathrm{Mat}(r, \mathbb{Z})$ is a permutation matrix, then*

$$\det(\Psi_{n,r}(\Phi_{n,r}P)) = \det(P)^{r+(-1)^n}.$$

Proof. This follows from the fact that a permutation matrix can be decomposed into a product of row-switching elementary matrices. Let P be a permutation matrix and $P = \prod_{k=1}^{s} E_k$, where E_k are row-switching elementary matrices. Then we have

$$\begin{aligned}\det \Psi_{n,r}(\Phi_{n,r}(P)) &= \det \Psi_{n,r}\left(\Phi_{n,r}\left(\prod_{k=1}^{s} E_k\right)\right) = \det \Psi_{n,r}(\Phi_{n,r}(E_1) * \cdots * \Phi_{n,r}(E_s)) \\ &= \det \prod_{k=1}^{s} \Psi_{n,r}(\Phi_{n,r}(E_k)) = \prod_{k=1}^{s} \det \Psi_{n,r}(\Phi_{n,r}(E_k)) = \prod_{k=1}^{s} (\det E_k)^{r-1} \\ &= \left(\prod_{k=1}^{s} \det E_k\right)^{r-1} = (\det P)^{r-1}\end{aligned}$$

by Proposition 5.3.1 and Lemma 5.4.3. □

Finally, we have a formula for how the determinant of an arbitrary matrix is modified by $\Psi_{n,r} \circ \Phi_{n,r}$.

Proposition 5.4.10. *Let $n \ge 2$, $r \ge 1$, and $M \in \mathrm{Mat}(r, \mathbb{Z})$. Then*

$$\det \Psi_{n,r}(\Phi_{n,r}^{M}) = \det(M)^{r+(-1)^n}.$$

Proof. Decompose M into a product of a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U. Then we have

$$\begin{aligned}\det \Psi_{n,r}(\Phi_{n,r}(M)) &= \det \Psi_{n,r}(\Phi_{n,r}(PLU)) = \det \Psi_{n,r}(\Phi_{n,r}(P) * \Phi_{n,r}(L) * \Phi_{n,r}(U)) \\ &= \det(\Psi_{n,r}(\Phi_{n,r}(P))\Psi_{n,r}(\Phi_{n,r}(L))\Psi_{n,r}(\Phi_{n,r}(U))) \\ &= (\det \Psi_{n,r}(\Phi_{n,r}(P)))(\det \Psi_{n,r}(\Phi_{n,r}(L)))(\det \Psi_{n,r}(\Phi_{n,r}(U))) \\ &= (\det P)^{r-1}(\det L)^{r+(-1)^n}(\det U)^{r+(-1)^n}\end{aligned}$$

by Proposition 5.3.1 and Lemma 5.4.3. Since $\det P = \pm 1$, $(\det P)^{r-1} = (\det P)^{r+(-1)^n}$. Thus

$$\begin{aligned}\det \Psi_{n,r}(\Phi_{n,r}(M)) =& (\det P)^{r+(-1)^n}(\det L)^{r+(-1)^n}(\det U)^{r+(-1)^n}\\ =& (\det(PLU))^{r+(-1)^n} = (\det M)^{r+(-1)^n}\end{aligned}$$

□

Now we have everything we need for proving the relationship between the invertibility of a matrix and that of its image under $\Psi_{n,r} \circ \Phi_{n,r}$.

Proof. This is a proof for Proposition 5.4.1. Suppose $M \in GL(r, \mathbb{Z})$, so $MM' = I_r$ for some $M' \in GL(r, \mathbb{Z})$, where I_r the identity matrix of dimension r. Then

$$\Phi_{n,r}(M) * \Phi_{n,r}(M') = \Phi_{n,r}(MM') = \Phi_{n,r}(I_r) = I_{\binom{r}{2}+rm}$$

by Proposition 5.3.1 and Lemma 5.3.6. Similarly, $\Phi_{n,r}(M') * \Phi_{n,r}(M) = I_{\binom{r}{2}+rm}$. Thus $\Phi_{n,r}(M)$ has a two-side inverse under $*$.
For the converse, suppose $N * \Phi_{n,r}(M) = I_{\binom{r}{2}+rm}$ for some $N \in \mathcal{M}_{n,r}$. Then

$$I_{\binom{r}{2}+r(n \bmod 2)} = \Psi_{n,r}(N * \Phi_{n,r}(M)) = \Psi_{n,r}(N)\Psi_{n,r}(\Phi_{n,r}(M)),$$

so $\Psi_{n,r}(\Phi_{n,r}(M)) \in \mathrm{Mat}(\binom{r}{2} + r(n \bmod 2), \mathbb{Z})$ is invertible under regular matrix multiplication. But this implies that $\det \Psi_{n,r}(\Phi_{n,r}(M)) = \pm 1$. Then by Proposition 5.4.10, $\det M = \pm 1$, so $M \in GL(r, \mathbb{Z})$. □

5.5 $\Phi_{n,r}$ and $(n, 2n)$-Cell Complexes

We now show how $\Phi_{n,r}$ is related to $(n, 2n)$-cell complexes. In particular, the following lemma shows how a $(n, 2n)$-cell map induces two change-of-basis matrices in $\pi_n(\bigvee^r S^n)$ and $\pi_{2n-1}(\bigvee^r S^n)$ that are linked by $\Phi_{n,r}$.

Lemma 5.5.1. *Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^m \mathbb{Z}_{d_k}\langle\alpha_k\rangle$, and $\hat{x} = (x_k)_{k=1}^r$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$. Moreover, let $A : \bigvee^r S^n \to \bigvee^r S^n$ be a continuous map. and let A_* be the induced Π-algebra homomorphism, so A_n induces some $M \in \mathrm{Mat}(r, \mathbb{Z})$ on the basis $\hat{x}$ and A_{2n-1} induces some $N \in \mathrm{Mat}(\binom{r}{2} + rm, \mathbb{Z})$ on the Hilton basis $\varphi_{n,r}(\hat{x})$ such that $A_n(\hat{x}) = M\hat{x}$ and $A_{2n-1}(\varphi_{n,r}(\hat{x})) = N\varphi_{n,r}(\hat{x})$. Then $\varphi_{n,r}(M\hat{x}) = N\varphi_{n,r}(\hat{x})$.*

Proof. Consider the element $\varphi_{n,r}(M\hat{x})_i$. If $\varphi_{n,r}(M\hat{x})_i = [(M\hat{x})_{i_1}, (M\hat{x})_{i_2}]$ for some $1 \leq i_1 < i_2 \leq r$, then

$$\varphi_{n,r}(M\hat{x})_i = [A_n(\hat{x})_{i_1}, A_n(\hat{x})_{i_2}] = [A_n(x_{i_1}), A_n(x_{i_2})].$$

By the axioms of Π-algebra homomorphism, the previous expression is equal to

$$A_{2n-1}([x_{i_1}, x_{i_2}]) = A_{2n-1}(\varphi_{n,r}(\hat{x})_i) = A_{2n-1}(\varphi_{n,r}(\hat{x}))_i.$$

so $\varphi_{n,r}(M\hat{x})_i = A_{2n-1}(\varphi_{n,r}(\hat{x}))_i$. A similar argument shows that this equality still holds when $\varphi_{n,r}(M\hat{x})_i = x_{i_1} \circ \alpha_{i_3}$ for some $1 \leq i_1 \leq r$ and $1 \leq i_3 \leq m$. Thus $\varphi_{n,r}(M\hat{x}) = N\varphi_{n,r}(\hat{x})$. □

Lemma 5.5.2. *Let $n \geq 2$, $r \geq 1$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle \alpha_k \rangle$, and $\hat{x} = (x_k)_{k=1}^{r}$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$. Moreover, let $A : \bigvee^r S^n \to \bigvee^r S^n$ be a continuous map, and let A_* be the induced Π-algebra homomorphism, so A_n induces some $M \in \mathrm{Mat}(r, \mathbb{Z})$ on the basis $\hat{x}$ and A_{2n-1} induces some $N \in \mathrm{Mat}(\binom{r}{2} + rm, \mathbb{Z})$ on the Hilton basis $\varphi_{n,r}(\hat{x})$ such that $A_n(\hat{x}) = M\hat{x}$ and $A_{2n-1}(\varphi_{n,r}(\hat{x})) = N\varphi_{n,r}(\hat{x})$. Then $N = \Phi_{n,r}(M)$.*

Proof. By Lemma 5.5.1, $\varphi_{n,r}(M\hat{x}) = N\varphi_{n,r}(\hat{x})$. Moreover, $\varphi_{n,r}(M\hat{x}) = \Phi_{n,r}(M)\varphi_{n,r}(\hat{x})$ by Lemma 5.2.4. Since $\varphi_{n,r}(\hat{x})$ is a basis, $N = \Phi_{n,r}(M)$. □

We also need the following lemma about the relationship between homotopy class and homotopy group endomorphisms in order to prove our main theorem.

Lemma 5.5.3. $[\bigvee^r S^n, \bigvee^r S^n] \cong \mathrm{End}(\pi_n(\bigvee^r S^n))$

Proof. We have isomorphisms

$$\left[\bigvee^r S^n, \bigvee^r S^n\right] \cong \bigoplus^r \left[S^n, \bigvee^r S^n\right] \cong \bigoplus^r \pi_n\left(\bigvee^r S^n\right) \cong \bigoplus^r \mathrm{Hom}\left(\pi_n(S^n), \pi_n\left(\bigvee^r S^n\right)\right)$$
$$\cong \mathrm{Hom}\left(\bigoplus^r \pi_n(S^n), \pi_n\left(\bigvee^r S^n\right)\right) \cong \mathrm{Hom}\left(\pi_n\left(\bigvee^r S^n\right), \pi_n\left(\bigvee^r S^n\right)\right).$$

Furthermore, this isomorphism is induced by the map

$$\left[\bigvee^r S^n, \bigvee^r S^n\right] \to \mathrm{Hom}\left(\pi_n\left(\bigvee^r S^n\right), \pi_n\left(\bigvee^r S^n\right)\right)$$

where $[f] \mapsto f_*$. □

5.6 Proof for the Main Theorem

After proving all the lemmas in this section, we are ready to present the proof for our main theorem.

Proof. This is a proof for Theorem 5.1.7. Since $\hat{y}$ is a Hilton basis, $\hat{y} = \varphi_{n,r}(\hat{x})$ for some basis $\hat{x}$ of $\mathcal{S}_{n,2n-1}$. First we uppose $(X, r, f) \simeq (Y, r, g)$. Then there exists a continuous map $A : \bigvee^r S^n \to \bigvee^r S^n$ such that $A \circ f \simeq g$, This A induces a Π-algebra homomorphism A_*. Moreover, A_n induces a $M \in \mathrm{Mat}(r, \mathbb{Z})$ on basis $\hat{x}$ and A_{2n-1} induces a $N \in \mathrm{Mat}(\binom{r}{2} + rm, \mathbb{Z})$ on basis $\hat{y}$ such that $A_n(\hat{x}) = M\hat{x}$ and $A_{2n-1}(\hat{y}) = N\hat{y}$. Meanwhile, $[g] = A_{2n-1}([f]) = A_{2n-1}(\hat{a}^T\hat{y})$. Since $A_3 \in End(\pi_3(\bigvee^r S^2))$, $A_{2n-1}(\hat{a}^T\hat{y}) = \hat{a}^T A_{2n-1}(\hat{y}) = \hat{a}^T N\hat{y}$, which in turn is equal to $\Phi_{n,r}(M)\hat{y}$ by Lemma 5.5.2. In addition, A_n is invertible since A is a homotopy equivalence, so M is invertible by Proposition 5.4.1.
For the converse, suppose $[g] = \hat{a}^T N\hat{y}$ for some $N \in \mathrm{Im}\,\Phi_{n,r} \cap \mathrm{GL}(\binom{r}{2} + rm, \mathbb{Z})$. Then N is invertible and there exists a $M \in \mathrm{Mat}(r, \mathbb{Z})$ such that $N = \Phi_{n,r}(M)$. By Proposition 5.4.1, $M \in GL(r, \mathbb{Z})$, so M on basis $\hat{x}$ corresponds to some element of $\mathrm{End}(\pi_{2n-1}(\bigvee^r S^n))$. Then

by Lemma 5.5.3, there exists a continuous map $A : \bigvee^r S^n \to \bigvee^r S^n$ such that A induces a Π-algebra homomorphism A_* and $A_n(\hat{x}) = M\hat{x}$. Furthermore, by Lemma 5.5.2, $A_{2n-1}(\hat{y}) = \Phi_{n,r}(M)\hat{y}$. Then

$$[g] = \hat{a}^T N\hat{y} = \hat{a}^T\Phi_{n,r}(M)\hat{y} = \hat{a}^T A_{2n-1}(\hat{y}) = A_{2n-1}(\hat{a}^T\hat{y}) = A_{2n-1}([f]),$$

which implies that $g \simeq A \circ f$. Therefore, by the homology long exact sequence argument and Whitehead's Theorem (Whitehead [6], pp. 181, 269), $(X, r, f) \simeq (Y, r, g)$. □

6 Examples for the Main Theorem from 4-Manifolds

We now illustrate the results from the previous sections by using specific 4-manifolds.
First of all, every $(2, 4)$-cell complex has an attaching map $f : S^3 \to \bigvee^r S^2$ with homotopy class

$$[f] = \sum_{1\le i_1<i_2\le r} a_{i_1i_2}[x_{i_1}, x_{i_2}] + \sum_{i=1}^{r} a_{ii}(x_i \circ \alpha_{2,1})$$

in $\pi_3(\bigvee^r S^2)$ for some integer coefficients $(a_{i_1i_2})_{1\le i_1\le i_2\le r}$, where $\alpha_{2,1}$, the Hopf fibration, is the generator of $\mathcal{S}_{2,3} \cong \mathbb{Z}$, and $(x_i)_{i=1}^r$ is a basis for $\pi_2(\bigvee^r S^2)$.
In particular, we have

1. The 4-sphere S^4: Here $f : S^3 \to *$, so $r = 0$.
2. The complex projective plane $\mathbb{C}P^2$: Here $f : S^3 \to S^2$ is the Hopf fibration, so that $r = 1$, $\hat{x} = (\iota)$, where ι, the identity map from S^2 to itself, is the generator of $\mathcal{S}_{2,2} \cong \mathbb{Z}$, and $\varphi_{2,1}(\hat{x}) = (\iota \circ \alpha_{2,1}) = \alpha_{2,1}$ in $\mathcal{S}_{2,3}$. Hence $\hat{a} = (a_{11}) = (1)$.
3. The sphere product $S^2 \times S^2$: Here $f : S^3 \to S^2 \vee S^2$ and $r = 2$. The attaching map f has homotopy class $[f] = [\iota_1, \iota_2]$, where ι_1 and ι_2, the identity maps from S^2 to itself, are the generators of $\pi_2(S^2 \vee S^2)$. Moreover, we have

$$\hat{x} = \begin{pmatrix}\iota_1\\ \iota_2\end{pmatrix} \text{ and } \varphi_{2,2} = \begin{pmatrix}\iota_1 \circ \alpha_{2,1}\\ [\iota_1, \iota_2]\\ \iota_2 \circ \alpha_{2,1}\end{pmatrix}, \text{ so } \hat{a} = \begin{pmatrix}a_{11}\\ a_{12}\\ a_{22}\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}.$$

Pick a $M \in \mathrm{GL}(2, \mathbb{Z})$, say

$$M = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}. \text{ Then } \Phi_{2,2}(M) = \begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} \text{ and } \hat{a}^T\Phi_{2,2}(M) = \begin{pmatrix}0\\ 1\\ 1\end{pmatrix}.$$

Then by Theorem 5.1.7, the 4-manifold derived from $[\iota_1, \iota_2] + \iota_2 \circ \alpha_{2,1}$ is of the same homotopy type as the sphere product $S^2 \times S^2$.

Now, if we pick a different attaching map, $[g] = \iota_1 \circ \alpha_{2,1}$, then we will have the coefficient vectors

$$\hat{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \hat{b}^T \Phi_{2,2}(M) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

which says that the 4-manifold derived from $\iota_1 \circ \alpha_{2,1} + 2[\iota_1, \iota_2] + \iota_2 \circ \alpha_{2,1}$ is of the same homotopy type as $\mathbb{C}P^2 \vee S^2$.

7 Further Problems

Our main theorem states that two $(n, 2n)$-cell complexes are equivalent if and only if the homotopy group elements that their attaching maps correspond to, when written in a Hilton basis, are different only by a change-of-basis matrix L that is in the image of $\Phi_{n,r}$ (see Theorem 5.1.7). One question to ask next is whether we can loosen the restriction on L from the image of $\Phi_{n,r}$ to the conjugates of the image of $\Phi_{n,r}$. This question is phrased in the following conjecture.

Conjecture 7.0.1. *Let $n \geq 2$, $r \geq 1$, and $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k} \langle \alpha_k \rangle$. Choose a Hilton basis $\hat{y}$ of $\pi_{2n-1}(\bigvee^r S^n)$. Moreover, let f, g be continuous maps from S^{2n-1} to $\bigvee^r S^n$ with $[f] = \hat{a}^T \hat{y}$ for some coefficient vector $\hat{a}$. Furthermore, let (X, r, f) and (Y, r, g) be the $(n, 2n)$-cell complexes derived from f and g. Suppose $[g] = \hat{a}^T L \hat{y}$ for some $L \in \left(K(\operatorname{Im}\Phi_{n,r})K^{-1}\right) \cap \mathrm{GL}(\binom{r}{2} + rm, \mathbb{Z})$, with some invertible $K \in \mathcal{M}_{n,r}$. Then $(X, r, f) \simeq (Y, r, g)$.*

If $\operatorname{Im}\Phi_{n,r}$ is normal, then the conjecture obviously holds. If not, then conjugation by an invertible matrix may have at least a chance of preserving the Hilton basis property, or do we need to add additional conditions to the conjecture to make it true? That is a question to be investigated.

A Further Examples

Example A.0.1. Let $n = 4$, so $\mathcal{S}_{4,7} \cong \mathbb{Z}\langle \alpha_{4,1} \rangle \oplus \mathbb{Z}_3 \langle \alpha_{4,2} \rangle \cong \mathbb{Z}_4 \langle \alpha_{4,3} \rangle$, where $H(\alpha_{4,1}) = \iota_7$, $H(\alpha_{4,2}) = H(\alpha_{4,3}) = 0$, and $[\iota_4, \iota_4] = 2\alpha_{4,1} - \alpha_{4,3}$ (see Appendix C). Moreover, let $r = 2$, so $\pi_4(\bigvee^2 S^4) \cong \mathbb{Z}^2$, and let $\hat{x} = (x_1, x_2)$ be an ordered basis of it. We use the ordering such that

$$\varphi_{4,2}(\hat{x}) = \begin{pmatrix} [x_1, x_2] \\ x_1 \circ \alpha_{4,1} \\ x_2 \circ \alpha_{4,1} \\ x_1 \circ \alpha_{4,2} \\ x_2 \circ \alpha_{4,2} \\ x_1 \circ \alpha_{4,3} \\ x_2 \circ \alpha_{4,3} \end{pmatrix}$$

Let $A, B \in \mathcal{M}_{4,2}$ and

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

Under normal matrix multiplication, we have

$$AB = \begin{pmatrix} 7 & 7 & 7 & 7 & 7 & 7 & 6 \\ 7 & 7 & 7 & 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 7 & 6 & 7 & 7 \\ 7 & 7 & 7 & 6 & 7 & 7 & 7 \\ 7 & 7 & 6 & 7 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 & 7 & 7 & 7 \\ 6 & 7 & 7 & 7 & 7 & 7 & 7 \end{pmatrix},$$

so under the matrix multiplication for $\mathcal{M}_{4,2}$, we have

$$A * B = \begin{pmatrix} 7 & 7 & 7 & 1 & 1 & 3 & 2 \\ 7 & 7 & 7 & 1 & 1 & 2 & 3 \\ 7 & 7 & 7 & 1 & 0 & 3 & 3 \\ 7 & 7 & 7 & 0 & 1 & 3 & 3 \\ 7 & 7 & 6 & 1 & 1 & 3 & 3 \\ 7 & 6 & 7 & 1 & 1 & 3 & 3 \\ 6 & 7 & 7 & 1 & 1 & 3 & 3 \end{pmatrix}.$$

Example A.0.2. Let $n = 4$, so $\mathcal{S}_{4,7} \cong \mathbb{Z}\langle\alpha_{4,1}\rangle \oplus \mathbb{Z}_3\langle\alpha_{4,2}\rangle\mathbb{Z}_4 \oplus \langle\alpha_{4,3}\rangle$ (see Appendix C). Moreover, let $r = 3$, so $\pi_4(\bigvee^3 S^4) \cong \mathbb{Z}^3$, and let $\hat{x} = (x_1, x_2, x_3)$ be an ordered basis of it. We use the ordering such that

$$\varphi_{4,3}(\hat{x}) = \begin{pmatrix} [x_1, x_2] \\ [x_1, x_3] \\ [x_2, x_3] \\ x_1 \circ \alpha_{4,1} \\ x_2 \circ \alpha_{4,1} \\ x_3 \circ \alpha_{4,1} \\ x_1 \circ \alpha_{4,2} \\ x_2 \circ \alpha_{4,2} \\ x_3 \circ \alpha_{4,2} \\ x_1 \circ \alpha_{4,3} \\ x_2 \circ \alpha_{4,3} \\ x_3 \circ \alpha_{4,3} \end{pmatrix}.$$

Suppose

$$N = \begin{pmatrix} 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 21 & 22 & 23 & 24 & 25 & 26 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 31 & 32 & 33 & 34 & 35 & 36 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 41 & 42 & 43 & 44 & 45 & 46 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 51 & 52 & 53 & 54 & 55 & 56 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 61 & 62 & 63 & 64 & 65 & 66 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 71 & 72 & 73 & 74 & 75 & 76 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 81 & 82 & 83 & 84 & 85 & 86 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 91 & 92 & 93 & 94 & 95 & 96 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 101 & 102 & 103 & 104 & 105 & 106 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 111 & 112 & 113 & 114 & 115 & 116 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 121 & 122 & 123 & 124 & 125 & 126 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \end{pmatrix},$$

which is in $\mathcal{M}_{4,3}$. Then

$$\Psi_{4,3}(N) = \begin{pmatrix} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{pmatrix}.$$

Example A.0.3. Let $n = 3$, so $\mathcal{S}_{3,5} \cong \mathbb{Z}_2\langle \eta_3^2 \rangle$ (see Appendix C). Moreover, let $r = 3$, so $\pi_3(\bigvee^3 S^3) \cong \mathbb{Z}^3$, and let $\hat{x} = (x_1, x_2, x_3)$ be an ordered basis of it. We use the ordering such that

$$\varphi_{3,3}(\hat{x}) = \begin{pmatrix} [x_1, x_2] \\ [x_1, x_3] \\ [x_2, x_3] \\ x_1 \circ \eta_3^2 \\ x_2 \circ \eta_3^2 \\ x_3 \circ \eta_3^2 \end{pmatrix}.$$

Suppose that

$$N = \begin{pmatrix} 11 & 12 & 13 & 1 & 1 & 1 \\ 21 & 22 & 23 & 1 & 1 & 1 \\ 31 & 32 & 33 & 1 & 1 & 1 \\ 41 & 42 & 43 & 1 & 1 & 1 \\ 51 & 52 & 53 & 1 & 1 & 1 \\ 61 & 62 & 63 & 1 & 1 & 1 \end{pmatrix},$$

which is in $\mathcal{M}_{3,3}$. Then

$$\Psi_{3,3}(N) = \begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix}.$$

Example A.0.4. We have $\mathcal{S}_{4,7} \cong \mathbb{Z}\langle\alpha_{4,1}\rangle \oplus \mathbb{Z}_3\langle\alpha_{4,2}\rangle \oplus \mathbb{Z}_4\langle\alpha_{4,3}\rangle$ (see Appendix C), and let $\hat{x} = (x_k)_{k=1}^4$ be an ordered basis of $\pi_7(\bigvee^r S^4)$. In the lexicographical ordering, the basis elements in $\varphi_{n,r}(\hat{x})$ that the rows and columns of $\Psi_{4,4}$ refer to, that is, $([x_{i_1}, x_{i_2}])_{1\le i_1<i_2\le 4} \cup (x_{i_1} \circ \alpha_{4,1})_{i_1=1}^4$, is ordered as

$$\begin{pmatrix} x_1 \circ \alpha_{4,1} \\ [x_1, x_2] \\ [x_1, x_3] \\ [x_1, x_4] \\ x_2 \circ \alpha_{4,1} \\ [x_2, x_3] \\ [x_2, x_4] \\ x_3 \circ \alpha_{4,1} \\ [x_3, x_4] \\ x_4 \circ \alpha_{4,1} \end{pmatrix}$$

Example A.0.5. We have $\mathcal{S}_{3,5} \cong \mathbb{Z}_2\langle\alpha_{3,1}\rangle$ (see Appendix C), and let $\hat{x} = (x_k)_{k=1}^4$ be an ordered basis of $\pi_5(\bigvee^r S^3)$. In the lexicographical ordering, the basis elements in $\varphi_{n,r}(\hat{x})$ that the rows and columns of $\Psi_{3,4}$ refer to, that is, $([x_{i_1}, x_{i_2}])_{1\le i_1<i_2\le 4}$, is ordered as

$$\begin{pmatrix} [x_1, x_2] \\ [x_1, x_3] \\ [x_1, x_4] \\ [x_2, x_3] \\ [x_2, x_4] \\ [x_3, x_4] \end{pmatrix}$$

B Proof for the Operation-Preserving Property of $\Phi_{n,r}$

Proof. This is a proof for Proposition 5.3.1. Let $n \ge 2$, $r \ge 1$, $M, M' \in \mathrm{Mat}(r, \mathbb{Z})$, $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^m \mathbb{Z}_{d_k}\langle\alpha_k\rangle$, and for $1 \le k \le m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^m c_k\alpha_k$. Moreover, let $\hat{x} = (x_k)_{k=1}^r$ be an ordered basis of $\pi_n(\bigvee^r S^n) \cong \mathbb{Z}^r$. We show that $\Phi_{n,r}(MM')_{ij} = (\Phi_{n,r}(M) * \Phi_{n,r}(M'))_{ij}$ for all $1 \le i, j \le \binom{r}{2} + rm$ in the following four cases.

- Case 1: $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$, $\varphi_{n,r}(\hat{x})_j = [x_{j_1}, x_{j_2}]$.

On one hand, we have

$$\begin{aligned}
\Phi_{n,r}(MM')_{ij} =&(MM')_{i_1j_1}(MM')_{i_2j_2} + (-1)^n (MM')_{i_1j_2}(MM')_{i_2j_1}\\
=&\sum_{k_1=1}^{r} M_{i_1k_1}M'_{k_1j_1}\sum_{k_2=1}^{r} M_{i_2k_2}M'_{k_2j_2} + (-1)^n \sum_{k_1=1}^{r} M_{i_1k_1}M'_{k_1j_2}\sum_{k_2=1}^{r} M_{i_2k_2}M'_{k_2j_1}\\
=&\sum_{1\le k_1,k_2\le r} M_{i_1k_1}M_{i_2k_2}M'_{k_1j_1}M'_{k_2j_2} + (-1)^n M_{i_1k_1}M_{i_2k_2}M'_{k_1j_2}M'_{k_2j_1}\\
=&\sum_{1\le k_1<k_2\le r} M_{i_1k_1}M_{i_2k_2}M'_{k_1j_1}M'_{k_2j_2} + (-1)^n M_{i_1k_1}M_{i_2k_2}M'_{k_1j_2}M'_{k_2j_1}\\
+&\sum_{1\le k_1<k_2\le r} M_{i_1k_2}M_{i_2k_1}M'_{k_2j_1}M'_{k_1j_2} + (-1)^n M_{i_1k_2}M_{i_2k_1}M'_{k_2j_2}M'_{k_1j_1}\\
+&\sum_{k_1=1}^{r}(1+(-1)^n)M_{i_1k_1}M_{i_2k_1}M'_{k_1j_1}M'_{k_1j_2}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\Phi_{n,r}(M) * \Phi_{n,r}(M'))_{ij}\\
=&\sum_{1\le k_1<k_2\le r}(M_{i_1k_1}M_{i_2k_2} + (-1)^n M_{i_1k_2}M_{i_2k_1})(M'_{k_1j_1}M'_{k_2j_2} + (-1)^n M'_{k_1j_2}M'_{k_2j_1})\\
+&\sum_{k_1=1}^{r}\sum_{k_3=1}^{m}(c_{k_3}M_{i_1k_1}M_{i_2k_1} \bmod d_{k_3})(t_{k_3}M'_{k_1j_1}M'_{k_1j_2})
\end{aligned}$$

We are left to check

$$\sum_{k_1=1}^{r}(1+(-1)^n)M_{i_1k_1}M_{i_2k_1}M'_{k_1j_1}M'_{k_1j_2} = \sum_{k_1=1}^{r}\sum_{k_3=1}^{m}(c_{k_3}M_{i_1k_1}M_{i_2k_1} \bmod d_{k_3})(t_{k_3}M'_{k_1j_1}M'_{k_1j_2})$$

If n is even, let $\alpha_{k'}$ be the only generator of $\mathcal{S}_{n,2n-1}$ that has an infinite order. Then by Lemma 5.3.4, the RHS is equal to

$$\begin{aligned}
&\sum_{k_1=1}^{r}(c_{k_3}M_{i_1k_1}M_{i_2k_1} \bmod d_{k_3})(t_{k_3}M'_{k_1j_1}M'_{k_1j_2})\\
=&\sum_{k_1=1}^{r}(c_{k'}M_{i_1k_1}M_{i_2k_1})(t_{k'}M'_{k_1j_1}M'_{k_1j_2})\\
=&\sum_{k_1=1}^{r}(c_{k'}t_{k'})M_{i_1k_1}M_{i_2k_1}M'_{k_1j_1}M'_{k_1j_2}
\end{aligned}$$

But Lemma 5.3.4 also shows that $c_{k'}t_{k'} = 2$. This gives the desired equality.
If n is odd, then the LHS is obviously equal to zero, and Lemma 5.3.5 shows that the RHS is equal to zero, too.

- Case 2: $\varphi_{n,r}(\hat{x})_i = [x_{i_1}, x_{i_2}]$, $\varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{j_3}$.
  On one hand, we have

$$\begin{aligned}\Phi_{n,r}(MM')_{ij} =& c_{j_3}(MM')_{i_1j_1}(MM')_{i_2j_1}\\ =& c_{j_3}\left(\sum_{k_1=1}^{r} M_{i_1k_1}M'_{k_1j_1}\right)\left(\sum_{k_2=1}^{r} M_{i_2k_2}M'_{k_2j_1}\right) && \mod d_{j_3}\\ =& c_{j_3}(\sum_{1\le k_1,k_2\le r} M_{i_1k_1}M_{i_2k_2}M'_{k_1j_1}M'_{k_2j_1}) && \mod d_{j_3}\\ =& c_{j_3}\sum_{1\le k_1<k_2\le r} M_{i_1k_1}M_{i_2k_2}M'_{k_1j_1}M'_{k_2j_1}\\ &+ c_{j_3}\sum_{1\le k_1<k_2\le r} M_{i_1k_2}M_{i_2k_1}M'_{k_1j_1}M'_{k_2j_1}\\ &+ c_{j_3}\sum_{k_1=1}^{r} M_{i_1k_1}M_{i_2k_1}(M'_{k_1j_1})^2 && \mod d_{j_3}\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\Phi_{n,r}(M)*\Phi_{n,r}(M'))_{ij}\\ =&\sum_{1\le k_1<k_2\le r}(M_{i_1k_1}M_{i_2k_2}+(-1)^nM_{i_1k_2}M_{i_2k_1})(c_{j_3}M'_{k_1j_1}M'_{k_2j_1} \mod d_{j_3})\\ &+\sum_{k_1=1}^{r}\sum_{k_3=1}^{m}(c_{k_3}M_{i_1k_1}M_{i_2k_1} \mod d_{k_3})\big(h_{k_3}c_{j_3}\binom{M'_{k_1j_1}}{2}+\theta_{k_3=j_3}M'_{k_1j_1} \mod d_{j_3}\big) && \mod d_{j_3}\\ =& c_{j_3}\sum_{1\le k_1<k_2\le r} M_{i_1k_1}M_{i_2k_2}M'_{k_1j_1}M'_{k_2j_1}+(-1)^nM_{i_1k_2}M_{i_2k_1}M'_{k_1j_1}M'_{k_2j_1}\\ &+\sum_{k_1=1}^{r}\sum_{k_3=1}^{m}(c_{k_3}M_{i_1k_1}M_{i_2k_1} \mod d_{k_3})(h_{k_3}c_{j_3}\binom{M'_{k_1j_1}}{2}+\theta_{k_3=j_3}M'_{k_1j_1}) && \mod d_{j_3}\end{aligned}$$

If n is even, we are left to check

$$\begin{aligned}& c_{j_3}\sum_{1\le k_1<k_2\le r} M_{i_1k_1}M_{i_2k_1}(M'_{k_1j_1})^2 && \mod d_{j_3}\\ =&\sum_{k_1=1}^{r}\sum_{k_3=1}^{m}(c_{k_3}M_{i_1k_1}M_{i_2k_1} \mod d_{k_3})(h_{k_3}c_{j_3}\binom{M'_{k_1j_1}}{2}+\theta_{k_3=j_3}M'_{k_1j_1}) && \mod d_{j_3}\end{aligned}$$

Let $\alpha_{k'}$ be the only generator of $\mathcal{S}_{n,2n-1}$ that has an infinite order. By Lemma 5.3.4, the RHS is equal to

$$\begin{aligned}
&\sum_{k_1=1}^{r}\sum_{k_3=1}^{m}(c_{k_3}M_{i_1k_1}M_{i_2k_1} \bmod d_{k_3})(h_{k_3}c_{j_3}\binom{M'_{k_1j_1}}{2}+\theta_{k_3=j_3}M'_{k_1j_1}) && \bmod d_{j_3}\\
=&c_{j_3}\sum_{k_1=1}^{r}M_{i_1k_1}M_{i_2k_1}h_{k'}c_{k'}\binom{M'_{k_1j_1}}{2}+c_{j_3}\sum_{k_1=1}^{r}M_{i_1k_1}M_{i_2k_1}M'_{k_1j_1} && \bmod d_{j_3}\\
=&c_{j_3}\sum_{k_1=1}^{r}M_{i_1k_1}M_{i_2k_1}M'_{k_1j_1}(M'_{k_1j_1}-1)+c_{j_3}\sum_{k_1=1}^{r}M_{i_1k_1}M_{i_2k_1}M'_{k_1j_1} && \bmod d_{j_3}
\end{aligned}$$

which is equal to the LHS.
If n is odd, then by Lemma 5.3.5,

$$\begin{aligned}
&(\Phi_{n,r}(M)*\Phi_{n,r}(M'))_{ij}\\
=&c_{j_3}\left(\sum_{1\le k_1<k_2\le r}M_{i_1k_1}M_{i_2k_2}M'_{k_1j_1}M'_{k_2j_1}-M_{i_1k_2}M_{i_2k_1}M'_{k_1j_1}M'_{k_2j_1} \bmod 2\right)\\
+&c_{j_3}\left(\sum_{k_1=1}^{r}M_{i_1k_1}M_{i_2k_1}M'_{k_1j_1} \bmod 2\right)\\
=&c_{j_3}\left(\sum_{1\le k_1<k_2\le r}M_{i_1k_1}M_{i_2k_2}M'_{k_1j_1}M'_{k_2j_1}+M_{i_1k_2}M_{i_2k_1}M'_{k_1j_1}M'_{k_2j_1} \bmod 2\right)\\
+&c_{j_3}\left(\sum_{k_1=1}^{r}M_{i_1k_1}M_{i_2k_1}(M'_{k_1j_1})^2 \bmod 2\right)\\
=&\Phi_{n,r}(MM')_{ij}
\end{aligned}$$

- Case 3: $\varphi_{n,r}(\hat{x})_i = x_{i_1}\circ\alpha_{i_3}$, $\varphi_{n,r}(\hat{x})_j=[x_{j_1},x_{j_2}]$.
  On one hand, we have

$$\begin{aligned}
\Phi_{n,r}(MM')_{ij} =&h_{i_3}(MM')_{i_1j_1}(MM')_{i_1j_2}\\
=&h_{i_3}\sum_{k_1=1}^{r}M_{i_1k_1}M'_{k_1j_1}\sum_{k_2=1}^{r}M_{i_1k_2}M'_{k_2j_2}\\
=&h_{i_3}\sum_{1\le k_1<k_2\le r}M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_2}\\
+&h_{i_3}\sum_{1\le k_1<k_2\le r}M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_2}\\
+&h_{i_3}\sum_{k_1=1}^{r}(M_{i_1k_1})^2M'_{k_1j_1}M'_{k_1j_2}
\end{aligned}$$

On the other hand,

$$(\Phi_{n,r}(M) * \Phi_{n,r}(M'))_{ij} = \sum_{1\le k_1<k_2\le r} (h_{i_3} M_{i_1k_1} M_{i_1k_2})(M'_{k_1j_1}M'_{k_2j_2} + (-1)^n M'_{k_1j_2}M'_{k_2j_1})$$
$$+ \sum_{k_1=1}^{r}\sum_{k_3=1}^{m} (h_{i_3}c_{k_3}\binom{M_{i_1k_1}}{2} + \theta_{i_3=k_3}M_{i_1k_1})(h_{k_3}M'_{k_1j_1}M'_{k_1j_2})$$

If n is even, let $\alpha_{k'}$ be the only generator of $\mathcal{S}_{n,2n-1}$ that has an infinite order. Then we have two cases. If $i_3 = k'$, then

$$\begin{aligned}(\Phi_{n,r}(M) * \Phi_{n,r}(M'))_{ij} =& h_{k'} \sum_{1\le k_1<k_2\le r} M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_2}\\ &+ h_{k'} \sum_{1\le k_1<k_2\le r} M_{i_1k_1}M_{i_1k_2}M'_{k_1j_2}M'_{k_2j_1}\\ &+ h_{k'} \sum_{k_1=1}^{r} (h_{k'}c_{k'}\binom{M_{i_1k_1}}{2} + M_{i_1k_1})M'_{k_1j_1}M'_{k_1j_2}\end{aligned}$$

But $h_{k'}c_{k'} = 2$ by Lemma 5.3.4, so $(\Phi_{n,r}(M) * \Phi_{n,r}(M'))_{ij} = \Phi_{n,r}(MM')_{ij}$. If $i_3 \neq k'$, then $h_{i_3} = 0$, so $(\Phi_{n,r}(M) * \Phi_{n,r}(M'))_{ij} = 0 = \Phi_{n,r}(MM')_{ij}$. This fact also applies to the case when n is odd, as described by Lemma 5.3.5.

- Case 4: $\varphi_{n,r}(\hat{x})_i = x_{i_1} \circ \alpha_{i_3}$, $\varphi_{n,r}(\hat{x})_j = x_{j_1} \circ \alpha_{j_3}$. We have

$$\Phi_{n,r}(MM')_{ij} = t_{i_3}c_{j_3}\binom{(MM')_{ij}}{2} + \theta_{i_3=j_3}(MM')_{i_1j_1} \qquad \mod d_{j_3}$$

and

$$\begin{aligned}&(\Phi_{n,r}(M) * \Phi_{n,r}(M')_{ij}\\ =& \sum_{1\le k_1<k_2\le r} (t_{i_3}M_{i_1k_1}M_{i_1k_2})(c_{j_3}M'_{k_1j_1}M'_{k_2j_1} \mod d_{j_3})\\ +& \sum_{k_1=1}^{r}\sum_{k_3=1}^{m} (t_{i_3}c_{k_3}\binom{M_{i_1k_1}}{2} + \theta_{i_3=k_3}M_{i_1k_1} \mod d_{k_3})\\ &(t_{k_3}c_{j_3}\binom{M'_{k_1j_1}}{2} + \theta_{k_3=j_3}M'_{k_1j_1} \mod d_{j_3}) \qquad \mod d_{j_3}.\end{aligned}$$

We further break down the situation into the following six cases.

  - Case a: n is even, $i_3 = i'$, $j_3 = i_3$.

On one hand, we have

$$\begin{aligned}&\Phi_{n,r}(MM')_{ij}\\ =&t_{i'}c_{i'}\binom{(MM')_{i_1j_1}}{2}+(MM')_{i_1j_1} && \mod d_{i'}\\ =&2\cdot\frac{1}{2}((MM')^2_{i_1j_1}-(MM')_{i_1j_1})+(MM')_{i_1j_1}\\ =&(MM')^2_{i_1j_1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\Phi_{n,r}(M)*\Phi_{n,r}(M')_{ij}\\ =&t_{i'}c_{i'}\sum_{1\le k_1<k_2\le r}M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_1}\\ &+\sum_{k_1=1}^{r}(t_{i'}c_{i'}\binom{M_{i_1k_1}}{2}+M_{i_1k_1})(t_{i'}c_{i'}\binom{M'_{k_1j_1}}{2}+M'_{k_1j_1}) && \mod d_{i'}\\ =&2\sum_{1\le k_1<k_2\le r}M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_1}+\sum_{k_1=1}^{r}(M_{i_1k_1})^2(M'_{k_1j_1})^2\\ =&(MM')^2_{i_1j_1}.\end{aligned}$$

Thus $\Phi_{n,r}(MM')_{ij}=(\Phi_{n,r}(M)*\Phi_{n,r}(M'))_{ij}$.

– Case b: n is even, $i_3=i'$, $j_3\neq i_3$.
On one hand, we have

$$\begin{aligned}&\Phi_{n,r}(MM')_{ij}\\ =&t_{i'}c_{j_3}\binom{(MM')_{i_1j_1}}{2} && \mod d_{j_3}\\ =&t_{i'}c_{j_3}\frac{1}{2}((\sum_{k_1=1}^{r}M_{i_1k_1}M'_{k_1j_1})^2-\sum_{k_1=1}^{r}M_{i_1k_1}M'_{k_1j_1}) && \mod d_{j_3}\\ =&t_{i'}c_{j_3}\frac{1}{2}(2\sum_{1\le k_1,k_2\le r}M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_1}\\ &+\sum_{k_1=1}^{r}(M_{i_1k_1})^2(M'_{k_1j_1})^2-\sum_{k_1=1}^{r}M_{i_1k_1}M'_{k_1j_1}2) && \mod d_{j_3}\\ =&t_{i'}c_{j_3}\sum_{1\le k_1,k_2\le r}M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_1}\\ &+t_{i'}c_{j_3}\frac{1}{2}\sum_{k_1=1}^{r}(M_{i_1k_1}M'_{k_1j_1})^2-M_{i_1k_1}M'_{k_1j_1} && \mod d_{j_3}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\Phi_{n,r}(M) * \Phi_{n,r}(M')_{ij} \\
=&t_{i'}c_{j_3} \sum_{1\leq k_1<k_2\leq r} M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_1} \\
&+ \sum_{k_1=1}^{r}(t_{i'}c_{i'}\binom{M_{i_1k_1}}{2} + M_{i_1k_1})(t_{i'}c_{j_3}\binom{M'_{k_1j_1}}{2}) \\
&+ \sum_{k_1=1}^{r}(t_{i'}c_{j_3}\binom{M_{i_1k_1}}{2})M'_{k_1j_1} && \mod d_{j_3} \\
=&t_{i'}c_{j_3} \sum_{1\leq k_1<k_2\leq r} M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_1} \\
&+t_{i'}c_{j_3}\frac{1}{2}\sum_{k_1=1}^{r}(M_{i_1k_1})^2((M'_{k_1j_1})^2 - M'_{k_1j_1}) \\
&+t_{i'}c_{j_3}\frac{1}{2}\sum_{k_1=1}^{r}(M^2_{i_1k_1} - M_{i_1k_1})M'_{k_1j_1} && \mod d_{j_3} \\
=&t_{i'}c_{j_3} \sum_{1\leq k_1<k_2\leq r} M_{i_1k_1}M_{i_1k_2}M'_{k_1j_1}M'_{k_2j_1} \\
&+t_{i'}c_{j_3}\frac{1}{2}\sum_{k_1=1}^{r}(M_{i_1k_1}M'_{k_1j_1})^2 - M_{i_1k_1}M'_{k_1j_1} && \mod d_{j_3}
\end{aligned}$$

Thus $\Phi_{n,r}(MM')_{ij} = (\Phi_{n,r}(M) * \Phi_{n,r}(M'))_{ij}$.

- Case c: n is even, $i_3 \neq i'$, $j_3 = i_3$.
  In this case, $h_{i_3} = 0$, so we simply have

$$\begin{aligned}
&\Phi_{n,r}(MM')_{ij} \\
=&(MM')_{i_1j_1} && \mod d_{j_3} \\
=&\sum_{k_1=1}^{r} M_{i_1k_1}M'_{k_1j_1} && \mod d_{j_3} \\
=&(\Phi_{n,r}(M) * \Phi_{n,r}(M')_{ij} && \mod d_{j_3}
\end{aligned}$$

- Case d: n is even, $i_3 \neq i'$, $j_3 \neq i_3$.
  In this case,

$$\Phi_{n,r}(MM')_{ij} = 0 = (\Phi_{n,r}(M) * \Phi_{n,r}(M'))_{ij}.$$

- Case e: n is odd, $i_3 = j_3$.
  The same as case c.

– Case f: n is odd, $i_3 \neq j_3$.
  The same as case d.

This completes the proof. □

C Values of Important Constants in $\mathcal{S}_{n,2n-1}$

Let $n \geq 2$, $r \geq 1$. Suppose $\mathcal{S}_{n,2n-1} \cong \bigoplus_{k=1}^{m} \mathbb{Z}_{d_k}\langle\alpha_k\rangle$, and for $1 \leq k \leq m$, $H_0(\alpha_k) = t_k$, $[\iota_n, \iota_n] = \sum_{k=1}^{m} c_k\alpha_k$. The table below shows the values of the constants for $2 \leq n \leq 11$.

n	d_k	t_k	c_k
2	0	2	1
3	2	0	0
4	$0, 3, 4$	$1, 0, 0$	$2, 0, -1$
5	2	0	1
6	0	2	1
7	2	0	0
8	$0, 3, 5, 8$	$1, 0, 0, 0$	$2, 0, 0, -1$
9	$2, 2, 2$	$0, 0, 0$	$1, 1, 1$
10	$0, 2, 2, 2$	$2, 0, 0, 0$	$1, 0, 0, 0$
11	$2, 2, 3$	$0, 0, 0$	$0, 1, 0$

For more details about the homotopy groups of spheres, see Toda [5].

References

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