Recursion in Topological Invariants of Twist and Rational Knot Exteriors

Christian Gorski
University of Notre Dame

Follow this and additional works at: https://scholar.rose-hulman.edu/rhumj

Recommended Citation
Available at: https://scholar.rose-hulman.edu/rhumj/vol17/iss2/4
Recursion in Topological Invariants of Twist and Rational Knot Exteriors

Christian Gorski\textsuperscript{a}

Volume 17, No. 2, Fall 2016

\textsuperscript{a}University of Notre Dame
Recursion in Topological Invariants of Twist and Rational Knot Exteriors

Christian Gorski

Abstract. We calculate explicit presentations for the knot group of twist knots, utilizing a recursive relation derived from the Wirtinger presentation. We also present a method for building rational knot exteriors by attaching two-handles to \((S^2 - ::) \times I\), the four-punctured sphere times an interval, and give an explicit algorithm for calculating the attaching curve for the two-handles in terms of the basis of the Kauffman bracket skein module of \((S^2 - ::) \times I\).

Acknowledgements: This paper was produced as part of the Research Experiences for Undergraduates program at Louisiana State University. The author would like to thank Professor Neal Stoltzfus for research guidance, and the basic idea behind constructing a knot exterior from \((S^2 - ::) \times I\); he would also like to thank his peers Larissa Sambel, Rose Kaplan-Kelly, and Dean Menezes for useful discussions and assistance in understanding material.
1 Introduction

Mathematical knots are essentially the same as physical knots, except that the ends of the “rope” are fused together. A fundamental problem in knot theory is determining whether two knots are equivalent, in the sense that the first can be manipulated in space to match the other, without cutting or gluing any rope in the process.

In order to help attack this problem, much of knot theory has been devoted to producing knot invariants—objects associated to a knot that only depend on its equivalence class. One such invariant is the knot exterior (or knot complement); roughly speaking, this is the shape of the space around the knot. We can then get more knot invariants by taking topological invariants of the knot complement. For example, the knot group (discussed in Section 1) is just the fundamental group of a knot complement. The Kauffman bracket skein module (discussed in Section 2) is a (perhaps less familiar) topological invariant that is designed to be applied to knot complements.

Ideally, these knot invariants should somehow express more information about a knot than the fact that it is not equivalent to some other knots. Specifically, it would be desirable if there were patterns in some knot invariants which reflected patterns in the construction of some families of knots. For example, the family of twist knots can be constructed recursively by successively adding more twists to a specific part of a specific knot diagram. Is this recursive construction reflected in, say, the fundamental groups of the twist knots? If we find patterns in knot invariants which correspond to patterns in the construction of knots, this may aid the computation of invariants, and, perhaps more importantly, help illuminate the precise relationship between the knot invariants and the knots themselves.

In Section 1 of this paper, we first make precise the definitions of knot and knot complement. We then define the family of twist knots. We discuss a method of producing a presentation (called the Wirtinger presentation) of the knot group of a knot and demonstrate that there is a (in some sense) recursively-defined presentation for the knot group of each twist knot. We also comment on another known presentation for the knot groups of twist knots which differs from the one presented here. In Section 2, we define the related concept of links and define the Kauffman bracket skein module or KBSM. We discuss some facts about the family of rational knots and then present a method of constructing an arbitrary rational knot exterior by attaching 2-handles to the “thickened four-punctured sphere.” We then proceed to give a (recursive) method for calculating an algebraic representation of the attaching curves in this construction, as a first step in relating the KBSM of rational knot exteriors to the better understood KBSM of the thickened four-punctured sphere.

2 The Fundamental Group of a Twist Knot Exterior

In this section, we examine the fundamental groups of twist knots. Before we define the twist knots, let us define exactly what we mean by “knot” and “knot exterior.”

Definition 2.1. Consider embeddings $h : S^1 \to S^3$ of the circle (1-sphere) into the 3-sphere.
Two such embeddings $g, h$ are called **ambient isotopic** if there is an isotopy $s_t : S^3 \to S^3, 0 \leq t \leq 1$ of $S^3$ such that $s_0$ is the identity on $S^3$ and $s_1 \circ g = h$. The property of being ambient isotopic is an equivalence relation. An **oriented knot** is an equivalence class of embeddings of $S^1$ into $S^3$ under this equivalence relation. An oriented knot is **tame** if it has a representative embedding which is polygonal, that is, whose image is a union of finitely many straight line segments in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. An **unoriented knot** is an equivalence class of oriented knots under the relation which identifies two knots which have representative embeddings whose images are identical. Usually, we will use the word “knot” to refer to a tame unoriented knot.

**Definition 2.2.** Let $K$ be a knot, $h : S^1 \to S^3$ a representative embedding, and $T(K)$ a tubular neighborhood (that is, a neighborhood homeomorphic to the solid torus $S^1 \times B^2$) of the image $h(S^1)$. The **knot complement** or **knot exterior** is the topological space $S^3 - T(K)$, and is sometimes denoted by $S^3 - K$. The **knot group** of $K$ is the fundamental group of the knot exterior, and is denoted $\pi_1(S^3 - T(K))$, $\pi_1(S^3 - K)$ or simply $\pi_1(K)$.

The twist knots are a family of knots characterized by a single “twist region” and two “clasps.” We will denote by $K_n$ the twist knot with $n$ twists. For example, the figure-eight knot is $K_2$, the twist knot with two twists. A diagram for $K_n$ is shown in Figure 1. In Figure 2 is the same knot, presented in “braid form.” The “braid form” of the knot diagram happens to be particularly useful for examining twist knots (and, in fact, two-bridge knots) systematically.

The idea is behind the braid form is that the knot can be thought of as a closed three-strand braid; the top strand on the left connects to the top strand on the right, and on both
Figure 3: To the left, we see the trefoil with its unbroken arcs labeled $a, b, c$. Moreover, we see the element of the fundamental group represented by $\gamma_c$. To the right, we see a single crossing in a diagram. The Wirtinger relation is obtained by composing the generators associated with the arcs in the crossing in counterclockwise order. An equivalent (via conjugation) relation could be found by moving clockwise (the inverse of the relation), or by starting with a different element (taking any cyclic permutation of the word).

sides the bottom two strands are connected. The twist knot with $n$ twists is then obtained from a braid which braids the top two strands $n$ times (positively), the bottom two 1 time (negatively), and then the top two 1 time (positively).

One of the most direct and intuitive ways to calculate the fundamental group of a knot complement is to use the so-called Wirtinger presentation. (For another treatment of the Wirtinger presentation, see Fox [2].) The construction is as follows. Take a reduced planar diagram for the knot. We assume the basepoint of the fundamental group lies somewhere high above the plane of the diagram. Traditionally, when one strand of the knot crosses under another, the undercrossing appears in the diagram to be broken into two arcs, while the overcrossing appears unbroken. To each arc $\alpha$ which appears as unbroken in the diagram, we assign a transverse orientation; we then associate to that arc a generator $\gamma_{\alpha}$ of our presentation. This generator will correspond to the element of the fundamental group which loops under the arc traveling in the direction of the assigned orientation. At each crossing, we have one relation, the four-letter word which represents looping under each strand of the crossing once in a counterclockwise (or clockwise) fashion. See Figure 3. (We shall refer to this relation as “the Wirtinger relation.”) Notice that in the Wirtinger presentation of any knot group, any one of the relations can be omitted without affected the group presented, since the product of all relations is trivial [2].

In the case of twist knots, there is a natural way to index the generators by integers which makes the presentation of the group easy to simplify and relate to the number of twists in the twist region. See Figure 4.

We begin by labeling the arcs in the top two strands of the braid, starting from the left. As we label these arcs, we assume that the transverse orientation is upward. In the first (that is, leftmost) twist of the twist region, the second strand crosses under the first; we label the generator associated to the leftmost arc of the second strand $\gamma_0$ and that associated to the leftmost arc of the first strand $\gamma_1$. We see that the arc associated with $\gamma_1$ is the overcrossing in the first twist of the twist region. Likewise, for the rest of the twist region, we denote the
generator associated to the overcrossing of the \(k\)-th twist region by \(\gamma_k\). At the \(n\)-th crossing, we denote the generator associated to the arc coming out from under the \(n\)-th overcrossing by \(\gamma_{n+1}\). At this point, we have labeled all the generators for the Wirtinger presentation of \(\pi_1(K_n)\). In Figure 4, we have written some generators more than once, in different parts of the diagram. This is to make it easier to compute the Wirtinger relation at crossings, and should not be interpreted to be the introduction of a new generator. Also notice that as we move \(\gamma_0\) and \(\gamma_{n+1}\) along their respective arcs, the transverse orientation may switch from pointing up to pointing down. When the orientation switches to point down, we still write an upward arrow, but we label it with the inverse of the original generator. Once again, this is not the introduction of a new generator, but a convenience.

We notice that at each crossing in the twist region, the Wirtinger relation takes on a particular form: for \(1 \leq k \leq n\),
\[
\gamma_{k+1} = \gamma_k^{-1} \gamma_{k-1} \gamma_k
\]
(see Figure 5). The remaining two crossings give the relations \(\gamma_0^{-1} \gamma_n = \gamma_{n+1}^{-1} \gamma_0^{-1}\) and \(\gamma_0^{-1} \gamma_{n+1} = \gamma_{n+1} \gamma_1\). Of course, as with any Wirtinger presentation, one of these may be discarded.

We can use the recursive relation (1) to write each generator \(\gamma_k\) as a word in \(\gamma_0\) and \(\gamma_1\). Since we only need one of the relations from the last two crossings, we can then simplify our presentation to one with two generators and one relation:
\[
\pi_1(K_n) \cong \langle \gamma_0, \gamma_1 \mid \gamma_{n+1}^{-1} \gamma_n = \gamma_{n+1} \gamma_1 \rangle
\]
where \(\gamma_{n+1}\) is considered as a word in \(\gamma_0\) and \(\gamma_1\), defined recursively by (1).

Notice that, considered as words in \(\gamma_0\) and \(\gamma_1\), the \(\gamma_k\) do not depend on the number of twists in a specific knot. Thus, by studying the behavior of the sequence \((\gamma_k)_{k=0}^\infty\) defined by
(1), we may gain some information about the family of twist knots as a whole. When we expand the recurrence relation, many terms cancel and a clear pattern emerges; the following expressions are equal (so long as \( k \) is large enough that they are defined):

\[
\gamma_k = \\
\gamma_{k-1} \gamma_{k-2} \gamma_{k-3} \gamma_{k-4} \gamma_{k-5} \\
\vdots
\]

In particular, we can use the above pattern to get a pattern in the expression of each \( \gamma_k \) as a word in \( \gamma_0 \) and \( \gamma_1 \). We could characterize this pattern in many ways; we prove one characterization of it now.

**Proposition 2.3.** For \( k \geq 0 \), we have

\[
\gamma_{2k} = (\gamma_0 \gamma_1)^{-k} \gamma_0 (\gamma_0 \gamma_1)^k
\]

and

\[
\gamma_{2k+1} = (\gamma_0 \gamma_1)^{-k} \gamma_1 (\gamma_0 \gamma_1)^k.
\]

**Proof.** We first prove the following recurrence relation: for \( k \geq 0 \),

\[
\gamma_{k+2} = (\gamma_0 \gamma_1)^{-1} \gamma_k (\gamma_0 \gamma_1).
\]

We proceed by induction on \( k \). By direct calculation (from (1)),

\[
\gamma_2 = \gamma_1^{-1} \gamma_0 \gamma_1 = \gamma_1^{-1} \gamma_0^{-1} \gamma_0 \gamma_0 \gamma_1 = (\gamma_0 \gamma_1)^{-1} \gamma_0 (\gamma_0 \gamma_1),
\]

so the relation is satisfied for \( k = 0 \). For \( k = 1 \), also by direct calculation,

\[
\gamma_3 = \gamma_2^{-1} \gamma_1 \gamma_2 = (\gamma_1^{-1} \gamma_0 \gamma_1)^{-1} \gamma_1 (\gamma_1^{-1} \gamma_0 \gamma_1) = \gamma_1^{-1} \gamma_0^{-1} \gamma_1 \gamma_1^{-1} \gamma_0 \gamma_1 = (\gamma_0 \gamma_1)^{-1} \gamma_1 (\gamma_0 \gamma_1).
\]

Now, for some \( k \geq 2 \), assume the relation holds for \( k - 1 \) and \( k - 2 \). Then by (1)

\[
\gamma_{k+2} = \gamma_{k+1}^{-1} \gamma_k \gamma_{k+1}
\]

which, by the inductive hypothesis, is equal to

\[
[(\gamma_0 \gamma_1)^{-1} \gamma_{k-1} (\gamma_0 \gamma_1)]^{-1} [(\gamma_0 \gamma_1)^{-1} \gamma_{k-2} (\gamma_0 \gamma_1)] [(\gamma_0 \gamma_1)^{-1} \gamma_{k-1} (\gamma_0 \gamma_1)] = (\gamma_0^{-1} \gamma_1)^{-1} \gamma_{k-1} [ (\gamma_0 \gamma_1)(\gamma_0 \gamma_1)^{-1} ] \gamma_{k-2} [(\gamma_0 \gamma_1)(\gamma_0 \gamma_1)^{-1}]^{-1} \gamma_{k-1} (\gamma_0 \gamma_1) = (\gamma_0 \gamma_1)^{-1} (\gamma_{k-1} \gamma_{k-2} \gamma_{k-1}) (\gamma_0 \gamma_1)
\]
which, by (1), is equal to $(\gamma_0\gamma_1)^{-1}\gamma_k(\gamma_0\gamma_1)$.

Thus, we have proved our first desired relation.

From here, it is easy to prove the equality given in the proposition. Clearly, the base case $k = 0$ holds, since $\gamma_0 = (\gamma_0\gamma_1)^{-0}\gamma_0(\gamma_0\gamma_1)^0$ and $\gamma_1 = (\gamma_0\gamma_1)^{-0}\gamma_1(\gamma_0\gamma_1)$. Assuming that the equality holds for some $k \geq 0$, we use the recurrence we proved above to see that

$$\gamma_{2(k+1)} = \gamma_{2k+2} = (\gamma_0\gamma_1)^{-1}\gamma_{2k}(\gamma_0\gamma_1)$$

$$= (\gamma_0\gamma_1)^{-1}(\gamma_0\gamma_1)^{-k}\gamma_0(\gamma_0\gamma_1)^k(\gamma_0\gamma_1) = (\gamma_0\gamma_1)^{-(k+1)}\gamma_0(\gamma_0\gamma_1)^{k+1}$$

and similarly,

$$\gamma_{2(k+1)+1} = \gamma_{(2k+1)+2} = (\gamma_0\gamma_1)^{-1}\gamma_{2k+1}(\gamma_0\gamma_1)$$

$$= (\gamma_0\gamma_1)^{-1}(\gamma_0\gamma_1)^{-k}\gamma_1(\gamma_0\gamma_1)^k(\gamma_0\gamma_1) = (\gamma_0\gamma_1)^{-(k+1)}\gamma_1(\gamma_0\gamma_1)^{k+1}.$$

Thus, we obtain an even more explicit form for the knot group of a twist knot:

**Corollary 2.4.** For $k \in \mathbb{Z}, k \geq 1$:

$$\pi_1(K_{2k-1}) \cong \langle \gamma_0, \gamma_1 | \gamma_0^{-1}(\gamma_0\gamma_1)^{-k}\gamma_0(\gamma_0\gamma_1)^k = (\gamma_0\gamma_1)^{-k}\gamma_0(\gamma_0\gamma_1)^k\gamma_1 \rangle$$

and

$$\pi_1(K_{2k}) \cong \langle \gamma_0, \gamma_1 | \gamma_0^{-1}(\gamma_0\gamma_1)^{-k}\gamma_1(\gamma_0\gamma_1)^k = (\gamma_0\gamma_1)^{-k}\gamma_1(\gamma_0\gamma_1)^k\gamma_1 \rangle.$$

**Remark 2.5.** We can get new presentations by substituting $y = \gamma_0\gamma_1$ and eliminating whichever original generator is most convenient. In particular,

$$\pi_1(K_{2k-1}) \cong \langle \gamma_0, y | \gamma_0 y^{k-1}\gamma_0 y y^{-k}\gamma_0^{-1}y^{-k-1}y^{-k} = 1 \rangle$$

and

$$\pi_1(K_{2k}) \cong \langle y, \gamma_1 | y^{-(k+1)}\gamma_1 y y^{-k}\gamma_1^{-1}y^{-k-1}y^{-k} = 1 \rangle.$$

Note that, geometrically, $\gamma_0\gamma_1$ is a loop around the top two strands of the braid.

The twist knots are part of a larger family of knots known as the rational knots. Each rational knot has a rational number $p/q$ associated to it. (Technically, there are several rational numbers associated to it, but we only need one to determine the knot uniquely.) Conventions vary from author to author as to whether the fraction is $p/q$ or $q/p$; we use the same convention used in Kauffman’s paper [4]. Under this convention, one of the rational numbers associated to a twist knot with $n$ twists is $\frac{2n+1}{2}$.

A theorem of Schubert gives a straightforward formula for the fundamental group of a rational knot [5].
Theorem 2.6. (Schubert) Let $K$ be a rational knot with fraction $p/q$. For each $i = 1, 2, ..., p - 1$, set
\[ \epsilon_i := (-1)^{\lfloor \frac{iq}{p} \rfloor}, \]
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. Let $w$ be the word in $a$ and $b$ given by
\[ w = b^{\epsilon_1}a^{\epsilon_2} \cdots a^{\epsilon_{p-1}}. \]
Then $\pi_1(K)$ has a presentation
\[ \langle a, b|aw = wb \rangle. \]

At first glance, the Schubert presentation bears great resemblance to the first we wrote out (in its first form, when $\gamma_{n+1}$ was not written out explicitly). If we set $\gamma_0 = a^{-1}$ and $\gamma_1 = b$, then the two presentations for the figure eight in particular look very similar; $\gamma_{n+1}$ differs from $w$ only in that $\gamma_{n+1}$ has an extra trailing $b$. (And, in fact, eliminating the final $b = \gamma_1$ from $\gamma_{n+1}$ always creates an equivalent presentation, as is seen from a little bit of simple manipulation.) However, this pattern does not hold in general. Although the two presentations present the same group, we could find no straightforward way to show this except by referring to the fact that they present the fundamental group of the same knot. Hopefully, the presentation just explicated may help further illuminate relationships among the twist knot family.

3 Skein Modules of Rational Knot Exteriors

Now we examine another knot invariant known as the Kauffman bracket skein module. In order to define this, we must introduce the related concept of links in a 3-manifold. One may think of links as collections of knots which may or may not be linked together in some way.

Definition 3.1. Let $M$ be a 3-manifold. Consider embeddings $h : (S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1) \times D^2 \to M$ of $k$ disjoint solid tori into the interior of $M$. Two such embeddings $g, h$ are called ambient isotopic if there is an isotopy $s_t : M \to M, 0 \leq t \leq 1$ of $M$ such that $s_0$ is the identity on $M$ and $s_1 \circ g = h$. We identify two such embeddings if they are ambient isotopic or if their images are identical. An equivalence class under this identification is called a $k$-component unoriented framed link in $M$.

Let $M$ be a 3-manifold. Denote by $L_{fr}$ the set of framed links in $M$, including an empty link. Let $R$ be a ring with an invertible element $A$. Consider the $R$-module on $L_{fr}$ (that is, the set of finite linear combinations of links in $L_{fr}$ whose coefficients are in $R$). If we take the quotient of this module by the relations
\[ \bigotimes - A \bigotimes - A^{-1} = 0, \]
\[ \bigcirc + A^2 + A^{-2} = 0, \]
then the result is called the **Kaufmann bracket skein module** (KBSM) on \( M \), and is denoted by \( \mathcal{S}_K(M) \). (The diagrams in the relations above represent links with representative embeddings which are identical except in the neighborhood which the diagram shows.)

In the case that \( M \) has the structure \( F \times I \), where \( F \) is a 2-manifold and \( I = [0, 1] \) is the unit interval, then the skein module admits a multiplicative structure, making it into a (generally non-commutative) algebra. The product of two links is given by placing the first link on top of the second. (See Figure 6.) Explicitly, if \( L_1 \) and \( L_2 \) are \( k \)-component and \( l \)-component links respectively with representative embeddings \( h_1 \) and \( h_2 \) respectively, and \( i_-, i_+ : F \times I \to F \times I \) map \( F \times I \) linearly onto \( F \times [0, \frac{1}{2}] \) and \( F \times [\frac{1}{2}, 1] \) respectively, the product \( L_1 \cdot L_2 \) is the \((k + l)\)-component link with representative embedding \( h = (i_+ \circ h_1) \sqcup (i_- \circ h_2) \). We then extend this product linearly to the whole module.

The skein module of \((S^2 - ::) \times I\), the four-punctured sphere times an interval, was studied by Bullock and Przytycki [1]. We wish to relate this module to the module of any rational knot. First, we show that any rational knot can be obtained from \((S^2 - ::) \times I\) by attaching 2-handles.

Before that, however, we review some necessary background about rational knots. We define a rational knot to be a knot which admits a planar diagram with only two overcrossings (a “two-bridge knot”). As alluded to before, each rational knot has rational numbers associated to it; each rational number corresponds to only one rational knot, but a rational knot has more than one rational number associated to it. We are largely unconcerned with the motivation behind this association for our construction. We will mostly make use of the fact that every rational knot has a braid form; that is, a diagram that consists of a three-strand braid, closed (as before) by fusing together the left and right ends of the top strand and fusing together the ends of the bottom two strands on the left and right sides [4]. In particular, there is a reduced diagram in braid form which is alternating; in such a diagram, one pair of strands braids only negatively, and the other braids only positively. We can determine the reduced braid form of a rational knot from its fraction as follows: use Euclid’s lemma to express the fraction as a continued fraction \([a_1, a_2, ..., a_n] := a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}\), where \( n \) is odd, none of the \( a_i \) have opposite sign, and none are 0, except possibly \( a_1 \). If \( \sigma_1 \) is a...
positive twist of the top two strands and \( \sigma_2 \) is a positive twist of the bottom two strands, then the braid form of the knot is the closure of the braid \( \sigma_1^{a_1} \sigma_2^{-a_2} \sigma_1^{a_3} \sigma_2^{-a_4} \cdots \sigma_1^{a_n} \).

With this in mind, we now explain how to build the knot exterior. (See Figure 7.) Given a rational knot, calculate the braid \( \sigma_1^{a_1} \sigma_2^{-a_2} \sigma_1^{a_3} \sigma_2^{-a_4} \cdots \sigma_1^{a_n} \) associated with the braid form. To build the knot exterior, start with \( (S^2-::) \times I \), and braid the bottom three tunnels according to the calculated braid \( \sigma_1^{a_1} \sigma_2^{-a_2} \sigma_1^{a_3} \sigma_2^{-a_4} \cdots \sigma_1^{a_n} \). Then, in order to “connect” the strands of the braid, attach four 2-handles over the ends of the top two and bottom two tunnels on either side. At this point, to obtain the entire exterior, technically we need to attach two 3-handles on the inside and outside of our constructed 3-manifold, but these 3-handles have no effect on the KBSM of the manifold, so we may as well consider the manifold without the 3-handles.

Any link in our constructed manifold can be isotoped away from the attached 2-handles, so the inclusion of the KBSM of the four-punctured sphere into its KBSM is surjective [1]. The kernel of this inclusion is generated by skeins of the form \( L - sl(L) \), where \( sl(L) \) is a link in \( S_K((S^2-::) \times I) \) obtained by sliding \( L \) along an attached 2-handle [3]. In general, calculating the kernel can take quite a bit of work, but keeping track of how the 2-handles are attached can be quite involved in itself; for the remainder of the paper, we attack this problem. Specifically, we show how to calculate the attaching curves of the two-handles in terms of the generators of \( S_K((S^2-::) \times I) \).

Figure 7: Building the knot exterior from \( (S^2-::) \times I \). Here, \( I \) is horizontal, and we view a cross-section where we can see the tunnels :: \( \times I \). The ellipses are a reminder that the manifold curves around and eventually meets itself again. Although not shown, to obtain the full exterior, we should technically attach 3-handles on the left and the right of the area shown.
Figure 8: Untwisting the trefoil. In the top row, we see the knot in a standard planar diagram. From this angle, one cannot see what is happening to the parts in red and blue as the knot is unbraided, so those parts are omitted in the later pictures. In the second row, we look from the right side of the knot, with our line of sight tangent to the plane of the original diagram. The third row shows much the same thing, except the attached 2-handles are drawn, rather than the just the rightmost part of the knot.

In order to put the attaching curves in the basis of $S_K((S^2-::) \times I)$, one can think of unbraiding the inner part of the knot complement and seeing what happens to the attaching curve. See Figure 8. As a preliminary simplification, notice that the attaching curves lying on $(S^1-::) \times \{1\}$ are isotopic, as are the attaching curves lying on $(S^1-::) \times \{0\}$. Therefore we only have to keep track of one attaching curve for each side of $(S^1-::) \times I$. Furthermore, our unbraiding only affects the attaching curve on $(S^1-::) \times \{1\}$, so we only really need to keep track of one curve.

In order to express this unbraiding algebraically, we need a basis for $S_K((S^2-::) \times I)$.

We have the result of Bullock and Przytycki [1]:

**Theorem 3.2.** (Bullock and Przytycki) The skein algebra of the four-punctured sphere is generated by the following curves:

Following Bullock and Przytycki, for convenience, we set $p_1 := a_1a_2 + a_3a_4$, $p_2 := a_2a_3 + a_4a_1$, $p_3 := a_1a_3 + a_2a_4$. 
Table 1: Action of braid group on $S_K((S^2-::) \times I)$. A skein resolution gives $z = A^2(x_1x_2 - p_3 - A^2x_3)$.

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$</td>
<td>$a_2$</td>
<td>$a_1$</td>
<td>$a_3$</td>
<td>$a_4$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$a_1$</td>
<td>$a_3$</td>
<td>$a_2$</td>
<td>$a_4$</td>
<td>$z$</td>
<td>$x_2$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>$\sigma_1^{-1}$</td>
<td>$a_2$</td>
<td>$a_1$</td>
<td>$a_3$</td>
<td>$a_4$</td>
<td>$x_1$</td>
<td>$z$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>$\sigma_2^{-1}$</td>
<td>$a_1$</td>
<td>$a_3$</td>
<td>$a_2$</td>
<td>$a_4$</td>
<td>$x_3$</td>
<td>$x_2$</td>
<td>$A^{-2}(x_3x_2 - p_1 - A^{-2}x_1)$</td>
</tr>
</tbody>
</table>

The boundary curves $a_1, a_2, a_3, a_4$ commute with every element (and therefore so do $p_1, p_2, p_3$). There are other relations, which were in fact worked out by Bullock and Przytycki, but writing them here would be cumbersome and unnecessary.

From here, we determine the action of the braid group on the generators of $S_K((S^2-::) \times I)$. In Figure 9, a diagram shows the action of $\sigma_1^{-1}$ (clockwise twist along $x_1$) and $\sigma_2$ (counterclockwise twist along $x_2$).

The results of the calculation for each kind of twist is summarized in Table 1. From here, we notice that all the attaching curves “look like” $x_2$ before any untwisting is done. The attaching curve that is unaffected by the twist will, of course, remain as $x_2$. However, we must take into account the effect of unbraiding on the other curve.

As an example of how to apply this, consider again the trefoil. (Refer back to Figure 8). This has braid continued fraction $[1,1,1]$ and braid $\sigma_1\sigma_2^{-1}\sigma_1$. To unbraid the curve, apply the inverse $\sigma_1^{-1}\sigma_2\sigma_1^{-1}$ to $x_2$. We calculate

$$\begin{align*}
\sigma_1^{-1}\sigma_2\sigma_1^{-1}(x_2) &= \sigma_1^{-1}\sigma_2(z) = \sigma_1^{-1}\sigma_2(A^2x_1x_2 - A^2p_3 - A^4x_3) \\
\sigma_1^{-1}(A^2z^2x_2 - A^2p_1 - A^4x_1) &= \sigma_1^{-1}(A^4x_1x_2^2 - A^2p_3x_2 - A^2x_3x_2 - A^2p_1 - A^4x_1) \\
&= A^4x_1z^2 - A^2p_2z - A^2x_2z - A^2p_1 - A^4x_1.
\end{align*}$$

The attaching curves for the trefoil are then $x_2$ and $A^4x_1z^2 - A^2p_2z - A^2x_2z - A^2p_1 - A^4x_1$.

Figure 9: Action of two elements of the braid group on $S_K((S^2-::) \times I)$. A skein resolution gives $z = A^2(x_1x_2 - p_3 - A^2x_3)$. The effect on the boundary curves is not shown.
Thus, with this method, we can calculate the attaching curve for any rational knot. The general procedure is: calculate the braid of the rational knot; apply the action of its inverse to $x_2$. The result is one of the attaching curves; the other is simply $x_2$. This accomplishes the first step in relating the skein module of a rational knot exterior to that of a four-punctured sphere times an interval. The next step, of course, is understanding the effect of handle-slides along these two-handles.

References


