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A Study of Sufficient Conditions for Hamiltonian Cycles

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ABSTRACT

A graph G is Hamiltonian if it has a spanning cycle. The problem of determining if a graph is Hamiltonian is well known to be NP-complete. While there are several necessary conditions for Hamiltonicity, the search continues for sufficient conditions. In their paper, "On Smallest Non-Hamiltonian Regular Tough Graphs" (Congressus Numerantium 70), Bauer, Broersma, and Veldman stated, without a formal proof, that all 4-regular, 2-connected, 1-tough graphs on fewer than 18 nodes are Hamiltonian. They also demonstrated that this result is best possible.

Following a brief survey of some sufficient conditions for Hamiltonicity, Bauer, Broersma, and Veldman's result is demonstrated to be true for graphs on fewer than 16 nodes. Possible approaches for the proof of the n=16 and n=17 cases also will be discussed.

1. Introduction

In this paper, we will investigate the conjecture that every 2-connected, 4-regular, 1-tough graph on fewer than 18 nodes is Hamiltonian. First, we investigate the historical development of sufficient conditions for Hamiltonicity as they relate to the notions of regularity, connectivity, and toughness. For notation and terminology not introduced consult [6] and [13].

A **graph** G consists of a finite nonempty set V = V(G) of n points called **nodes**, together with a prescribed set X of e unordered pairs of distinct nodes of V. Each pair x = $\{u,v\}$ of nodes in X is an **edge** of G, and x is said to **join** u to v. We write x = uv or x = *vu* and say that *u* and *v* are **adjacent** nodes, and x is **incident** on *u* and *v*. The **order** of a graph G is the number of nodes in V(G). In our discussion, we will deal only with **simple** graphs, i.e., a graph with no loops or multiple edges.

The **degree** of a node *v*, in a graph G, is denoted deg (*v*), and is defined to be the number of edges incident with *v*. Closely related to the concept of degree is that of the neighborhood. The neighborhood of a node *u* is the set N(*u*) consisting of all nodes *v* which are adjacent to *u*. In simple graphs, deg $(u) = |N(u)|$. The **minimum degree** of a graph G is denoted by δ, and the **maximum degree** is denoted by Δ. If $\delta = \Delta = r$ for any graph G, we say G is a **regular** graph of degree r, or simply, G is an **r-regular** graph, i.e. all nodes have degree r. Figure 1.1 contains a 4-regular graph with V(G) = 16.

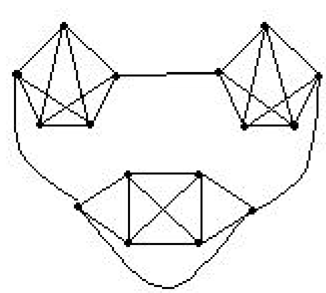


Figure 1.1

We define a **walk** to be an alternating sequence of nodes and edges, beginning and ending with nodes, in which each edge is incident on the two nodes immediately preceding and following it. A walk is called a **trail** if all the edges are distinct, and a **path** if all the nodes are distinct. A path is called a **cycle** if it begins and ends with the same node. A **spanning cycle** is a cycle that contains all the nodes in V(G), and a graph is **connected** iff every pair of nodes is joined by a path.

2. Hamiltonian Cycles

A graph is said to be **Hamiltonian** if it contains a spanning cycle. The spanning cycle is called a **Hamiltonian cycle** of G, and G is said to be a **Hamiltonian graph** (the graph in Figure 1.1 is also a Hamiltonian graph). A **Hamiltonian path** is a path that contains all the nodes in V(G) but does not return to the node in which it began. No characterization of Hamiltonian graphs exists, yet there are many sufficient conditions.

We begin our investigation of sufficient conditions for Hamiltonicity with two early results. The first is due to Dirac, and the second is a result of Ore. Both results consider this intuitive fact: the more edges a graph has, the more likely it is that a Hamiltonian cycle will exist. Many sources on Hamiltonian theory treat Ore’s Theorem as the main result that began much of the study of Hamiltonian graphs, and Dirac’s result a corollary of that result. Dirac's result actually preceded it, however, and in keeping with the historical intent of this paper, we will begin with him.

Theorem 1.1 (Dirac, 1952, [6], [7]): If *G is a graph of order n ³ 3 such that d ³ n/2, then G is Hamiltonian.*

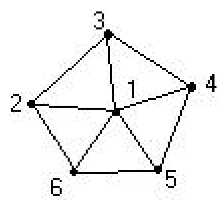


Figure 1.2

As an illustration of Dirac’s Theorem, consider the wheel on six nodes, W_6 (Figure 1.2). In this graph, $d = 3 \geq \frac{6}{2}$, so it is Hamiltonian. Traversing the nodes in numerical order 1-6 and back to 1 yields a Hamiltonian cycle.

Theorem 1.2 (Ore, 1960, [24]): *If G is a graph of order n ³ 3 such that for all distinct nonadjacent pairs of nodes u and v, deg (u) + deg (v) ³ n, then G is Hamiltonian.*

The wheel, W_6, also satisfies Ore’s Theorem. The sum of the degrees of nonadjacent nodes (i.e., deg(2) + deg (5), or deg(3) + deg (6), etc.) is always 6, which is the order of the graph.

Before we discuss the results of Nash-Williams and Chvatal and Erdos, we must first define the notions of connectivity and independence.

The **connectivity** $\kappa = \kappa(G)$ of a graph G is the minimum number of nodes whose removal results in a disconnected graph. For $\kappa \geq k$, we say that G is **k-connected**. We will be concerned with 2-connected graphs, that is to say that the removal of fewer than 2 nodes will not disconnect the graph. For $\kappa = k$, we say that G is **strictly k-connected**. For clarification purposes, consider the following. Let G be any simple graph, κ=3. Then G is 3-connected, 2-connected, and strictly 3-connected.

A set of nodes in G is **independent** if no two of them are adjacent. The largest number of nodes in such a set is called the **independence number** of G, and is denoted by β. The following result by Nash-Williams builds upon the two previous results by adding the condition that G be 2-connected and using the notion of independence.

Theorem 1.3 (Nash-Williams, 1971, [22]): *Let G be a 2-connected graph of order n with* $\delta(G) \geq \max\{(n+2)/3, \beta\}$. Then G is Hamiltonian.

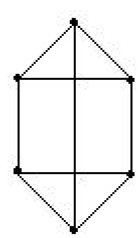


Figure 1.3

The graph in Figure 1.3 demonstrates the Nash-Williams result. In this 2-connected graph on six nodes, $d = 3, b = 2,$ and $d \geq \max\left\{\frac{6+2}{3}, 2\right\}$, implying Hamiltonicity.

In the same paper, Nash-Williams presents another very useful result. Note that a cycle C is a **dominating cycle** in G if V(G – C) forms an independent set.

Theorem 1.4 (Nash-Williams, 1971, [22]): *Let G be a 2-connected graph on n vertices with d ³ (n+2)/3. Then every longest cycle is a dominating cycle.*

Another sufficient condition uses the notion of a forbidden subgraph, i.e., a graph that cannot be a subgraph of any graph under consideration. A **subgraph** of a graph G is a graph having all of its nodes and edges in G. The following result by Goodman and Hedetniemi introduces the connection between certain subgraphs and the existence of Hamiltonian cycles. A **bipartite graph** G is a graph whose node set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2. If G contains every possible edge joining V_1 and V_2, then G is a **complete bipartite graph**. If V_1 and V_2 have m and n nodes, we write $G = K_{m,n}$ (see Figure 1.4)

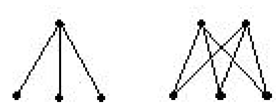


Figure 1.4: $K_{1,3}$ and $K_{2,3}$ (or $K_{3,2}$)

Goodman and Hedetniemi connected $\{K_{1,3}, K_{1,3} + x\}$-free graphs and Hamiltonicity in 1974. A $\{K_{1,3}, K_{1,3} + x\}$-free graph is a graph that does not contain a $K_{1,3}$ or a $K_{1,3} + x$ (see Figure 1.5) as an induced subgraph. (i.e., the maximal subgraph of G with a given node set S of V(G).)

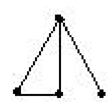


Figure 1.5: $K_{1,3} + x$

Theorem 1.5 (Goodman and Hedefniemi, 1974, [12]): *If G is a 2-connected $\{K_{1,3}, K_{1,3} + x\}$-free graph, then G is Hamiltonian.*

The wheel, W_6, in Figure 1.2, is an example of a graph that is $\{K_{1,3}, K_{1,3} + x\}$-free. The subgraph formed by node 1 and any three consecutive nodes on the cycle is $K_{1,3}$ plus 2 edges.

A year after Nash-Williams's result, Chvatal and Erdos proved a sufficient condition linking the ideas of connectivity and independence.

Theorem 1.6 (Chvatal and Erdos, 1972, [7]): *Every graph G with n ³ 3 and k ³ b has a Hamiltonian cycle.*

Chvatal and Erdos's result can be demonstrated by the graph in Figure 1.6. In this graph, κ=2 and β=2.

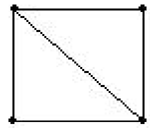


Figure 1.6

Theorem 1.6 contains, as a special case, the following result:

Theorem 1.7 (Haggkvist and Nicoghossian, 1981, [14]): *Let G be a 2-connected graph of order n with d ³ (n+ k) /3. Then G is Hamiltonian.*

By requiring that G be 1-tough (which implies 2-connectedness), Bauer and Schmeichel where able to lower the minimum degree condition found in Theorem 1.7. Let ω(G) denote the number of components of a graph G. Then the **toughness** [20] of G, denoted by τ, is defined as follows:

$$t(G) = \min_{X \subseteq V(G), w(G-X)>1} \left\{ \frac{|X|}{w(G-X)} \right\}.$$

We say G is **t-tough** for t ≥ τ(G). It is important to note that all Hamiltonian graphs are 1-tough, but the converse is not true. The Petersen Graph (see Figure 1.7) is a 1-tough, non-Hamiltonian graph.

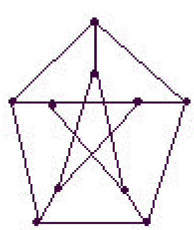


Figure 1.7: The Petersen Graph

Theorem 1.8 (Bauer and Schmeichel, 1991, [2]): *Let G be a 1-tough graph of order n with d(G) ³ (n+ k - 2)/3. Then G is Hamiltonian.*

Theorem 1.8 is best possible if κ = 2 (see Figure 1.8).

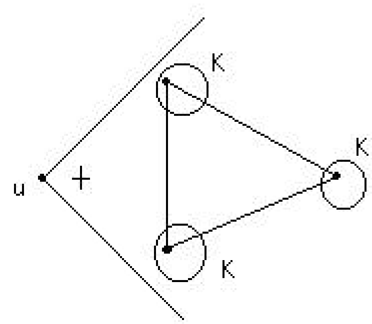


Figure 1.8

Figure 1.8 is comprised of 3 K_r, r ≥ 2, joined with a single node u. In this case, G is a 2-connected, 1-tough graph and δ = r = (n+κ-3)/3 (i.e., *d* < *(n+ k - 2)/3).* By relaxing the minimum degree requirements, we lose Hamiltonicity.

Fan later introduced distance as a contributing factor for Hamiltonicity. The **distance,** d(u,v), between two nodes u and v is the length of the shortest path joining them. Theorem 1.9 builds upon Dirac's result by adding a distance condition.

Theorem 1.9 (Fan, 1984, [9]): *Let G be a 2-connected graph of order n. If for all nodes u,v with d(u,v) = 2 we have max {deg (u), deg (v)} ³ n/2, then G is Hamiltonian.*

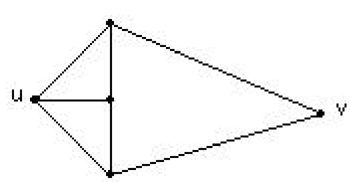


Figure 1.9

In Figure1.9 above, nodes u and v have distance 2.

$$\max\{\deg(u), \deg(v)\} = \max\{3, 2\} \geq \frac{5}{2}.$$

Thus, G is Hamiltonian.

We can consider Dirac's Theorem as a neighborhood condition on one node. By requiring the connectivity to be 2, Fraudee, Gould, Jacobsen, and Schelp were able to consider the neighborhood union of 2 nodes.

Theorem 1.10 (Fraudee, Gould, Jacobsen, Schelp, 1989, [11]): *If G is a 2-connected graph such that for every pair of nonadjacent nodes u and v,*

çN (u) È N (v) ç ³ (2n-1) /3,

then G is Hamiltonian.

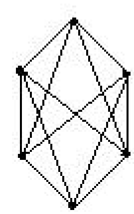


Figure 1.10

In Figure 1.10 above,

$$\left|N(u) \cup N(v)\right| = 4 \geq \frac{2n-1}{3} = \frac{11}{3}$$

Similarly, every pair of nonadjacent nodes satisfies the conditions of Theorem 1.11and G is Hamiltonian.

Fraisse further expanded the set of nonadjacent nodes by requiring a higher connectivity.

Theorem 1.11 (Fraisse, 1986, [10]): *Let G be a k-connected graph of order n ³ 3. If there exists some t £ k such that for every set S of t mutually nonadjacent nodes,*

çN (S)ç > t (n-1) / (t+1),

then G is Hamiltonian.

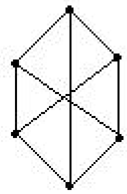


Figure 1.11

In Figure 1.11 above, k = 3. Let t = 1. Then,

$$\left|N(S)\right| = 3 \geq \frac{t(n-1)}{(t+1)} = \frac{5}{2}$$

Thus, G is Hamiltonian.

Closely related to neighborhood unions are degree sum conditions. These often lead to less strict conditions since the degree sum counts certain nodes twice, unlike the neighborhood conditions. For k ≥ 2, we define [3]

$$s_k = \min\left\{\sum_{i=1}^{k} \deg(v_i) \middle| v_1,\dots,v_k \textit{ is an independent set of nodes}\right\}$$

To demonstrate this, consider the following graph (Figure 1.12).

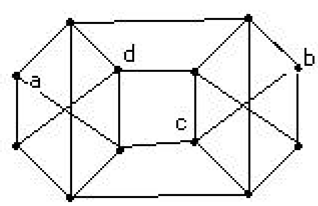


Figure 1.12

Consider σ_2 using nodes c and d. In this case,

$$\deg(c) + \deg(d) = 8.$$

Is this the minimum, however? If we consider nodes a and b, then

$$\deg(a) + \deg(b) = 6.$$

We find that 6 is the minimum. Thus,

$$s_2 = 6.$$

Theorem 1.12 (Jung, 1978, [20]): *Let G be a 1-tough graph of order n ³ 11 with* *s_2 (G) ³ n – 4. Then G is Hamiltonian.*

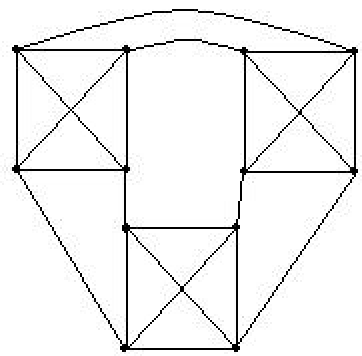


Figure 1.13

In Figure 1.13 above,

$$s_2 = 8 \geq n - 4 = 8.$$

Thus, G is Hamiltonian.

A year later Bigalke and Jung proved a result linking independence and minimum degree on 1-tough graphs.

Theorem 1.13(Bigalke and Jung, 1979, [4]): *Let G be a 1-tough graph of order n ³ 3 with d ³ max{n/3, b -1}. Then G is Hamiltonian.*

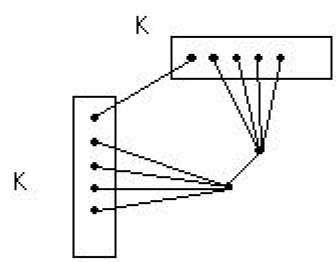


Figure 1.14

Consider Figure 1.14 above. This graph contains 12 nodes, δ=5, and β(G)=3. Therefore,

$$\max\left\{\frac{n}{3}, b(G)-1\right\} = \max\{4,2\}$$
$$d = 5 \geq \max\{4,2\}.$$

Thus, G is Hamiltonian.

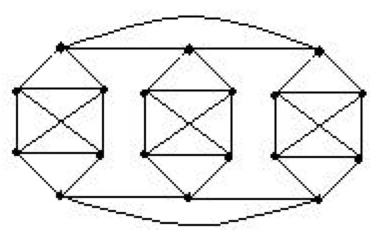
3. Hamiltonicity in 4-regular, 1-tough Graphs

Statement

Bauer, Broersma, and Veldman in [1] consider the problem of finding the minimum order of a non-Hamiltonian, k-regular, 1-tough graph. We will attempt to prove the following conjecture:

Conjecture 2.1: *Let G be a 1-tough, 2-connected, 4-regular graph of order £ 17. Then G is Hamiltonian.*

Define an **(n, k)-graph** to be a non-Hamiltonian, k-regular, 1-tough graph on n nodes. By **f(k)** we denote the minimum value of n for which there exists an (n, k)-graph. Conjecture 2.1 is best possible for n = 17, since there exists an (18, 4)-graph (see Figure



2.1).

Fig**ure 2.1: An (18, 4)-graph**

Thus, we can restate Conjecture 2.1 as:

Conjecture 2.1 (Bauer, Broersma, and Veldman, 1990, [1]): f*(k) = 18.*

Bauer, Broersma, and Veldman investigated this conjecture in [1]. They convinced themselves, through a lengthy distinction of classes, that the conjecture holds. No formal proof exists, however.

In our attempt to prove this conjecture, we shall divide the graphs into subcases based on the number of nodes.

Case 1: 5 £ n £ 8

Note that the first simple class of graphs, which satisfies the conditions of the conjecture, is of order 5. More specifically, G is K_5. Thus we must consider graphs where $5 \leq n \leq 8$.

Dirac's Theorem (Theorem 1.1) proves this case. Since G is 4-regular, $\delta = 4$. Thus, if $n \leq 8$, G is Hamiltonian.

Case 2: 8 £ n £ 12

Several results prove the existence of Hamiltonian cycles in this class of graphs. The following three theorems prove the conjecture for graphs on exactly 9, exactly 12, and up to 9 nodes, respectively.

Theorem 2.2 (Nash-Williams, 1969, [23]): *Let G be a k-regular graph on 2k + 1 nodes. Then G is Hamiltonian*.

Theorem 2.3 (Erdos and Hobbs, 1978, [8]): *Let G be a 2-connected, k-regular graph on 2k + 4 nodes, where k ³ 4. Then G is Hamiltonian.*

Theorem 2.4 (Bollobas and Hobbs, 1978, [5]): *Let G be a 2-connected, k-regular graph on n nodes, where 9k/4 ³ n. Then G is Hamiltonian.*

Note that Theorem 2.2 and Theorem 2.3 solve our problem only for graphs on exactly 9 and 12 nodes respectively. Thus we need to consider graphs on 10 or 11 nodes. In 1980, the most inclusive result appeared. Jackson's result satisfies our problem for graphs where $n \leq 12$.

Theorem 2.5 (Jackson, 1980, [18]): *Let G be a 2-connected, k-regular graph on at most 3k nodes. Then G is Hamiltonian.*

In our problem, all the graphs are 2-connected and 4-regular. Thus 3(4) = 12 is the maximum number of nodes for which the result holds.

Case 3: 12 £ n £ 15

Case 3 of the conjecture is proven by a 1986 result of Hilbig.

Theorem 2.6 (Hilbig, 1986, [17]): *Let G be a 2-connected, k-regular graph on at most 3k+3 nodes. Then G satisfies one of the following properties:*

1) *G is Hamiltonian;*
2) *G is the Petersen graph, P (Figure 1.6);*
3) *G is* P'*—the graph obtained by replacing one node of P by a triangle.*

For our problem 3(4) + 3 = 15, so all graphs up to those on 15 nodes (1-tough, 4-regular, 2-connected) are Hamiltonian by Hilbig's result.

Case 4: n = 16, 17

This leads us to the consideration of 4-regular, 1-tough, 2-connected graphs on 16 and 17 nodes. We began our investigation of this case by generating graphs of this type and separating them into six cases. For ease of notation, we define **[v,k]-graphs** to be all Hamiltonian, 1-tough, 4-regular graphs on v nodes that are strictly k-connected. Appendix A contains examples of graphs of each of the six types: [16,2], [16,3], [16,4], [17,2], [17,3], and [17,4].

We continued our investigation by examining the topology of the generated graphs. Independence number, planarity, and toughness were all considered. These results are enumerated in Appendix B.

All planar [16,4] and [17,4]-graphs are Hamiltonian by the following result of Tutte.

Theorem 2.7 (Tutte, 1956, [25]): Every *4-connected planar graph has a Hamiltonian cycle.*

Consider the following graphs:

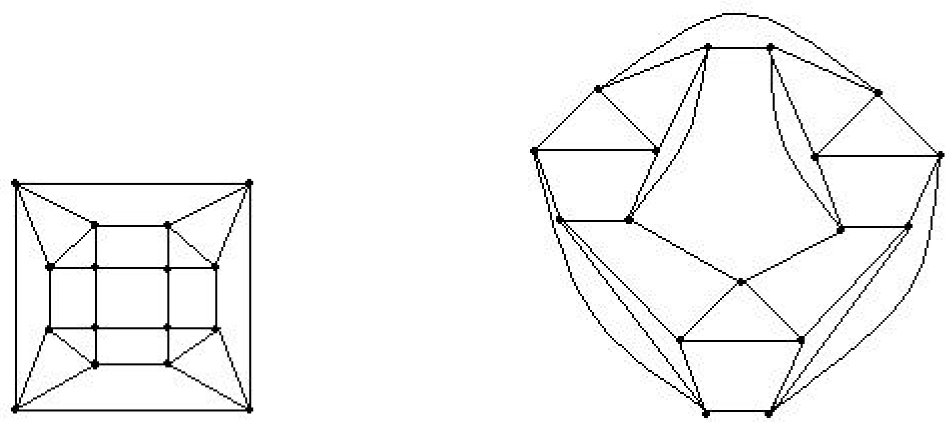


Figure 2.2: A [16,4]-graph and a [17,4]-graph

Both these graphs are 4-connected (by definition, also 2-connected), 1-tough, and 4-regular. By Tutte's Theorem, they are also Hamiltonian.

The following two observations could lead to a constructive method of proof of Conjecture 2.1.

<u>Observation 1</u>: It is interesting to note that the presence of a K_4 subgraph in G prevents planarity in 4-regular graphs. See Figure 2.3.

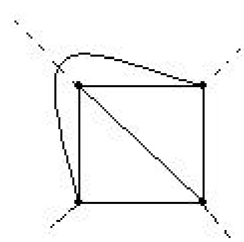
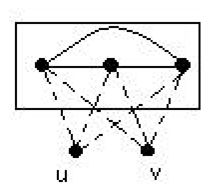


Figure 2.3

<u>Observation 2</u>: There is a minimum size of the components obtained by the removal of a κ-set in a [16,2]-graph.

Proposition 2.8: *Let G be a [16,2]-graph. Then ∃ a k-set of order 2 whose removal leaves all components of G of at least order 5.*

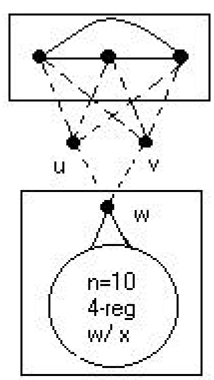
Proof: By the regularity of G, the minimum order of a component must be 3, so let the smallest component be a K_3, since the removal of one edge makes the component easier to disconnect.



G_1

This gives rise to 2 cases:

<u>Case 1</u>: u and v are adjacent to the same node, w, in G_2.

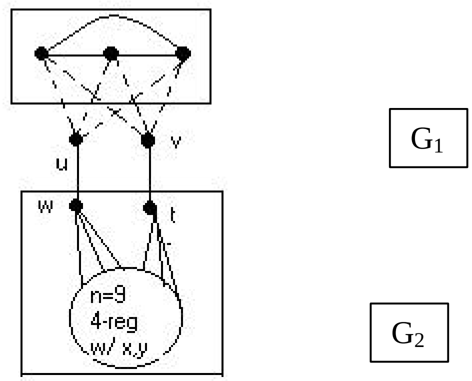


G_1

G_2

However, w is a cut-point, and thus G is 1-connected, which is a contradiction.

Case 2: u and v are adjacent to two distinct nodes in G_2.



In this case, we can choose our cut set as {t,w} and force the order of G_1 to be 5. If t and w are adjacent, then we arrive at the same results.

We conclude that G_1 and G_2 are of order at least 5.■

The research into this problem has led us to believe that a constructive approach to a proof of Conjecture 2.1 is the direction in which to head. Further study is needed for cases [16,2], [16,3], [17,2], and [17,3]. Proposition 2.9 may prove helpful and an adaptation may exist for [17,2].

If proven, Conjecture 2.1 may aid in proving the following related open problems:

Conjecture 2.9 (Haggkvist, [18]): *If G is an m-connected, k-regular graph on at most (m+1)k nodes, then G is Hamiltonian.*

Conjecture 2.10 (Haggkvist, 1976, [15]): *If G is 2-connected, k-regular, bipartite graph on at most 6k nodes, then G is Hamiltonian.*

Conjecture 2.11 (Jackson, 1979, [18]): *If G is a 2-connected graph on at most 3k + 2 vertices with degree sequence (k, k, …, k +1, k + 1) then G is Hamiltonian.*

Conjecture 2.12 (Jackson and Jung, 1992, [19]): *For $k^{3}4$, all 3-connected, k-regular graphs on at most 4k vertices are Hamiltonian.*

Acknowledgment

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