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Sprials, Partial Sums and Continuous Images

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1 Introduction

In this paper a continuum is a compact connected subset of the plane. When we consider two continua X and Y, one of the basic questions we ask is whether there exists a continuous map of X onto Y. More generally, when we consider two collections of continua, we ask when there exists a continuous map of a member of one collection onto a member of the other collection.

We consider a collection of continua each of which is the union of the unit circle and a ray spiralling down upon the circle in a way to be defined later. The purpose of this paper is to determine which of these continua is the continuous image of a nonseparating continuum, i. e., a continuum that has a connected complement in the plane.

This work is a special case of more general work done by David Bellamy. In [2] Bellamy proves the theorem for a larger class of "compactifications of (0,1] with remainder a circle" that are not assumed to be subsets of the plane, and he determines that the nonseparating continuum can be required to have the special property of being "chainable." Naturally his proof is more involved. We believe that our special case conveys the idea behind this result in a way accessible to undergraduate mathematics majors.

2 Definitions

Let B be the annulus of inner radius 1 and outer radius 2, and let S^1 denote the unit circle. Let M be the unit circle and two sprials, each starting at (2,0) and spiralling in towards the unit circle, one counterclockwise and one clockwise. More specifically, using polar coordinates define the subset $M \subset B$ to be $M = S^1 \cup \{(1+e^{-\theta}, \theta) \mid \theta \geq 0\} \cup \{(1+e^{-\theta}, -\theta) \mid \theta \geq 0\}$.

Let $A^s = (a_0, a_1, \ldots, a_n, \ldots)$ be an infinite sequence such that $a_n = 1$ or $a_n = -1$. For each term a_n, there exists a corresponding subset $U_n \subset M$ where

$$U_n = \begin{cases} \{(1+e^{\theta}, \theta) \mid 2\pi n \leq \theta \leq 2\pi(n+1)\}, & \text{if } a_n = 1\ ; \\ \{(1+e^{-\theta}, -\theta) \mid 2\pi n \leq \theta \leq 2\pi(n+1)\}, & \text{if } a_n = -1. \end{cases}$$

For each A^s, define the subset $A \subset M$ to be $A = S^1 \cup (\bigcup U_n)$. As mentioned above, a topological space $X \subseteq \mathbb{R}^2$ is nonseparating if $\mathbb{R}^2 - X$ is connected, and X is a continuum if it is a compact, connected metric space. The purpose of this paper is to determine which A's are the continuous image of a nonseparating plane continuum. This is a special case of more general work done by David Bellamy in [2].

3 Examples

To understand the relationship between A^s and A, consider the following two examples.

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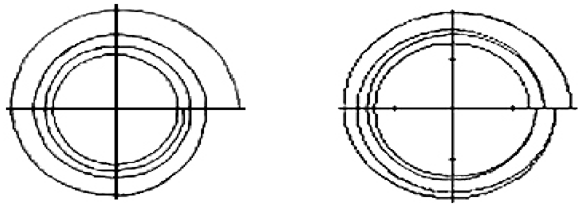
Example *Let $A^s = (1, 1, 1, \ldots)$. Then A is the union of S^1 and a counter-clockwise spiral as shown in Figure 1.1.*

Example *If $A^s = (1, -1, 1, -1, \ldots)$, then A is sketched in Figure 1.2.*

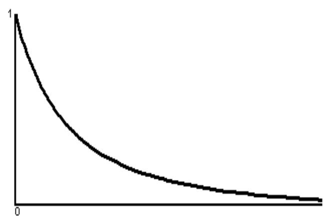
Figure 1.1 and Figure 1.2



Notice that in Example 2, the spiral alternates between counterclockwise and clockwise. Define a covering map of B by $\pi : \mathbb{R}\times I \to B$ with $\pi(s,t) = (t+1)e^{2\pi i s}$. We are interested in the preimage of each A under π in $\mathbb{R}\times I$. Because π is a covering map, $\pi^{-1}(U_n)$ consists of pairwise disjoint arcs each of which is homeomorphic to $[0,1]$. We will assume that $\pi^{-1}(U_0)$ begins at $(0,1)$. For each consecutive U_n chose an arc from $\pi^{-1}(U_n)$ to begin where $\pi^{-1}(U_{n-1})$ ends. This constructs a connected subset of $\bigcup \pi^{-1}(U_n)$ homeomorphic to $[0,\infty)$, call it Z. Let $[a,b]$ be a subset of $\mathbb{R}\times 0$, $a, b \in \mathbb{Z}$. Then π will map $Z \cup [a,b]$ onto A.

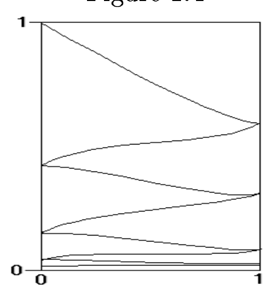
Example *Using A^s from Example 1, Z will be*

Figure 1.3



Example *Using A^s from Example 2, Z will be*

Figure 1.4

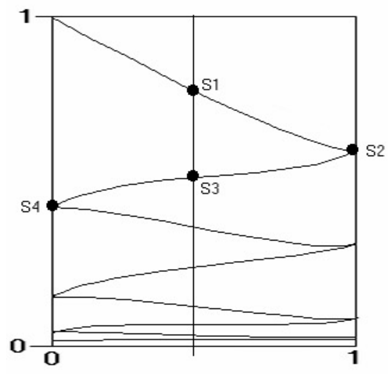


4 Partial Sums

From here on we refer to the set of points $\{(x,0) \mid (x,0) \in \mathbb{R}\times I\}$ as $\mathbb{R}$. By the above method of constructing Z, one notices that the n^{th} partial sum, S_n, of the sequence A^s is the first coordinate of the endpoint of the n^{th} arc. We call this coordinate s_n.

Example *In the following diagram, the corresponding endpoints are shown for the first four partial sums of the sequence* $A^s = (1, 1, -1, -1, 1, 1, -1, -1, \ldots)$

Figure 1.5



With this background we are now prepared to prove the following theorem that classifies all infinite sequences A^s.

5 Conclusion

Theorem *Let* A^s *and* A *be defined as above. There exists a nonseparating continuum* X *and a continuous map* f *from* X *onto* A *if and only if the sequence of partial sums of* A^s *is bounded.*

For the proof we will require the following result. The notation $[X, Y] = 0$ for topological spaces X and Y means that all maps from X into Y are homotopic to a constant map. Two maps $f, g : X \to Y$ are homotopic if there exists a one parameter family of continuous maps from $X \to Y$ starting at f and ending at g. In other words, two maps are homotopic if you can continually deform one into the other. For a further explanation of homotopy between maps see [1, pages 87-108].

Lemma *A continuum* $X \subset \mathbb{R}^2$ *is nonseparating if and only if* $[X, S^1] = 0$.

See [3] for a proof.

We are now ready to proceed with the proof of the Theorem. Assume the sequence of partial sums is bounded. Then there exist integers n and m such that, for all i, $S_n \leq S_i \leq S_m$. Therefore Z is bounded since all points in Z are contained in the rectangle $[s_n, s_m] \times [0, 1]$. Because all the limit points of Z are contained in $\mathbb{R}$, $Z \cup [s_n, s_m]$ is closed. Hence $Z \cup [s_n, s_m]$ is compact.

Any closed interval of $\mathbb{R}$ is connected and Z is connected, thus it suffices to show that $\bar{Z} \cap [s_n, s_m] \neq \emptyset$ in order to show that $Z \cup [s_n, s_m]$ is connected. Because $[s_n, s_m]$is bounded there are a finite number of integers contained in $[s_n, s_m]$. One of these integers must be a limit point for Z because there is an infinite number of endpoints for Z in $[s_n, s_m] \times I$, and as $t \to 0$, $Z \to [s_n, s_m]$. Thus $\bar{Z} \cap [s_n, s_m] \neq \emptyset$, and therefore $Z \cup [s_n, s_m]$is connected.

Clearly $Z \cup [s_n, s_m]$ is nonseparating. Hence $Z \cup [s_n, s_m]$is a nonseparating continuum and π maps $Z \cup [s_n, s_m]$ onto A.

Next, assume the sequence of partial sums is not bounded. Then for each n there exists $m > n$ such that $| S_n | < | S_m |$. Hence, given any endpoint s_n of Z there exists an endpoint s_m such that $| s_n | < | s_m |$. Therefore, Z is not bounded and thus not compact. Because Z is not compact, all components of $\pi^{-1}(A - S^1)$ are also not compact because they are all homeomorphic to Z. Let X be a nonseparating continuum, and $f : X \to A$ a map. Then by the Lemma we have $[X, S^1] = 0$ and since S^1 is homotopy equivalent to $S^1 \times I$ we know $[X, S^1 \times I] = 0$. Because B is homeomorphic to $S^1 \times I$ it is also homotopy equivalent; therefore $[X, B] = 0$. Thus $f \underset{H}{\simeq} c$, where $c : X \to B$ is a constant map. By the homotopy lifting property of covering spaces [4, Theorem 3, page 67], there exists a map $\tilde{H} : X \times I \to \mathbb{R} \times I$ such that $\pi \circ \tilde{H} = H$. Hence there exists $\tilde{f} : X \to \mathbb{R} \times I$ such that $\pi \circ \tilde{f} = f$. Since X is connected, the image of $\tilde{f}$ must be connected and so it must be contained in either Z, $\mathbb{R}$, or, if connected, $Z \cup \mathbb{R}$. If the image of $\tilde{f}$ is contained in either Z or $\mathbb{R}$, then f cannot map onto A. If the image of $\tilde{f}$ is contained in $Z \cup \mathbb{R}$ then it must contain all of Z for f to map onto A. But, since X is compact the image of $\tilde{f}$ must be compact and hence bounded. Thus it cannot contain Z. Therefore there cannot exist a map from a nonseparating continuum onto A if the sequence of partial sums is not bounded.

References

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