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Turning the Lights Out in Three Dimensions

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Tiger electronics now has an entire Lights Out puzzle series. The original version, solved by means of Linear Algebra by Feil and Anderson [1] in October 1998, is a five by five grid of lights. Pressing a button results in a change of parity of that button and a change in parity of the north, south, east and west neighbors of that light (if such neighbors exist). The object of the game is to get all of the lights turned off. Later, Tiger released another version of the mind puzzle, Lights Out Cube, a cube in which the sides are three by three grids of lights. The parity-changing rule still applies, except this time if a light lies on the edge of a face, pressing it will change all of its neighbors, including those on adjacent faces. Thus, in Lights Out Cube, pressing any button will always result in the change of parity of five buttons, itself and its four neighbors. Again, the game presents the user with a configuration of lights, some off and some on, and the objective is to turn all the lights out.

In Section 1, we will present a summary of Anderson and Feil's solution to the original Lights Out puzzle. We will see that with the five by five grid, representationed by a very convenient matrix, the solution is easily obtained using some basic Linear Algebra. In Section 2, we will discuss Lights Out Cube, and we will present a complete solution. We will observe for the cube that the representation is not so "convenient." Thus, the solution for Lights Out Cube becomes a bit more problematic.

1. Five by Five Lights Out.

The original Lights Out is a 5×5 array of buttons, each of which are either lit or unlit. The board will be represented by a 5×5 matrix with entires in the integers modulo 2. For a given board state, if a button (i, j) is lit, then the game matrix B has a 1 in the (i, j) spot; if a button (i, j) is off, B has a 0 in the (i, j) position. Pressing a button changes the light's status (on to off or off to on) and the status of each of the light's vertical and horizontal neighbors. The goal of the game is, given an initial configuration of lights, to find a winning strategy, i.e. find which buttons need to be pressed in order to turn all of the lights off. Feil and Anderson begin by making two initial observations:

1. Pushing a button twice is equivalent to not pushing it at all. Hence, for any given configuration, we need consider only strategies in which each button is pushed at most once.

2. The state of a button depends only on how often (whether even or odd) it and its neighbors have been pushed. Hence, the order in which the buttons are pushed is immaterial.

Together, these two observations imply that zeroing out a configuration involves pressing the same buttons as starting from a zero matrix and arriving at that configuration. Thus, given a configuration

$$B = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ b_6 & b_7 & b_8 & b_9 & b_{10} \\ b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{16} & b_{17} & b_{18} & b_{19} & b_{20} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{pmatrix},$$

we can begin with a zero matrix and find which buttons need to be pressed in order to arrive at B. More specifically, we must solve for λ_i $(1 \leq i \leq 25)$ in

$$\lambda_1 \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \cdots$$

$$+\lambda_{25} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ b_6 & b_7 & b_8 & b_9 & b_{10} \\ b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{16} & b_{17} & b_{18} & b_{19} & b_{20} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{pmatrix}$$

or

$$\sum_{i=1}^{25} \lambda_i A_i = B,$$

where each A_i represents the matrix formed by pressing button i on a game board with all of the lights turned off, and each λ_i is an integer modulo 2 which represents either the act of pressing button i or the act of not pressing button i. (Recall that all arithmetic is being done in the integers modulo 2.) The numbering used to solve this problem is as follows: row 1 has buttons 1 through 5 in order, row 2 has buttons 5 through 10 in order, ... , row 5 has buttons 21 through 25 in order. If we find that $\lambda_i = 1$, then the winning strategy involves pressing button i; if we find that $\lambda_i = 0$, then the winning strategy involves not pressing button i. A winning strategy is a solution, $\vec{x}$, of

$$A\vec{x} = \vec{b}$$

where

$$\vec{x} = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_{25})^T$$

$$\vec{b} = (b_1, b_2, b_3, \cdots, b_{25})^T$$

$$A = \begin{pmatrix} C & I & O & O & O \\ I & C & I & O & O \\ O & I & C & I & O \\ O & O & I & C & I \\ O & O & O & I & C \end{pmatrix};$$

here I is the 5×5 identity matrix, O is the 5×5 matrix of all zeros, and C is the matrix

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

This is easy to verify, and we leave it up to the reader. Therefore, a configuration $\vec{b}$ is winnable, (that is there exists a set of lights that when pressed will result in turning all of the lights off), if and only if it belongs to the column space of the matrix A, denoted $\mathrm{Col}(A)$. Performing Gauss-Jordan elimination on A results in $RA = E$, where E is the Gauss-Jordan echelon form, and R is the product of the elementary matrices which perform the reducing row operations.

The rank of the matrix E turns out to be 23, and the two free variables are λ_{24} and λ_{25}. To verify this, we recommend using a computer algebra system such as *Maple*. To find which configurations have solutions, we must determine which vectors $\vec{x}$ are in Col(A). However, A is symmetric, so Col(A) = Row(A), the row space of A, and Row(A) is the orthogonal complement of the null space of A, Null(A) = Null(E). A basis for Null(E) can be found by examining the last two columns of E:

$$\vec{n}_1 = (0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,1,0)^T$$

$$\vec{n}_2 = (1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,1)^T.$$

Formally stated, we have

Theoerm 1 [Feil and Anderson]. *A configuration $\vec{b}$ is winnable if and only if $\vec{b}$ is orthogonal to the two vectors $\vec{n}_1$ and $\vec{n}_2$.*

Since the dimension of the null space of E is 2, it follows that only one fourth of all possible configurations are winnable. Moreover, if $\vec{b}$ is winnable and $\vec{x}$ is a winning strategy, then $\vec{x}+\vec{n_1}$, $\vec{x}+\vec{n_2}$, and $\vec{x}+\vec{n_1}+\vec{n_2}$ are also winning strategies. (E.g. $A\vec{x}=\vec{b} \Rightarrow A(\vec{x}+\vec{n}_1) = A\vec{x}+A\vec{n}_1 = A\vec{x}+\vec{0}=\vec{b}$.)

We now attempt, given a winnable configuration, to find a winning strategy, i.e. given $\vec{b}$, find $\vec{x}$ such that $A\vec{x}=\vec{b}$. The key observation about the matrix E (one that will make the solution quite convenient) is that if $\vec{b}$ is a winnable configuration, then λ_{24} and λ_{25} can be set to zero in any winning strategy, $\vec{x}$, and E acts like identity matrix on such a vector; that is $\vec{x}=E\vec{x}$. (We are free to set $\lambda_{24}=\lambda_{25}=0$ because they are free variables.) Making some substitutions, we have $\vec{x}=E\vec{x}=RA\vec{x}=R\vec{b}$. We now have a solution to a winnable configuration:

Theorem 2 [Feil and Anderson]. *Given $\vec{b}$, a winnable configuration, the winning strategies are $R\vec{b}$, $R\vec{b}+\vec{n_1}$, $R\vec{b}+\vec{n_2}$,$R\vec{b}+\vec{n_1}+\vec{n_2}$, where R is the product of elementary matrices that reduce A to Gauss-Jordan echelon form.*

2. Three Dimensional Lights Out.

In this section, we will describe a method for finding the solution to Lights Out Cube, a 3-dimensional version of the original game.

In Lights Out Cube, 3×3 grids of lights comprise the six faces of the cube. Again, the object of the game is to turn all the lights out. However, this time when a button is pressed, its parity is changed along with the parity of each of its vertical and horizontal neighbors, including those on adjacent faces. This "wrap around" effect gives Lights Out Cube a different flavor than the original game. The analysis begins just as in Section 1, but the matrix representation of the 5×5 puzzle is visually more compelling than the solution to the matrix for the cube. The matrix representing a 5×5 configuration "looks like" it would on the game board; the matrix for Lights Out Cube does not.

We begin by giving each light a number, i ($1 \le i \le 54$). Refer to Appendix B for the location of the lights. Next we form the matrix A in the same way that we did in Section 1, where the i^{th} row of A contains a 1 in the j^{th} position if light j is changed by pressing light i, and a 0 otherwise. Making the same two observations made by Anderson and Feil, we come to the same conclusions: solving a configuration $\vec{b}$ is equivalent to beginning with a zero matrix and arriving at $\vec{b}$. Once again, we must solve for $\vec{x}$ in $A\vec{x} = \vec{b}$, where $\vec{b}$ is the configuration presented by the game.

Using a computer algebra package such as Maple, and tediously typing in the matrix A, we find that A is indeed a symmetric matrix (as we would expect). So, Col(A)=Row(A), Row(A) is the orthogonal complement of Null(A), and the dimension of the null space for A turns out to be 6. A basis for Null(A) is given by:

$$\vec{n}_1 = (1,1,1,1,0,1,0,1,0,0,1,0,1,0,1,1,1,1,0,1,0,1,0,1,1,1,1,1,1,1,$$
$$1,0,1,0,1,0,0,0,1,0,0,0,1,0,0,1,0,0,0,0,0,0,0,1)^T$$
$$\vec{n}_2 = (1,0,1,0,0,0,1,0,1,1,0,1,0,0,0,1,0,1,1,0,1,0,0,0,1,0,1,1,0,1,$$
$$0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)^T$$
$$\vec{n}_3 = (0,0,1,0,0,0,0,0,1,1,1,0,1,0,1,0,1,1,0,0,1,0,0,0,0,0,1,0,1,1,$$
$$1,0,1,1,1,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0,0,1,0,0)^T$$

$$
\begin{aligned}
\vec{n}_4 = (&0,1,1,1,0,1,1,1,0,1,0,0,0,0,0,1,0,0,1,1,0,1,0,1,0,1,1,1,0,0,\\
&0,0,0,1,0,0,1,0,1,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0)^T\\
\vec{n}_5 = (&0,1,1,0,0,0,0,1,1,1,0,0,1,0,1,1,0,0,0,1,1,0,0,0,0,1,1,1,0,0,\\
&1,0,1,1,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,0,0,0,1,0)^T\\
\vec{n}_6 = (&0,0,1,1,0,1,0,0,1,1,1,0,0,0,0,1,1,0,0,0,1,1,0,1,0,0,1,1,1,0,\\
&0,0,0,1,1,0,0,0,0,1,0,1,0,0,0,0,0,0,1,0,1,0,0,0)^T.
\end{aligned}
$$

This, of course, is a property of how the buttons were numbered.

We now state the following:

Theorem 3. *A configuration $\vec{b}$ is winnable if and only if $\vec{b}$ is orthogonal to the six vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$, $\vec{n}_4$, $\vec{n}_5$, $\vec{n}_6$.*

Since the dimension of the null space of A is 6, only 1 out of every $2^6 = 64$ possible configuration is winnable. Moreover, each winnable configuration $\vec{b}$, has not just one winning strategy, but 64 different winning strategies.

As we attempt to find a vector $\vec{x}$ that will serve as a winning strategy for $\vec{b}$ (i.e. $A\vec{x} = \vec{b}$), we examine E, the Gauss-Jordan echelon form of A. Recall that this matrix in Section 1 for the 5×5 case had a convenient property: the free variables were the last two, and setting those variables equal to zero resulted in $\vec{x} = E\vec{x}$. Stated differently, the upper left 23×23 matrix was the identity matrix. Unfortunately, this property does not hold for E in the three dimensional case because the free variables are not the last 6. In fact, the free variables are $\lambda_{36}, \lambda_{48}, \lambda_{51}, \lambda_{52}, \lambda_{53}$, and λ_{54}. This doesn't hinder us from finding a solution by setting the free variable equal to zero, but it does make it a bit more time consuming to do by hand, and the solution procedure is not nearly as concise as the solution procedure found by Anderson and Feil. It is best to write a procedure using a computer algebra package to actually perform this series of operations.

The *Maple* algorithm presented on the following pages in Appendix A1 simulates (after the deletion of some rows) the property $\vec{x} = E\vec{x}$ after setting each of the free variables $\lambda_{36}, \lambda_{48}, \lambda_{51}, \lambda_{52}, \lambda_{53}$, and λ_{54} to 0. We begin by obtaining the matrix R, the product of the elementary matrices used to perform the row reducing operations on A in order to arrive at E. We then multiply R by the vector $\vec{b}$, our initial configuration; call this result $\vec{w}$. If the i^{th} component of $\vec{w}$ is a 1, the we begin at the (i,i) coordinate in E and march across the i^{th} row until we hit a 1 in the (i,j) position. The j button is one that will need to be pressed. We repeat this process for the first 48 coordinates of $\vec{w}$. In most cases, there will be a 1 in the (i,i) position, thus we can stop there with $j = i$; however, in some cases, the first 1 will be offset by 1 $(j = i + 1)$ or 2 $(j = i + 2)$. If you look closely at E, this will become apparent. Once a solution is found, any linear combination of that solution and n_i $(1 \leq i \leq 6)$ is also a solution, just as in the 5×5 puzzle.

Although this alogrithm will produce a winning strategy for a configuration $\vec{b}$, we would like to have a process analogous to that of the 5×5 puzzle, one with the same "nice" mathematics. If we could renumber the lights on the cube so that the free variables of its associated matrix E came out to be the last 6, then we could present a theorem for Lights Out Cube analogous to Theorem 2. Thus, we would like to permute the numbering of the lights so that lights 36 and 49 swap and lights 48 and 50 swap. Then the free variables of E would permute from $\lambda_{36}, \lambda_{48}, \lambda_{51}, \lambda_{52}, \lambda_{53}$, and λ_{54} to $\lambda_{49}, \lambda_{50}, \lambda_{51}, \lambda_{52}, \lambda_{53}$, and λ_{54}. If we multiply A by the permutation matrix P with 1's along the diagonal except for $P[36,36] = P[49,49] = P[48,48] = P[50,50] = 0$, $P[36,49] = P[49,36] = P[48,50] = P[50,48] = 1$, and zeros elsewhere, we will obtain a new matrix A^* that is analogous to the old A with a swap of buttons 36 and 49 and buttons 49 and 50. (This amounts to renumbering four buttons.) Performing Gauss-Jordan elimination modulo 2 results in $R^*A^* = E^*$, where E^* is the Gauss-Jordan ecehlon form, and R^* is the product of elementary matrices which perform the reducing row operations. A basis for Null(A^*)=Null(E^*) is:

$$\vec{n}_1^* = (1,1,1,1,0,1,0,1,0,0,1,0,1,0,1,1,1,1,0,1,0,1,0,1,1,1,1,1,1,1,$$
$$1,0,1,0,1,0,0,0,1,0,0,0,1,0,0,1,0,0,0,0,0,0,0,1)^T$$
$$\vec{n}_2^* = (1,0,1,0,0,0,1,0,1,1,0,1,0,0,0,1,0,1,1,0,1,0,0,0,1,0,1,1,0,1,$$
$$0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0)^T$$
$$\vec{n}_3^* = (0,1,1,1,0,1,1,1,0,1,0,0,0,0,0,1,0,0,1,1,0,1,0,1,0,1,1,1,0,0,$$
$$0,0,0,1,0,0,1,0,1,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0)^T$$

$$\begin{aligned}\vec{n}_4^* = (&0,0,1,0,0,0,0,0,1,1,1,0,1,0,1,0,1,1,0,0,1,0,0,0,0,0,1,0,1,1,\\ &1,0,1,1,1,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0,0,1,0,0)^T\\ \vec{n}_5^* = (&0,0,1,1,0,1,0,0,1,1,1,0,0,0,0,1,1,0,0,0,1,1,0,1,0,0,1,1,1,0,\\ &0,0,0,1,1,1,0,0,0,1,0,1,0,0,0,0,0,0,0,0,1,0,0,0)^T\\ \vec{n}_6^* = (&0,1,1,0,0,0,0,1,1,1,0,0,1,0,1,1,0,0,0,1,1,0,0,0,0,1,1,1,0,0,\\ &1,0,1,1,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,0,0,0,1,0)^T.\end{aligned}$$

We now state the following:

Theorem 4. *Under the new numbering of lights as described above, a configuration $\vec{b}$ is winnable if and only if $\vec{b}$ is orthogonal to the six vectors $\vec{n}_1^*$, $\vec{n}_2^*$, $\vec{n}_3^*$, $\vec{n}_4^*$, $\vec{n}_5^*$, $\vec{n}_6^*$.*

We conclude our discussion by presenting the following:

Theorem 5. *Under the new numbering of lights as described about, given $\vec{b}$ a winnable configuration, the winning strategies are $R^*\vec{b}$ and the result of adding $R^*\vec{b}$ with one or more of $\vec{n}_i^*$ $(1 \le i \le 6)$.*

The *Maple* code for the improved Solution to Lights Out Cube can be found in Appendix A2.

REFERENCES

1. Anderson, Marlow and Feil, Todd, *Turning Lights Out with Linear Algebra*, Math Magazine (4) **71** (1998), 300–303.