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# Basins of Roots and Periodicity in Newton's Method for Cubic Polynomials

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## Introduction

Newton's method is generally introduced in Calculus I courses as a useful tool for finding the roots of functions when analytical methods fail. This method works well because, if the initial guess is close enough to the actual root, iterations will converge quickly to the root. The student is usually warned against picking a point where the derivative is zero because the function used in Newton's method is undefined at that point. Polynomials used for Calculus I problems don't tend to have any further complications, but polynomials with interesting behavior do exist.

If we expand our study of the dynamics of Newton's method to the complex plane, we find lots of interesting properties. Fractals, chaos, attracting periodic cycles, Mandelbrot sets, and other phenomena are present, depending on what type of functions we study. We will focus on cubic polynomials with real coefficients for the actual analysis in Sections III and IV, but we begin with some background on the properties we will use for analytic functions in Section I and rational functions in Section II.

### I. Iteration and Newton's method

A sequence of points can be created by *functional iteration* which, given a function  $g: \mathbf{C} \rightarrow \mathbf{C}$  and an initial value  $p_0$ , is defined by the successive evaluation of the results of the function starting with the initial value. The sequence obtained is of the form  $\{p_0, p_1=g(p_0), \dots, p_i=g(p_{i-1}), \dots\}$ . It is possible for the sequence to yield a periodic  $n$ -cycle where  $g^{(n)}(p) = p$  for some  $p \in \mathbf{C}$ , where  $g^{(n)}$  denotes the  $n$ th iteration of  $g$ . One special case of this is a *fixed point*, which occurs when  $n = 1$ . A periodic point  $p$  can be classified depending on the value of  $\lambda = (g^{(n)})'(p)$ .

Definition: ([Cr], p. 203)

The periodic point  $p$  where  $g^{(n)}(p) = p$  is

*superattracting* if  $\lambda = 0$ ,

*attracting* if  $|\lambda| < 1$ ,

*neutral* if  $|\lambda| = 1$ ,

*repelling* if  $|\lambda| > 1$ . •

Since 
$$\lambda = (g^{(n)})'(p) = \prod_{i=1}^n g'(p_i)$$

by the chain rule, each point in the periodic cycle will have the same  $\lambda$ . Thus, the cycle can be described with the above terminology.

If a point  $p$  is attracting or superattracting, then there is a collection of points such that functional iteration of  $g$  with any of these points as the initial value will converge to  $p$ .

Definition:

The *basin of attraction* for a fixed point  $p$  is

$$A(p) = \{z \in \mathbf{C} : g^{(n)}(z) \rightarrow p \text{ as } n \rightarrow \infty\}.$$

The *basin of attraction* for a periodic cycle  $P = \{p_1, p_2, \dots, p_n\}$  of length  $n$  is

$$A(P) = \{z \in \mathbf{C} : g^{(in)}(z) \rightarrow p_k \text{ for some } k \in \{1, 2, \dots, n\} \text{ as } i \rightarrow \infty\}. \bullet$$

We would like to be assured that attracting periodic and fixed points, based on the definition using  $\lambda$ , actually have basins of attraction.

Theorem 1:

Suppose  $p$  is an attracting or superattracting periodic point for an iterative function,  $g$ . Then there exists a neighborhood  $U$  of  $p$  such that  $U \subseteq A(p)$ .

Proof:

Suppose a fixed point,  $p$ , is attracting or superattracting, so that  $|\lambda| < 1$  (recall  $\lambda = g'(p)$ ). (If  $p$  is a periodic point, a similar argument could be applied to  $g^{(n)}(p)$ .)

We can expand  $g$  about  $p$  using a Taylor approximation.

$$g(z) = g(p) + g'(p)(z-p) + \frac{g''(p)}{2!}(z-p)^2 + \dots$$

For values of  $z \in \mathbf{C}$  near  $p$

$$\begin{aligned} g(z) &\approx g(p) + g'(p)(z-p) \\ |g(z) - g(p)| &\approx |g'(p)||z-p| = |\lambda||z-p| \end{aligned}$$

This shows that  $g(z)$  and  $g(p) = p$  are approximately closer together than  $z$  and  $p$  since  $|\lambda| < 1$ , so the iteration would eventually converge to  $p$ . However, we would like this to be more rigorous.

Let  $\mu \in \mathbf{R}$  such that  $|\lambda| < \mu < 1$ .

$$\text{Now } \lim_{z \rightarrow p} |(g(z) - g(p))/(z-p)| = |g'(p)| = |\lambda| < \mu < 1$$

By the definition of the derivative,  $\exists \delta$  such that  $|z-p| < \delta$  implies

$$|(g(z) - g(p))/(z-p)| < \mu$$

Let  $p_i \in \mathbf{C}$  such that  $|p_i - p| < \delta$ , i.e.  $p_i \in U_\delta(p)$  (the neighborhood of points within a distance of  $\delta$  from  $p$ ) and recall  $p_{i+1} = g(p_i)$ .

$$\text{Then } |(g(p_i) - g(p))/(p_i - p)| < \mu \Rightarrow |p_{i+1} - p| = |g(p_i) - g(p)| < \mu |p_i - p|$$

Since  $\mu < 1$ ,  $p_{i+1} \in U_\delta(p)$  and the same argument holds.

Now let  $p_0 \in U_\delta(p)$ . Then,

$$\begin{aligned}
|p_{i+1}-p| &< \mu|p_i-p| = \mu|g(p_{i-1})-p| \\
&< \mu^2|p_{i-1}-p| \\
&< \mu^3|p_{i-2}-p| \\
&\dots \\
&< \mu^{i+1}|p_0-p|.
\end{aligned}$$

Since  $\mu < 1$ , this value approaches zero as  $i$  goes to infinity, so  $p_0 \in A(p)$ .

Therefore,  $U_\delta(p) \subseteq A(p)$ .  $\blacklozenge$

Now we would like to understand the difference between attracting and superattracting. Suppose a fixed point  $p$  is superattracting so  $\lambda = 0$ . Again, we can use Taylor's approximation to expand  $g$  about  $p$ .

$$g(z) = g(p) + g'(p)(z-p) + \frac{g''(p)}{2!}(z-p)^2 + \dots$$

For values of  $z \in \mathbf{C}$  near  $p$

$$\begin{aligned}
g(z) &\approx g(p) + 0 + g''(p)(z-p)^2 / 2! \\
|g(z) - g(p)| &\approx |g''(p) / 2!| |(z-p)|^2 = M |z-p|^2
\end{aligned}$$

where  $M = |g''(p)/2!|$ .

Suppose  $p_0 \in \mathbf{C}$  and  $|p_0-p| < 1/M$ , and  $p_{i+1} = g(p_i)$  for  $i = 0, 1, 2, \dots$  then

$$|p_1-p| = |g(p_0)-g(p)| \approx M|p_0-p|^2 < 1/M.$$

Now

$$\begin{aligned}
|p_{i+1}-p| &\approx (1/M)(M|p_i-p|)^2 \\
&\approx (1/M)(M|p_{i-1}-p|)^4 \\
&\dots
\end{aligned}$$

$$\approx (1/M)(M|p_0-p|)^{2^{i+1}} < 1/M.$$

This is converging to zero, but faster than in the attracting case. Unfortunately, the above heuristic argument is more difficult to prove. If we were working in  $\mathbf{R}$ , then we could have used Taylor's Theorem to get  $|g(z)-g(p)| = |g''(x)/2!||z-p|^2$  for some  $x$  between  $z$  and  $p$ . However, the simplest case of Taylor's Theorem, the Mean Value Theorem, is not valid for complex valued functions ([Co], p. 305). A more rigorous argument can be found in Proposition 8 in ([K], p.64).

We can classify the two types of convergence without relying on the value of  $\lambda$ .

Definition:

A sequence  $[p_n]$  *converges linearly* to a point  $p$  if there exists an  $M < 1$  such that, for all  $i = 0, 1, 2, \dots$ ,  $|p_{i+1}-p| < M |p_i-p|$ . •

Definition:

A sequence  $[p_n]$  *converges quadratically* to a point  $p$  if there exists an  $M \in \mathbf{R}$  such that, for all  $i = 0, 1, 2, \dots$ ,  $|p_{i+1}-p| < M |p_i-p|^2$ . •

Again, quadratic convergence is faster than linear convergence, which explains the term “superattracting.”

Newton's method is an iterative function with quadratic convergence that is used to find roots of an analytic function  $f$ .

Definition:

For an analytic function  $f: \mathbf{C} \rightarrow \mathbf{C}$  and some  $z \in \mathbf{C}$ , Newton's method is the functional iteration of

$$N_f(z) = z - f(z)/f'(z)$$



(The subscript will be omitted when it is clear from context.) •

Theorem 2:

If  $p$  is a simple root of  $f$ , then  $p$  is a superattracting fixed point of  $N_f$ .

Proof:

Suppose that  $p$  is a simple root of  $f$  so that  $f(p) = 0$  and  $f'(p) \neq 0$ .

$$N_f(p) = p - \frac{f(p)}{f'(p)} = p - 0 = p$$
$$N_f'(p) = 1 - \frac{[f'(p)]^2 - f(p)f''(p)}{[f'(p)]^2} = 1 - 1 + 0 = 0$$

So  $p$  is a superattracting fixed point for  $N_f$ . ♦

This is the property that makes Newton's method work so well. Once you get inside a certain neighborhood of the root of  $f$ , the iterations of  $N_f$  will converge quickly to the root. When we study the basins of attraction for Newton's method, we find some quite interesting behavior.

## II. Rational functions and Julia sets

The study of Newton's method, and all rational functions, is interesting because the attracting basins are not the only sets of points present in the plane.

Theorem:

For any attracting fixed point  $p$  of a continuous function  $g$ , the basin,  $A(p)$  is open.

Proof:

From Theorem 1 we know that there is a neighborhood  $U$  of  $p$  such that  $U \subseteq A(p)$ .

Since  $U$  is a neighborhood, it is open. Since  $g$  is continuous,  $g^{-1}(U)$  is open and  $g^{-1}$

$g^{-n}(U)$  is open for all  $n = 1, 2, \dots$ . Then

$$\bigcup_{n=1}^{\infty} g^{-n}(U)$$

is open. Suppose we have a point  $u$  in this union. Then iterating  $g$  with  $u$  as the initial point will eventually yield a point in  $U$ , and thus will converge to  $p$ .

Therefore, 
$$\bigcup_{n=1}^{\infty} g^{-n}(U) \subseteq A(p)$$

Now suppose we have a point  $a$  in  $A(p)$ . Then there exists an  $n \in \{1, 2, \dots\}$  such that  $g^{(n)}(a) \in U$ . Then 
$$a \in \bigcup_{n=1}^{\infty} g^{-n}(U)$$

Therefore, 
$$\bigcup_{n=1}^{\infty} g^{-n}(U) \supseteq A(p)$$

Therefore, the union and the basin are equal, so  $A(p)$  is open. ◆

Then, the complement of  $A(p)$  must be closed.

There are points in the complex plane that are not members of basins of attraction for attracting fixed points of a rational function  $R$ . Based on algebra, there are solutions to  $R^{(n)}(p) = p$ , yielding periodic points. These points can not be in the fixed point basins, because that would imply that they converge to the fixed point, contradicting the periodicity. It is difficult to find these periodic points because they usually are repelling, so that  $|\lambda| > 1$ . The closure of these repelling periodic points forms the complement of the attracting basins.

Definition: ([K], p.64)

The *Julia set*  $J_R$  for the rational function  $R: \mathbf{C} \rightarrow \mathbf{C}$  is the closure of the repelling periodic points.

(The subscript will be omitted when it is clear from context.) •

Theorem 3: ([PS], p. 208)

For a rational function  $R$ , that has attracting periodic points,  $z_1, z_2, \dots, z_k$ ,

$$J_R = \partial A(z_i), i = 1, 2, \dots, k.$$

This tells us that, if we are on the boundary of one basin of attraction, then we must be on the boundary of all basins of attraction. We can bring much of what we have discussed so far together by looking at Figure 1. (See Appendix for programs used to create figures.)

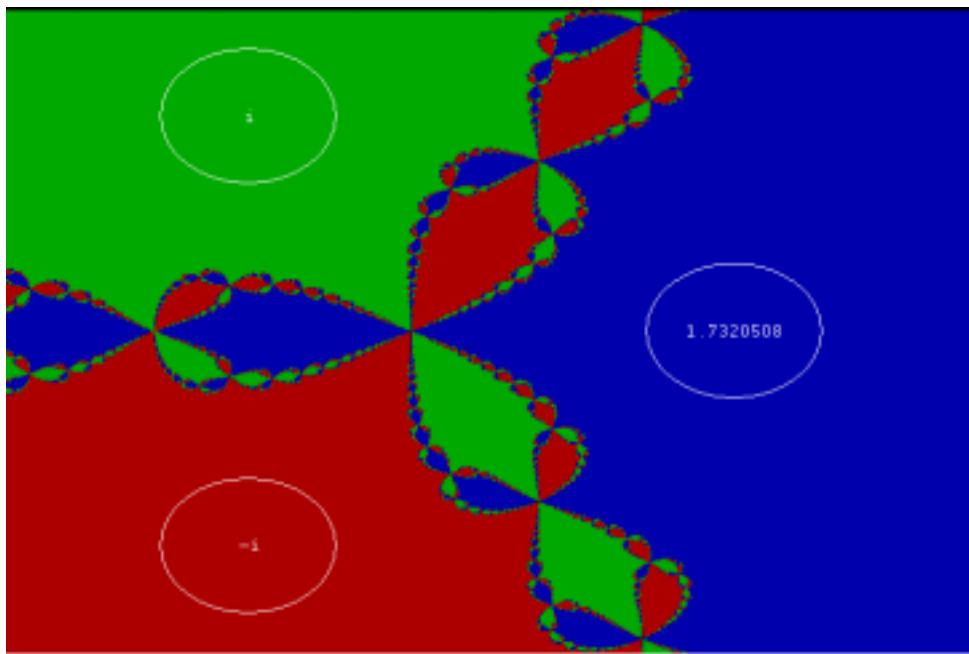


Figure 1  
 $q(z) = (z-i)(z+i)(z-1.7320508)$

This is the dynamic plane of a polynomial  $p$  with roots at  $i$ ,  $-i$ , and  $1.7320508$  (approximately  $\sqrt{3}$ ) in the complex plane. Each pixel is used as an initial value for iterating  $N_p$ . Once iteration yields a value within the white circle around a root, the initial value pixel is colored according to that root. Green represents  $A(i)$ , red represents  $A(-i)$ , and blue represents  $A(1.7320508)$ . We can see that the boundary of each basin touches each of the other basins. The boundary is also where we find the Julia set, which has even more interesting behavior.

Definition: ([Cr], p. 149-50)

Let  $g: X \rightarrow X$  be an iterative function where  $X$  is a metric space  $(X, d)$ . The function  $g$  is *chaotic* if

- $g$  has sensitive dependence on initial conditions:

$\exists \delta > 0$  such that if  $x \in X$  and  $U$  is an open set containing  $x$ , then  $\exists n > 0$  and  $y \in U$  such that  $d(g^{(n)}(x), g^{(n)}(y)) > \delta$ ,

- $g$  is transitive:

$\forall$  nonempty open  $U, V, \exists n \geq 0$  such that  $g^{(n)}(U) \cap V \neq \emptyset$ ,

- periodic points of  $g$  are dense in  $X$ :

$\forall x \in X, \forall$  neighborhood  $U$  of  $x, \exists$  at least one periodic point  $p$  such that  $p \in U$ . •

Theorem 4: ([K], p. 62)

If  $J_R$  is the Julia set for a rational function  $R$  and  $z \in J_R$ , then  $\cup_n R^{(-n)}(z)$  is dense in  $J_R$ .

We will not prove Theorem 4, but use it to prove chaos.

Theorem 5:

If  $J_R$  is the Julia set for a rational function  $R$ , then  $R$  is chaotic on  $J_R$ .

Proof:

Let  $R: J_R \rightarrow J_R$  be a rational function.

By Theorem 4,  $\forall z \in J_R \cup_n R^{(-n)}(z)$  is dense in  $J_R$ , i.e.  $\forall$  neighborhood  $U, \exists$  at least one point  $x \in \cup_n R^{(-n)}(z)$  such that  $x \in U$ .

- $R$  has sensitive dependence on initial conditions:

By Proposition 4 in ([K], p. 62), there are infinitely many repelling periodic points in the Julia set. Choose two distinct repelling periodic cycles,  $C_1 = \{c_{11}, c_{12}, \dots, c_{1r}\}$  and  $C_2 = \{c_{21}, c_{22}, \dots, c_{2s}\}$ .

Define  $\delta < [\min\{|c_{1a} - c_{2b}|: 1 \leq a \leq r, 1 \leq b \leq s\}]/2$ .

Let  $x \in J_{\mathbb{R}}$  and  $U$  be a neighborhood of  $x$ , and let  $j \in C_1$  and  $k \in C_2$ . Then,  $\exists n$  such that  $R^{(-n)}(j) \in U$  and  $\exists m$  such that  $R^{(-m)}(k) \in U$ .

Let  $u = R^{(-n)}(j)$  and  $v = R^{(-m)}(k)$ . Without loss of generality, suppose  $m > n$ .

Now consider  $R^{(m)}(v)$  and  $R^{(n+(m-n))}(u) = R^{(m)}(u)$ .

$R^{(m)}(u) \in C_1$  and  $R^{(m)}(v) \in C_2$  so  $|R^{(m)}(u) - R^{(m)}(v)| > 2\delta$ .

Now consider  $R^{(m)}(x)$ .

Suppose, for contradiction,  $|R^{(m)}(u) - R^{(m)}(x)| \leq \delta$  and  $|R^{(m)}(v) - R^{(m)}(x)| \leq \delta$ .

Then, by the Triangle Inequality,

$$|R^{(m)}(u) - R^{(m)}(v)| \leq |R^{(m)}(u) - R^{(m)}(x)| + |R^{(m)}(v) - R^{(m)}(x)| \leq 2\delta.$$

Contradiction:  $2\delta < 2\delta$ .

Therefore, either  $|R^{(m)}(u) - R^{(m)}(x)| > \delta$  or  $|R^{(m)}(v) - R^{(m)}(x)| > \delta$ .

- $g$  is transitive:

Let  $U, V$  be nonempty open sets in  $J_{\mathbb{R}}$  and let  $x \in V$ . Then,  $\exists$  at least one point  $u$

$\in \cup_n R^{(-n)}(x)$  such that  $u \in V$ . So  $x = R^{(m)}(u)$  for some  $m \in \{1, 2, \dots\}$

implies  $x \in R^{(m)}(U)$ . Therefore,  $R^{(m)}(U) \cap V \neq \emptyset$ .

- periodic points of  $g$  are dense in  $J_{\mathbb{R}}$ :

Let  $j \in J_{\mathbb{R}}$  and  $U$  is a neighborhood of  $j$ . Then, by definition of  $J_{\mathbb{R}}$ ,  $\exists$  at

least one point  $x$  such that  $x \in U$  and is a repelling periodic point.  $\blacklozenge$

Now that we know some information about the behavior of repelling periodic points, is it possible to have attracting periodic points? It turns out that it is possible, so how can we find them?

Theorem 6: ([K], p.66).

If  $R$  is a rational function with an attracting periodic cycle, then the basin of attraction of the periodic cycle contains at least one critical point of  $R$ .

Now we can use this information to study the dynamics of the rational iterative function used in Newton's method, and specifically where we can find attracting periodic cycles.

### III. Cubic polynomials with real coefficients

We will focus on the dynamics of cubic polynomials with real coefficients. By the Intermediate Value Theorem, every polynomial of odd degree with real coefficients has at least one real root. This allows cubic polynomials to be written in the form  $q(z) = (z-r_1)(az^2+bz+c)$ , where  $r_1$  is the real root and  $a, b, c \in \mathbf{R}$ . Using the quadratic formula, the remaining roots are

$$r_2, r_3 = (-b \pm \sqrt{b^2 - 4ac}) / 2a$$

The cubic can have three distinct real roots, two real roots where one has multiplicity two, or one real root with multiplicity three. Since any polynomial with real coefficients and a complex root must also have a root at the complex conjugate, the only other possibility is one real root and two complex conjugate roots.

An affine mapping can be used to write the polynomial in a form with only one parameter instead of four.

Theorem 7:

For any cubic polynomial,  $q(z)$ , with real coefficients, there exists  $\alpha, \beta, \gamma \in \mathbf{R}$  and  $\alpha, \gamma \neq 0$  such that  $\gamma q(\alpha z + \beta)$  is equal to one of

$$q(z) = z^3,$$

$$q(z) = z^2(z-1),$$

$$q(z) = z(z-1)(z-c) \text{ for } 0 < c < 1,$$

$$q(z) = (z-i)(z+i)(z-c) \text{ for } c \in \mathbf{R}.$$

Proof:

Case 1:  $q(z)$  has one real root with multiplicity three.

Then,  $q(z) = k(z-r)^3$  for some  $k, r \in \mathbf{R}$ .

Let  $\gamma = 1/k, \alpha = 1, \beta = r$ .

$$\text{Then, } \gamma q(\alpha z + \beta) = \frac{1}{k} q(z+r) = \frac{k}{k} (z+r-r)^3 = z^3$$

Case 2:  $q(z)$  has one real root with multiplicity two and one real root with multiplicity one.

Then,  $q(z) = k(z-r)^2(z-s)$  for some  $k, r, s \in \mathbf{R}$ .

Let  $\gamma = 1/(k(s-r)^3), \alpha = s-r, \beta = r$ .

Then,

$$\begin{aligned}
\gamma q(\alpha z + \beta) &= \frac{1}{k(s-r)^3} q((s-r)z+r) \\
&= \frac{k}{k(s-r)^3} ((s-r)z+r-r)^2 ((s-r)z+r-s) \\
&= \left(\frac{s-r}{s-r} z\right)^2 \left(\frac{s-r}{s-r} z + \frac{r-s}{s-r}\right) = z^2 (z-1)
\end{aligned}$$

Case 3:  $q(z)$  has three distinct real roots.

Then,  $q(z) = k(z-r)(z-s)(z-t)$  for some  $k, r, s, t \in \mathbf{R}$  where  $r < s < t$  (without loss of generality).

Let  $\gamma = 1/(k(t-r)^3)$ ,  $\alpha = t-r$ ,  $\beta = r$ ,  $c = (s-r)/(t-r)$ .

Then,

$$\begin{aligned}
\gamma q(\alpha z + \beta) &= \frac{1}{k(t-r)^3} q((t-r)z+r) \\
&= \frac{k}{k(t-r)^3} ((t-r)z+r-r)((t-r)z+r-s)((t-r)z+r-t) \\
&= \left(\frac{t-r}{t-r} z\right) \left(\frac{t-r}{t-r} z + \frac{r-s}{t-r}\right) \left(\frac{t-r}{t-r} z + \frac{r-t}{t-r}\right) \\
&= z(z-c)(z-1)
\end{aligned}$$

Case 4:  $q(z)$  has two complex conjugate roots and one real root.

Then,  $q(z) = k(z-(r+si))(z-(r-si))(z-t)$  for some  $k, r, s, t \in \mathbf{R}$ .

Let  $\gamma = 1/(ks^3)$ ,  $\alpha = s$ ,  $\beta = r$ ,  $c = (t-r)/s$ .

Then,

$$\begin{aligned}
\gamma q(\alpha z + \beta) &= \frac{1}{ks^3} q(sz+r) \\
&= \frac{k}{ks^3} (sz+r-r-si)(sz+r-r+si)(sz+r-t) \\
&= \left(\frac{sz-si}{s}\right) \left(\frac{sz+si}{s}\right) \left(\frac{sz+r-t}{s}\right) \\
&= (z-i)(z+i)(z-c)
\end{aligned}$$

These four cases exhaust all possibilities.  $\blacklozenge$



The dynamics of Newton's method on a given analytical function are preserved by affine mappings.

Theorem 8:

If  $f$  is analytic and  $g(z) = \gamma f(\alpha z + \beta)$  where  $\alpha, \beta, \gamma \in \mathbf{R}$  and  $\alpha, \gamma \neq 0$ , then

$$T \circ N_g \circ T^{-1} = N_f$$

where  $T(z) = \alpha z + \beta$ .

Proof:

Let  $f$  be analytic and  $g(z) = \gamma f(T(z)) = \gamma f(\alpha z + \beta)$  where  $\alpha, \beta, \gamma \in \mathbf{R}$  and  $\alpha, \gamma \neq 0$ .

$$\begin{aligned} T \circ N_g \circ T^{-1}(z) &= \alpha \left[ T^{-1}(z) - \frac{g(T^{-1}(z))}{g'(T^{-1}(z))} \right] + \beta \\ &= \alpha \left[ \frac{z - \beta}{\alpha} - \frac{\gamma f(T(T^{-1}(z)))}{\gamma (f \circ T)'(T^{-1}(z))} \right] + \beta \\ &= \alpha \left[ \frac{z - \beta}{\alpha} - \frac{f(z)}{f'(T(T^{-1}(z))) \cdot T'(T^{-1}(z))} \right] + \beta \\ &= \alpha \left[ \frac{z - \beta}{\alpha} - \frac{f(z)}{f'(z) \cdot \alpha} \right] + \beta \\ &= z - \beta - \frac{f(z)}{f'(z)} + \beta \\ &= z - \frac{f(z)}{f'(z)} = N_f(z) \end{aligned}$$



Since the dynamics of Newton's method are unchanged by the affine transformation and scaling, we can use our new parameterizations to study each of the four cases of cubic polynomials with real coefficients. This will make our job much easier since we only have to deal with one parameter,  $c$ , for the last two cases, and no parameters for the first two cases. We can begin by looking at some representative examples for each case.

Let us first consider the case in which a polynomial has one real root with multiplicity three. We know from Theorems 7 and 8 that we only need to study the behavior of  $q(z) = z^3$ . Newton's method for this polynomial gives us

$$N(z) = z - \frac{z^3}{3z^2} = \frac{2z}{3}$$

So, using any point in the complex plane as an initial value for  $N(z)$ , iteration will converge to the root at zero.

For the case of a polynomial with a real root with multiplicity two and a real root with multiplicity one, we can study the dynamics of  $q(z) = z^2(z-1)$ . As we can see from Figure 2, the dynamic plane for this  $q(z)$ , there is a curve that separates the basins  $A(0)$  and  $A(1)$ .

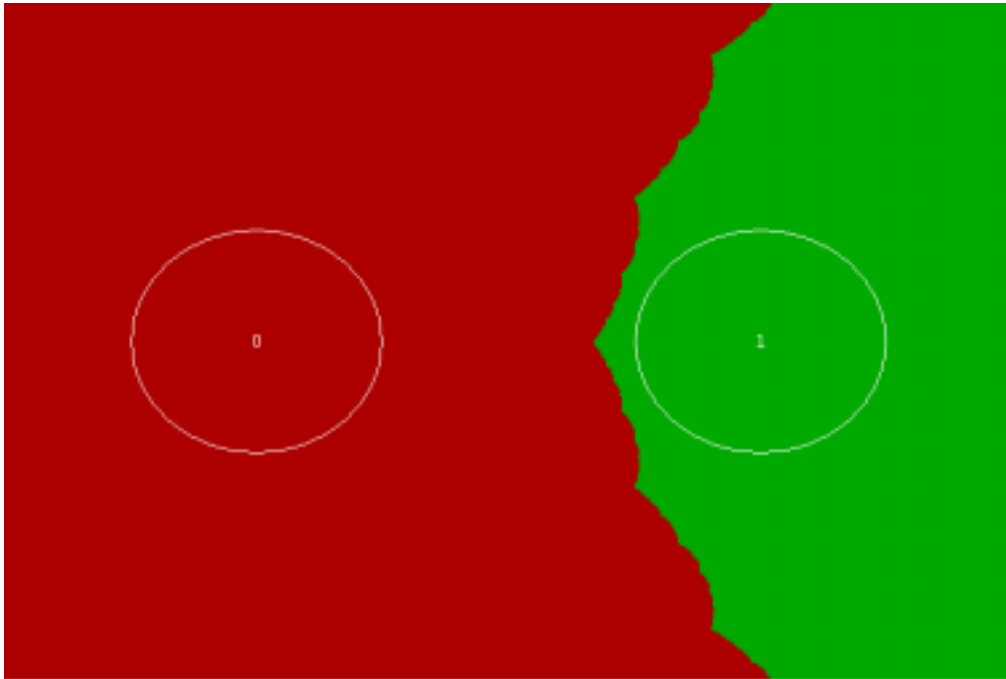


Figure 3  
 $q(z) = z^2(z-1)$

It appears that all points belong to one of these two basins or to the Julia set that forms the dividing curve.

An example of the dynamic plane of a polynomial with three distinct real roots is given in Figure 3 where  $q(z) = z(z-1)(z-c)$  and  $c = .5$ .

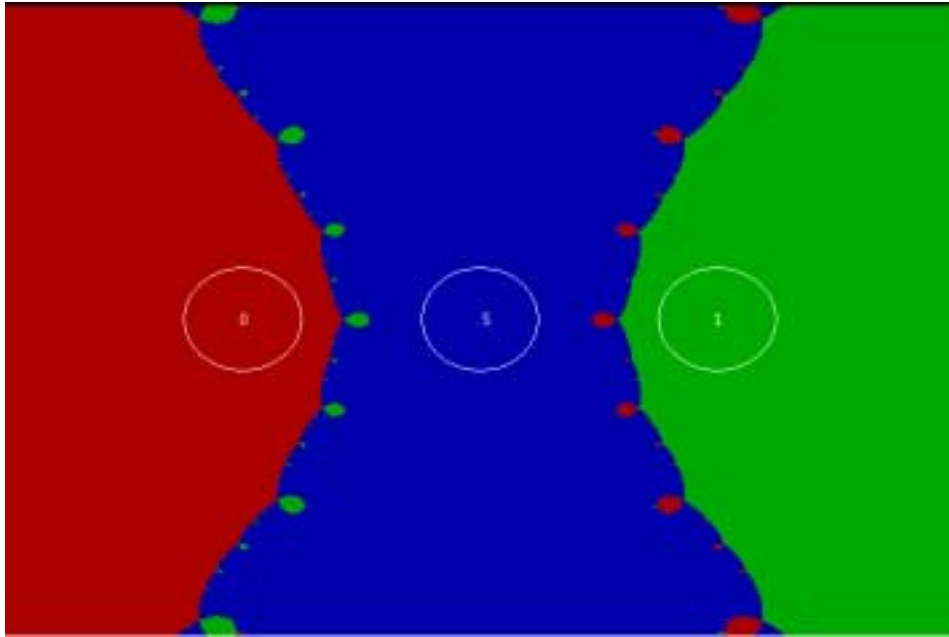


Figure 3  
 $q(z) = z(z-1)(z-0.5)$

Along the boundaries, there are “bulbs” of a given basin, which are covered in smaller bulbs of another basin. Zooming in on these bulbs, we find that there is an infinite pattern of “bulbs upon bulbs” of alternating basins. We know from Theorem 3 that these bulbs result from the shared boundary of the three basins.

We saw one example of the case with two complex conjugate roots and a real root in Figure 1. Another example of  $q(z) = (z-i)(z+i)(z-c)$  is given in where  $c = 2.7$ .

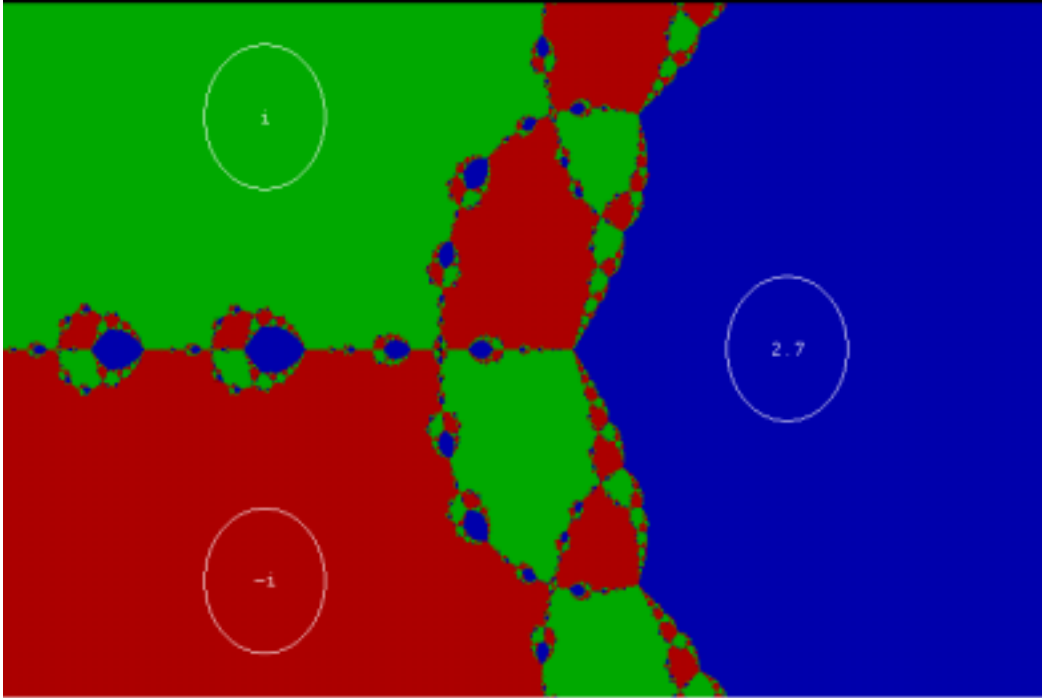


Figure 4  
 $q(z) = (z-i)(z+i)(z-2.7)$

Again we notice the result of Theorem 3 that  $\partial(A(i)) = \partial(A(-i)) = \partial(A(2.7))$ . A student using Newton's method to find a root would like to know that, upon picking a real initial value and calculating iterations, they would not obtain something strange, such as one of the complex roots. Newton's method is closed under the reals. With probability one, picking a real value to iterate under Newton's method will yield convergence to the real root. It is possible that there is a point on the real line that is a member of the Julia set. However, these points are repelling and the Julia set has measure zero, so they do not present a problem ([B] p. 145).

What other behavior could a student encounter when using Newton's method? We will find in the next section that some polynomials have attracting periodic cycles. Instead of just looking at examples, we will try to find these cycles analytically.

#### IV. Attracting periodic cycles and critical points

We know that the only cubic polynomials with real coefficients that could have attracting periodic cycles fall under either the case with three distinct real roots or the case with two complex conjugate roots and one real root. Now we need to determine which values of  $c$  for each case give polynomials with attracting periodic cycles. Since any attracting periodic cycle must attract at least one critical point by Theorem 6, we need only to find whether the critical points of the Newton's method function for that polynomial attract to a periodic point or a root.

$$N_q(z) = z - \frac{q(z)}{q'(z)}$$
$$N_q'(z) = 1 - \frac{q'(z)q'(z) - q(z)q''(z)}{[q'(z)]^2}$$
$$N_q''(z) = \frac{q(z)q''(z)}{[q'(z)]^2}$$

The critical points of the Newton's method function are where  $q(z) = 0$  or  $q''(z) = 0$ . The condition  $q(z) = 0$  occurs at each of the roots, which are fixed points of the Newton's method function, so this will not yield an attracting periodic cycle. Thus we only need to consider the inflection point where  $q''(z) = 0$  in order to find which polynomials have attracting periodic points.

First, we can assure ourselves that the case with one root of multiplicity two and one root of multiplicity one does not have any attracting periodic cycles, as we concluded from Figure 2. The form for this case is  $q(z) = z^2(z-1)$ .

$$q(z) = z^2(z-1) = z^3 - z^2$$

$$q'(z) = 3z^2 - 2z$$

$$q''(z) = 6z - 2 = 0$$

$$z_0 = \frac{1}{3}$$

Newton's method yields the sequence { .3333, .1111, .0519, .0252, .0124, .0062, .0031, .0015, .0008, .0004, .0002, .0001, 0 } which converges to zero. (These values are rounded to four decimal places from values obtained using Mathematica.)

For polynomials with three real roots, we have the form  $q(z) = z(z-1)(z-c)$ .

$$q(z) = z(z-1)(z-c) = z^3 - z^2 - cz^2 + cz$$

$$q'(z) = 3z^2 - 2z - 2cz + c$$

$$q''(z) = 6z - 2 - 2c = 0$$

$$z_0 = \frac{c+1}{3}$$

Using this  $z_0$  as our initial point in Newton's method, if it converges to a root, then there are no attracting periodic cycles.

Theorem 9:

For all polynomials of the form  $q(z) = z(z-1)(z-c)$  where  $0 < c < 1$ , there are no attracting periodic cycles having period greater than one. (i.e. The only attracting periodic cycles are the fixed points at the roots.)

Lemma:

For all polynomials of the form  $q(z) = z(z-1)(z-c)$  where  $0 < c < 1$ , the sequence obtained using Newton's method with the initial value  $(c+1)/3$  will converge to  $c$ .

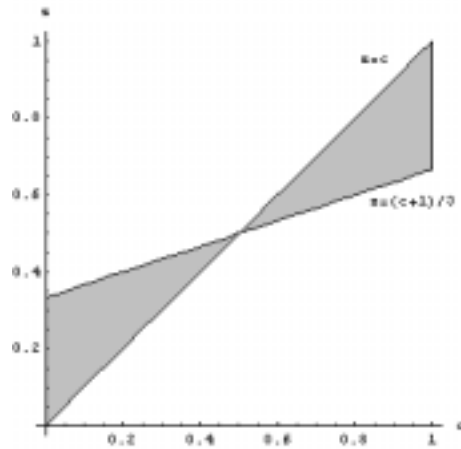
Proof:

Let  $q(z) = z(z-1)(z-c)$  where  $0 < c < 1$ . We want to show  $|N_q'(z)| < 1$  for  $z$  between (and including)  $c$  and  $(c+1)/3$  so that the region falls into the neighborhood of  $c$  described in Theorem 1 and is, therefore, contained in  $A(c)$ .

We must show  $-1 < N_q'(z) < 1$ .

$$N_q'(z) = \frac{2z(z-1)(c^2 - 4zc + c + 3z^2 - z)}{(-2zc + c + 3z^2 - 2z)^2}$$

The region of interest,  $0 < c < 1$  and  $z$  between (and including)  $c$  and  $(c+1)/3$  is shown:



Left inequality:

The inequality  $-1 < N'(q)$  is equivalent to

$$0 < 15z^4 - 20cz^3 - 20z^3 + 6c^2z^2 + 24cz^2 + 6z^2 - 6c^2z - 6cz + c^2$$

We define this quantity as  $\text{hopePos}(c, z)$  and look for its critical points. Using Mathematica, we find that the only real-valued critical points on this region are at the points  $(0,0)$ ,  $(\frac{1}{2}, \frac{1}{2})$ , and  $(1,1)$ . We find  $\text{hopePos}(0,0) = 0$ ,  $\text{hopePos}(\frac{1}{2}, \frac{1}{2}) = 1/16$ , and  $\text{hopePos}(1,1) = 0$ . Since any relative extremum on the region must be a critical point of  $\text{hopePos}(c, z)$  by Theorem 12.16 in ([LHE], p.887), we know that

extreme values must occur on the boundary, which includes the three points above. Our region only includes  $0 < c < 1$ , so having  $\text{hopePos}(0,0) = \text{hopePos}(1, 1) = 0$  is not a problem for having strictly positive values on the region. Now we must check the borders of the region.

Let  $c = z$ . Then  $\text{hopePos}(z) = z^2 - 2z^3 + z^4$ . The critical points are at  $z = 0, \frac{1}{2}$ , and  $1$ , which correspond to the points found above.

Let  $c = 3z - 1$ . Then  $\text{hopePos}(z) = (3z^2 - 3z + 1)^2$ . The only real critical point is at  $z = \frac{1}{2}$ . Again, this corresponds to  $c = 3(\frac{1}{2}) - 1 = \frac{1}{2}$ .

Let  $c = 0$ . Then  $\text{hopePos}(z) = 6z^2 - 20z^3 + 15z^4$ . For this border,  $0 < z < \frac{1}{3}$ . The critical point in this range is  $z = 0.276393$  and we calculate  $\text{hopePos}(0.276393) = 0.123607$ . We also must check the endpoint at  $z = \frac{1}{3}$ , and we find  $\text{hopePos}(\frac{1}{3}) = \frac{1}{9}$ .

Let  $c = 1$ . Then  $\text{hopePos}(z) = 1 - 12z + 36z^2 - 40z^3 + 15z^4$ . For this border,  $\frac{2}{3} < z < 1$ . The critical point in this range is  $z = 0.723607$  and we find  $\text{hopePos}(0.723607) = 0.123607$ . Again, we must check the endpoint at  $z = \frac{2}{3}$  and we find  $\text{hopePos}(\frac{2}{3}) = \frac{1}{9}$ .

Since the polynomial only reaches a minimum of zero at  $(0, 0)$  and  $(1, 1)$  on the boundary surrounding the region, we know that  $\text{hopePos}(c, z) > 0$  for all values in our region.

Right inequality:

The inequality  $N'(q) < 1$  is equivalent to

$$0 > -3z^4 + 4cz^3 + 4z^3 - 2c^2z^2 - 4cz^2 - 2z^2 + 2c^2z + 2cz - c^2$$



We define this quantity as  $\text{hopeNeg}(c, z)$ . The roots of  $\text{hopeNeg}(c, z)$  have  $c$  values equal to  $(2z^3 - 2z^2 + z \pm \sqrt{(-2z^6 + 6z^5 - 7z^4 + 4z^3 - z^2)}) / (2z^2 - 2z + 1)$ . The expression under the radical simplifies to  $-(z-1)^2 z^2 (2z^2 - 2z + 1)$ , and the latter quadratic polynomial is positive for all real  $z$ . This shows that the expression under the radical is negative for  $0 < z < 1$ , so the roots in our region are complex. Thus,  $\text{hopeNeg}(c, z)$  always has the same sign on our region. We can choose the value of  $\text{hopeNeg}(c, z)$  at any point in our region and values at all points will have the same sign. We find  $\text{hopeNeg}((c+1)/3, c) = -(1-c+c^2)^2/9$ , so  $\text{hopeNeg}(c, z) < 0$  for all values in our region.

Therefore,  $|\mathbb{N}_q'(z)| < 1$  on the desired region. ◆

Proof of Theorem 9:

Let  $q(z) = z(z-1)(z-c)$  where  $0 < c < 1$ . By the Lemma, Newton's method with the initial value  $(c+1)/3$  converges to the root  $c$ . By Theorem 6,  $q(z)$  has no attracting periodic cycles. ◆

Since no polynomials with three distinct real roots have attracting periodic cycles, the dynamic planes for all  $0 < c < 1$  will have similar behavior to Figure 3.

For polynomials with two complex conjugate roots and one real root, the form is  $q(z) = (z-i)(z+i)(z-c)$ .

$$\begin{aligned}
 q(z) &= (z-i)(z+i)(z-c) = z^3 - cz^2 + z - c \\
 q'(z) &= 3z^2 - 2cz + 1 \\
 q''(z) &= 6z - 2c = 0 \\
 z^0 &= \bar{c}
 \end{aligned}$$

We can use this  $z_0$  as an initial point in Newton's method in order to create a parameter axis of all values of  $c$  so that we can find which polynomials of this type have periodic cycles. Figure 5 is one such parameter axis for values of  $c$  between  $-6$  and  $6$ .

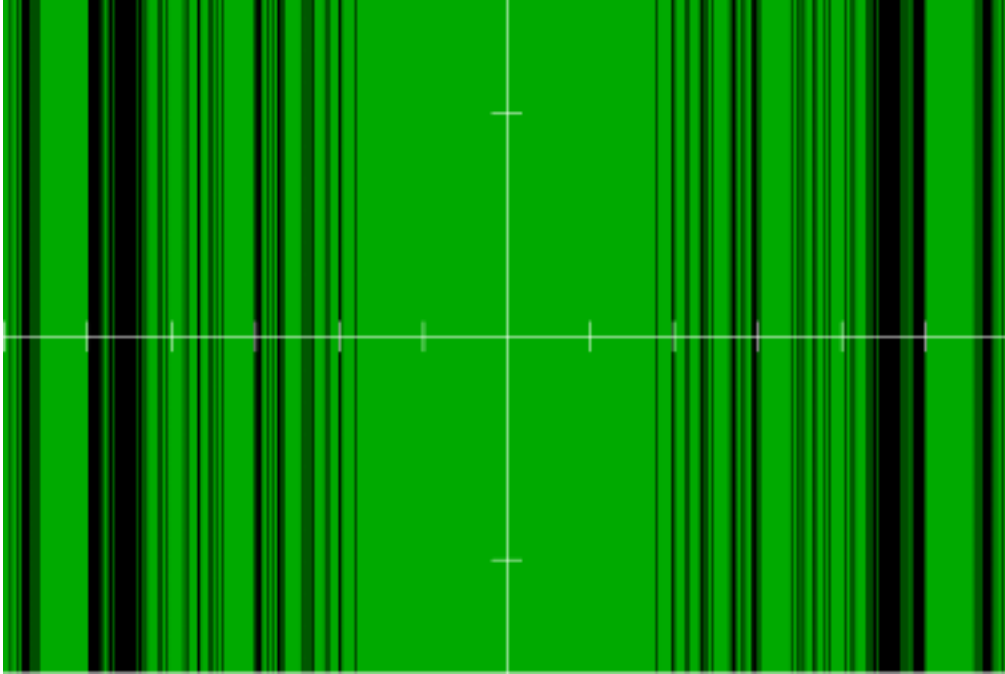


Figure 5  
c-axis range  $(-6, 6)$

For each value of  $c$  on the axis, the point  $c/3$  is used as the initial value in Newton's method on the polynomial with the given  $c$ . If this converges to a root, the line at  $c$  is colored green. If no root is reached after a given number of iterations, it may converge to an attracting periodic cycle, and the line at  $c$  remains black.

From the parameter axis, we have chosen two  $c$  values that fall in the black area for examples. Figure 6 shows the dynamic plane for Newton's method on the polynomial  $q(z) = (z-i)(z+i)(z-c)$  where  $c = 2.95$ .

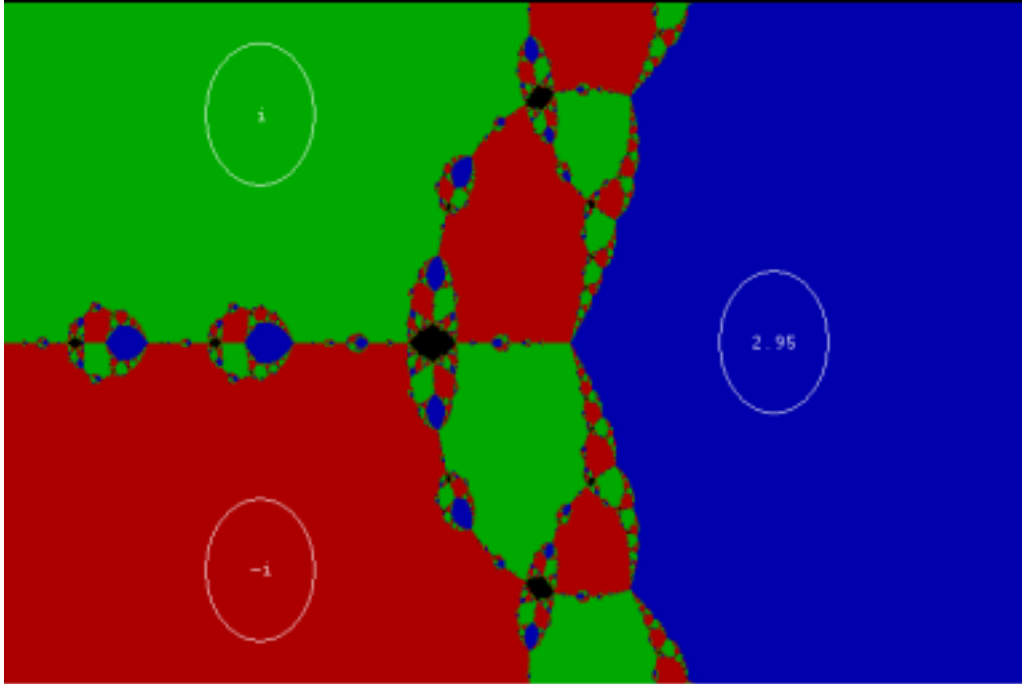


Figure 6  
 $q(z) = (z-i)(z+i)(z-2.95)$

Again, each color represents the basin of attraction for a given root. If a given pixel used as an initial value in Newton's method does not converge to one of the roots after a certain number of iterations, that pixel remains black. We can see that the areas of black intersect the real axis, yielding intervals of points that can be chosen as initial values for Newton's method that will not converge to a root. This shows that, if an initial value is chosen at random from some bounded region on the real axis, the probability of that value being in a black region is greater than zero. In Figure 7, we have zoomed in so we can see more of these black areas. (The large black area towards the right is the same as the black area toward the center of Figure 6.)

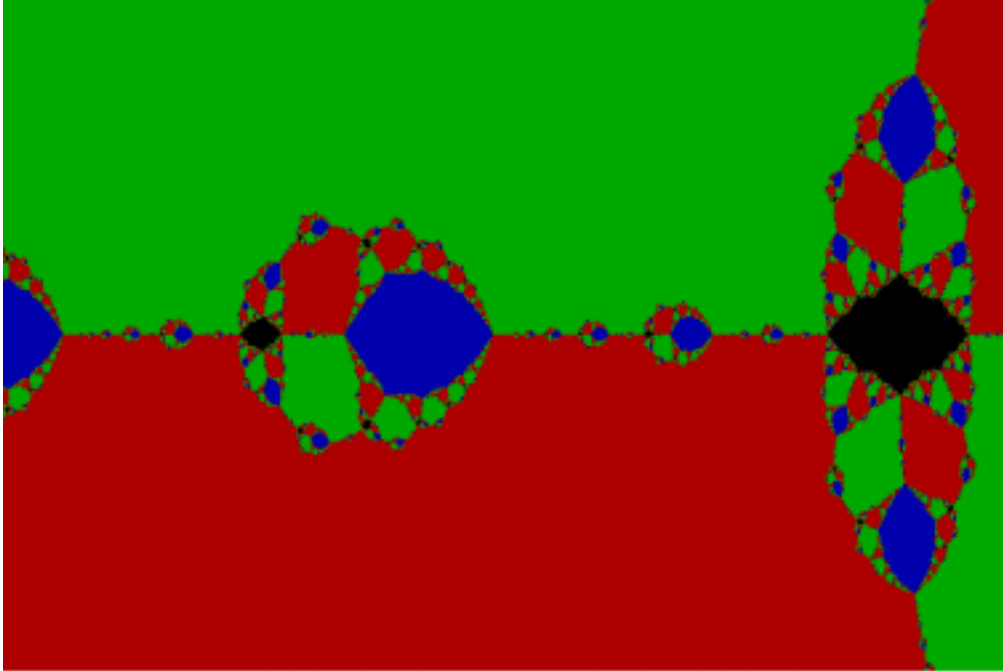


Figure 7  
 $q(z) = (z-i)(z+i)(z-2.7)$   
 zoomed

We can calculate the periodic cycle that the critical point attracts to. Using  $z_0 = c/3 \approx 0.98333$ , we obtain a sequence which converges to the period-3 cycle  $\{1.014, -1.05479, -0.253687\}$  after about 65 iterations.

The second  $c$  value that we chose from the parameter axis is  $c = 4.6$ . The dynamic plane for the polynomial  $q(z) = (z-i)(z+i)(z-4.6)$  can be seen in Figure 8.

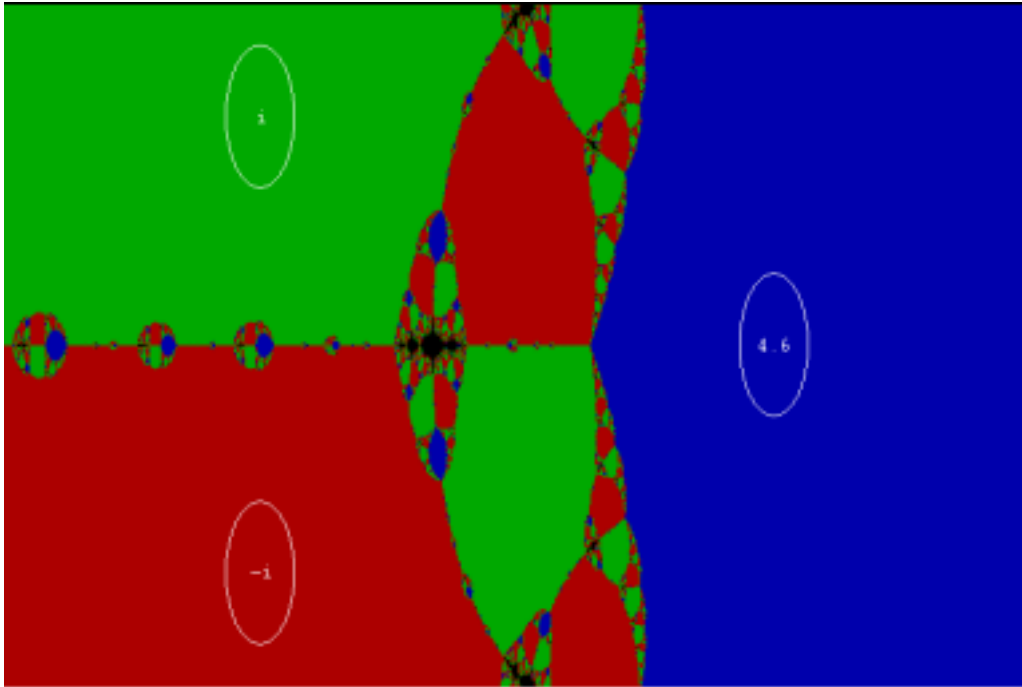


Figure 8  
 $q(z) = (z-i)(z+i)(z-4.6)$

Again we see areas of black that intersect the real axis. Here we see an interesting pattern formed by the largest area of black. We calculate the sequence using the critical point  $z_0 = c/3 \approx 1.5333$  and obtain the period-4 cycle  $\{1.71915, -0.196079, 1.50999, -0.164821\}$  after about 80 iterations.

We can get a better idea of what periodic cycles will occur for a given  $c$  value in the case  $q(z) = (z-i)(z+i)(z-c)$  if we create a bifurcation diagram. We create this diagram by plotting iterates of Newton's method using  $c/3$  as the initial value. At each iteration, for  $i = 101$  to  $200$ , we receive the value  $z_i$ , and plot the points  $(c, z_i)$ . This diagram can be compared to Figure 5, expecting the green areas to correspond with a plot of the line  $y = c$  and the black areas to correspond with points representing a periodic cycle.

We only need to study values where  $c \geq 0$  due to symmetry. Consider the polynomial  $q_c(z) = (z-i)(z+i)(z-c)$  and the iteration of Newton's method for  $q_c(z)$  with the

initial value  $z_0$ . Now consider the polynomial  $q_c(z) = (z-i)(z+i)(z+c)$  and the iteration under Newton's method for  $q_c(z)$  with initial value  $-z_0$ .

$$N_{q_c}(z_0) = z_0 - \frac{z_0^3 - z_0^2 c + z_0 - c}{3z_0^2 - 2z_0 c + 1}$$

$$N_{q_{-c}}(-z_0) = -z_0 - \frac{-z_0^3 + z_0^2 c - z_0 + c}{3z_0^2 - 2z_0 c + 1}$$

$$= -N_{q_c}(z_0)$$

This shows that the dynamics for the polynomial with parameter  $-c$  will be the same as those for  $c$ . Figure 9 shows the bifurcation axis for  $0 \leq c \leq 6$ .

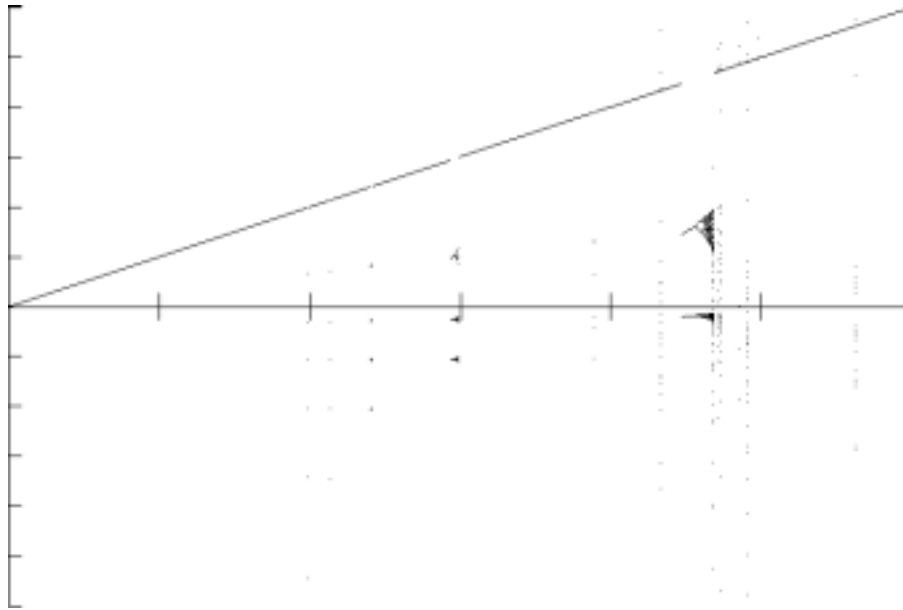


Figure 9  
Bifurcation axis (0, 6)

Using the zoom function of the program, we look at the area surrounding  $c = 2.95$  as shown in Figure 10.

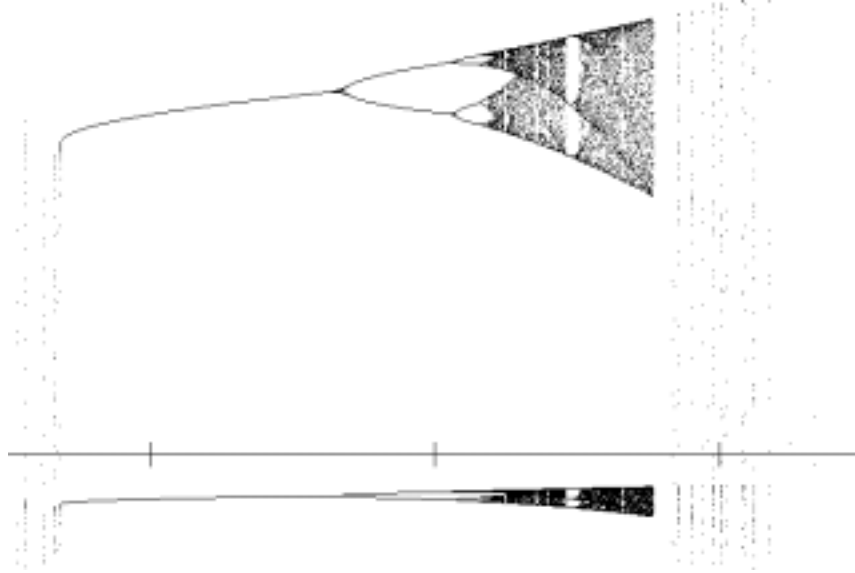


Figure 10  
Bifurcation axis (2.89, 3.01)

We see the three points that represent the attracting periodic 3-cycle that we found by iteration. Figure 11 shows the area around  $c = 4.6$ .

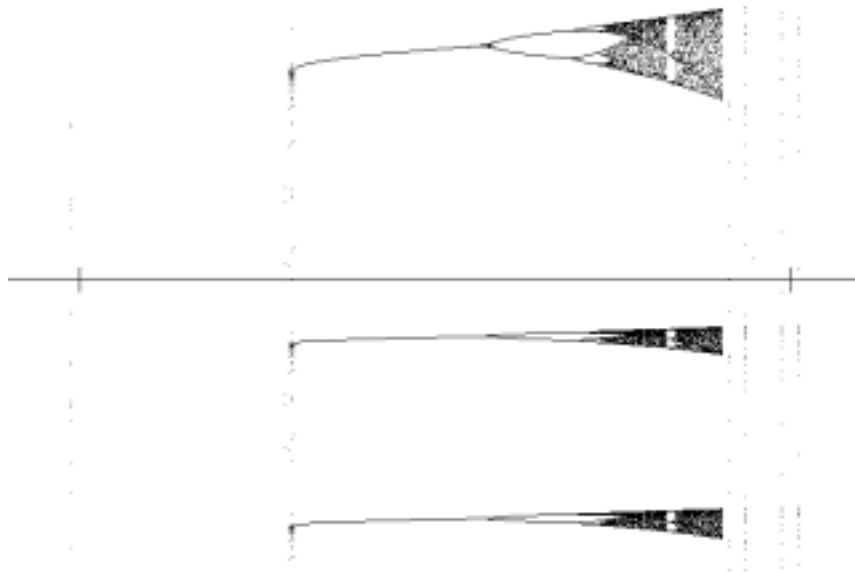


Figure 11  
Bifurcation axis (4.45, 4.75)

Again, we get the expected periodic 4-cycle. We see that there are intervals of  $c$  values that share the same period in the attracting cycle. These split (or bifurcate) which leads to a classic diagram of period doubling to chaos. These figures can be used to study more

properties of the polynomials  $q(z) = (z-i)(z+i)(z-c)$ , their attracting periodic cycles, and more occurrences of chaos.

### **Conclusion and further study**

When asked to find the roots of a cubic polynomial with real coefficients, we do not need to worry when picking an initial value if all three roots are real. The periodic points are always repelling and they comprise a Julia set that has measure zero. However, when there are two complex conjugate roots and a real root, picking an initial value is not so simple. Depending on the value of our parameter  $c$ , we do have to consider the possibility of attracting periodic points. Figure 9 helps us to determine which of these  $c$  values have attracting periodic cycles.

It would also be very interesting to study the behavior of the dynamic plane for different values of  $c$  in the case with two complex conjugate roots and one real root. As we saw from Figure 8, some  $c$  values give more interesting regions of points that converge to the attracting periodic cycle. Further study could discover why this happens and if different “bulbs” of the black region have different meanings. The bifurcation diagram may give us more insight into this and also help us to discover more interesting behavior.

We could remove the limits of real coefficients from our cubic polynomials. This would allow us to have a parameter plane of possible  $c$  values. The study in [B] does just this, and he finds that the parameter plane contains Mandelbrot sets and discusses why this occurs. Study could also be expanded to other polynomials, enabling us to compare and contrast both the parameter and dynamic planes among the polynomials.



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**NOTE:** Computer programs in App.doc (see table of contents of the journal) were written using TrueBasic. Following each dynamic plane program is a Mathematica printout which shows the method by which we found the formulas used in that program. Figures 1, 4, and 6-8 were created using newt2complex.tru. Figure 2 was created using newtmult2.tru. Figure 3 was created using newt3real.tru. Figure 5 was created using parameteraxis.tru. Figures 9-11 were created using bifuraxis.tru.