Rose-Hulman Undergraduate Mathematics Journal

Volume 3
Issue 2

Article 3

RISKy Business: An In-Depth Look at the Game RISK

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1 Introduction

Games have been of interest to mathematicians for many years because they can be mathematical models of simple real life situations. Baris Tan used this concept in his article entitled "Markov Chains and the *RISK* Board Game." In his paper, a model is presented for the board game *RISK* using modified rules. Tan defined the loser as the person who has zero armies. In order for this condition to occur, the attacker is allowed to attack with only one army, which cannot happen in the actual game of *RISK*. Baris Tan also assumed that multiple rolls in a battle are independent events, when in fact they are not. When the dice are rolled, they are then placed in descending order and thus are no longer independent. This paper will introduce a new model that is consistent with the rules of the game and discards the assumption of independent events. We will explore several mathematical aspects of the game *RISK* using mathematical tools, such as Markov chains and probability theory, to answer two main questions: *What is the probability that if you attack a territory, you will capture that territory?* and *If you engage in war, what is the expected number of losses based on the number of defending armies on that territory?* Before these questions are answered, the rules of the game need to be explained.

2 Rules of the Game

As the slogan for the game claims, *RISK* is the "game of global domination." The objective of the game is to attack and capture enemy territories in order to wipe out the opponent and become the ruler of the world. This game is played on a board that is divided up into 42 territories and each player is given multiple tokens where each token represents one army. The number of armies is determined at the beginning of each turn and is based on the number of territories that the player occupies. In order to win the game, a player must conquer all the territories. On each turn, a player may attack an adjoining territory controlled by the defender. The attacker must have at least two armies in order to attack so that at least one army remains in the currently occupied territory and one army can attack the defending army. An attacker may choose to withdraw from war at any time but if they remain in battle until the end, there are two possible outcomes. The first is that the defending army will be destroyed and the attacker will now have a new territory in its possession. The second option is that the attacker will only have one army left and will be unable to engage in battle, resulting in a failure to conquer the territory.

In order to determine if a battle is won or lost by the attacking army, both players roll dice. The number of dice depends on the number of armies the player has. If the defender has one army, one die is rolled but if the defender has two or more armies then two dice may be rolled. An attacker rolls either one, two, or three dice depending on the number of armies he/she has. The attacker must have at least one more army in the territory than the number of dice to be rolled. For example, if an attacker has three armies, he/she can roll a maximum of two dice. A player does have the option of rolling less than their maximum number of die allotted. This would decrease the potential number of armies lost but would reduce the odds of winning. In this paper, the number of dice rolled will be the maximum allowed to a player on that turn.

Once the dice are rolled, they are placed in descending order. The attacker's highest die is then compared to the defender's highest and if each player has at least two dice then the second largest for both sides is compared. The attacker loses one army for every die in the compared pairs that is less than or equal to the defender's die and the defender loses one army for every compared die that is less than the attacker's die. Armies are removed from the board when they are lost and the dice are rolled again until one side can no longer engage in battle. Throughout the course of a single battle, armies are lost and can never be gained. For more information, refer to the rules of *RISK*. [5]

There are several interesting mathematical questions that come to mind when studying the game of *RISK*. As a player of the game, it would be helpful to know the probability of winning and expected loss of armies and so a method for determining the answers to these questions will be shown. It would also be interesting to see if the attacker's probability of winning or expected loss would be affected if a different size die is used. The answers to all of these questions asked are important in developing a strategy to win the game of *RISK*.

3 Markov Chains and State-Space Models

A Markov chain is a mathematical model used to describe an experiment that is performed multiple times in the same way where the outcome of each trial of the experiment falls into one of several specified possible outcomes. A Markov chain possesses the property that the outcome of one trial depends only on the trial immediately preceding that trial[4]. Markov chains are very useful when developing a state-space model. This model is a sequence of probability vectors that describe the state of the system at a given time. Before answers can be found for the questions asked above, a state-space model needs to be developed for a single battle. Let A be the total number of attacking armies and D be the total number of defending armies. The state of the system at the beginning of each battle is described in terms of A and D. Let X_n be the state of the system at the beginning of the nth turn in the battle where a_n is the number of attacking armies and d_n is the number of defending armies:

$$X_n = (a_n, d_n), 1 \leq a_n \leq A, 0 \leq d_n \leq D.$$

The initial state of the system is $X_0 = (A, D)$. If the number of armies for each side is known at the beginning of a given turn, the probability of moving from state $X_n = (a_n, d_n)$ to state $X_{n+1} = (a_{n+1}, d_{n+1})$, referred to as the transition probability, can be found using the Markov property:

$$P[X_{n+1} = (a_{n+1}, d_{n+1}) | X_n = (a_n, d_n), X_{n-1} = (a_{n-1}, d_{n-1}), ..., X_0 = (A, D)]$$

$$= P[X_{n+1} = (a_{n+1}, d_{n+1}) | X_n = (a_n, d_n)] \ .$$

Therefore, $X_n : n = 0, 1, 2, ...$ is a Markov chain and has a state space of $(a, d) : 1 \leq a \leq A, 0 \leq d \leq D$. The transition probabilities for a Markov chain can also be formulated as a transition matrix, P. In matrix P, the rows are the possible states at the beginning of turn n (X_n) and the columns are the possible states at the beginning of the next turn (X_{n+1}). Also, the entry $P_{i,j}$ represents the probability of moving from the state in the ith row to the state in the jth column. For more information on Markov chains, refer to [3] or [4].

A battle ends when one of the following states is reached: $X_n = (a_n, 0)$ and $a_n \geq 2$ (attacker wins) or $X_n = (1, d_n)$ and $d_n > 0$ (defender wins). These states are called absorbing states. In general, a state in the ith row of P is absorbing if $P_{i,i} = 1$. Since one side has to either win or lose a battle, the state (1,0) can never be reached. All states that are not one of the two types of states given above are called transient states.

The total number of possible states is $A \cdot D + A$ - 1 where $A \cdot D - D$ are all the transient states and $A + D$ - 1 are the absorbing states. For example, let $A = 3$ and $D = 2$. Using the formula above, there should be 8 possible states for this example. The possible states of this system given in the form (a,d) are (2,1), (2,2), (3,1), (3,2), (2,0), (3,0), (1,1), and (1,2). The first four states listed are all transient and the last four states are absorbing.

The transition matrix has the form $P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$ where P has $A \cdot D + A$ - 1 rows and $A \cdot D + A$ - 1 columns. Q is a matrix whose entries are the probabilities from moving from one transient state to another transient state. Using the example above, it was stated that the transient states are (2,1), (2,2), (3,1), and (3,2). To form Q, these states are the rows and columns. The rows are the possible transient states at the beginning of turn n (X_n) and the columns are the possible transient states that can be reached at the beginning of next turn (X_{n+1}). Matrix Q will take the form given below with ?'s representing probabilities of successfully moving from one transient state to another and 0's where it is impossible to move to another transient state. The method to find these probabilities are given in the next section.

$Q =$

	(2,1)	(2,2)	(3,1)	(3,2)
(2,1)	0	0	0	0
(2,2)	?	0	0	0
(3,1)	?	0	0	0
(3,2)	?	0	0	0

R is a matrix whose entries are the probability of changing from a transient state to an absorbing state. Using the example above, the rows of this matrix are the possible transient states at the beginning of the turn and the columns are the possible absorbing states at the beginning of the next turn. As with matrix Q, ?'s and 0's are going to be used to show where it is and where it is not possible to move from a state on a row to a state on a column.

$R =$

	(2,0)	(3,0)	(1,1)	(1,2)
(2,1)	?	0	?	0
(2,2)	0	0	0	?
(3,1)	0	?	0	0
(3,2)	0	?	0	?

Matrix 0 contains the probability of moving from an absorbing state to a transient state and thus all of the entries of this matrix are 0's. Matrix I contains the probabilities that a state moves from absorbing to absorbing and so the identity matrix is produced.

Now that each entry of P is known, let's see all of the entries put together using the same example with $A = 3$ and $D = 2$.

$P =$

	(2,1)	(2,2)	(3,1)	(3,2)	(2,0)	(3,0)	(1,1)	(1,2)
(2,1)	0	0	0	0	?	0	?	0
(2,2)	?	0	0	0	0	0	0	?
(3,1)	?	0	0	0	0	?	0	0
(3,2)	?	0	0	0	0	?	0	?
(2,0)	0	0	0	0	1	0	0	0
(3,0)	0	0	0	0	0	1	0	0
(1,1)	0	0	0	0	0	0	1	0
(1,2)	0	0	0	0	0	0	0	1

One can use matrix Q or R to determine the probability of going from a specific transient state to a specific transient or absorbing state respectively. If a state becomes absorbing on the nth turn, then the previous $n-1$ turns were states that remained transient and thus they are elements of matrix Q. The nth turn is an element of matrix R since the state went from transient to absorbing. Thus, the probabilities of reaching an absorbing state on the nth turn is found by $Q^{n-1}R$. Since it is possible to enter an absorbing state at any turn n, the probability of a turn ending in an absorbing state is found by summing the products of $Q^{n-1}R$ from $n = 1$ to ∞ . Therefore, the entries of the matrix $S = \sum_{n=1}^{\infty} Q^{n-1}R$ represent the probability of starting at a specific transient state and ending at a specific absorbing state. The rows of S will be the possible transient states and the columns will be the possible absorbing states.

Theorem. *If* $max\{|\lambda| : \lambda$ *is an eigenvalue of* $Q\} < 1$, *then* $S = \sum_{n=1}^{\infty} Q^{n-1}R = (I-Q)^{-1}R$.

For a proof of this theorem, see [1].

4 Determining the Probabilities

The transition between states depends on the number of dice rolled by the attacker and defender, which depends on the number of armies controlled by the attacker and the defender. There are six possible cases for a state (a,d). The first case is $a = 2$ and $d = 1$. The second case is that $a = 3$ and $d = 1$. Case 3 is that $a \geq 4$ and $d = 1$. The fourth case occurs when $a = 2$ and $d \geq 2$. Case 5 is when $a = 3$ and $d \geq 2$. The final case occurs when $a \geq 4$ and $d \geq 2$. Since two is the maximum number of armies that either side can lose, a state (a,d) can change to one of the possible states $(a-2,d)$,$(a-1,d)$,$(a,d-1)$, $(a,d-2)$, and $(a-1,d-1)$ if both a and d are greater than or equal to two.

For example, consider Case 5 where the attacker rolls two dice and the defender rolls two dice. Let Y_1 and Y_2 be the outcomes of the attacker's dice and Z_1 and Z_2 be the outcomes of the defender's dice. Let $Y^{(1)}$ and $Y^{(2)}$ be the maximum and second largest, respectively, of the attacker's dice and similarly $Z^{(1)}$ and $Z^{(2)}$ be the maximum and second largest of the defender's dice.

For each value of $Y^{(1)}$ and $Y^{(2)}$, the probability that $Y^{(1)} = Y^{(2)}$ is $\frac{1}{36}$. However, if $Y^{(1)} > Y^{(2)}$, then the probability is $\frac{2}{36}$. For example, the event $(Y_1 = 5,\ Y_2 = 4)$ has the same probability as the event $(Y_1 = 4,\ Y_2 = 5)$ and so the event $(Y^{(1)} = 5,\ Y^{(2)} = 4)$ has the probability of $\frac{2}{36}$. Thus the probability distribution for $Y^{(1)}$, $Y^{(2)}$ is

$$f\left(Y^{(1)}, Y^{(2)}\right) = \begin{cases} \frac{1}{36} & Y^{(1)} = Y^{(2)}, \\ \frac{2}{36} & Y^{(1)} > Y^{(2)}. \end{cases}$$

Likewise, the probability distribution for the defender is

$$f\left(Z^{(1)}, Z^{(2)}\right) = \begin{cases} \frac{1}{36} & Z^{(1)} = Z^{(2)}, \\ \frac{2}{36} & Z^{(1)} > Z^{(2)}. \end{cases}$$

Recall that in the game of *RISK*, the attacker's and defender's dice are compared in order to determine the transition state on every turn. Therefore, a joint distribution is needed to compare the attacker's and defender's dice. There are four possible conditions to consider:

$$Y^{(1)} = Y^{(2)} \text{ and } Z^{(1)} = Z^{(2)}, \tag{1}$$
$$Y^{(1)} = Y^{(2)} \text{ and } Z^{(1)} > Z^{(2)}, \tag{2}$$
$$Y^{(1)} > Y^{(2)} \text{ and } Z^{(1)} = Z^{(2)}, \tag{3}$$
$$Y^{(1)} > Y^{(2)} \text{ and } Z^{(1)} > Z^{(2)}. \tag{4}$$

Since the rolling of two sets of dice is an independent event, the probability of condition 1 is

$$\begin{aligned} P\left(Y^{(1)} = Y^{(2)} \text{ and } Z^{(1)} = Z^{(2)}\right) &= P\left(Y^{(1)} = Y^{(2)}\right) \cdot P\left(Z^{(1)} = Z^{(2)}\right) \\ &= \frac{1}{36} \cdot \frac{1}{36} \\ &= \frac{1}{1296}. \end{aligned}$$

Using the same process for conditions 2, 3, and 4, the joint distribution is

$$f\left(Y^{(1)}, Y^{(2)}, Z^{(1)}, Z^{(2)}\right) = \begin{cases} \frac{1}{1296} & Y^{(1)} = Y^{(2)} \text{ and } Z^{(1)} = Z^{(2)}, \\ \frac{2}{1296} & Y^{(1)} = Y^{(2)} \text{ and } Z^{(1)} > Z^{(2)}, \\ \frac{2}{1296} & Y^{(1)} > Y^{(2)} \text{ and } Z^{(1)} = Z^{(2)}, \\ \frac{4}{1296} & Y^{(1)} > Y^{(2)} \text{ and } Z^{(1)} > Z^{(2)}. \end{cases}$$

Now that the joint distribution is known, let's look at how this applies to the game *RISK*. In Case 5, where $a = 3$ and $d \geq 2$, there are three possible outcomes. When the rolling of the dice occurs and $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$, then the defender loses two armies. To find the probability of this outcome, we need to count the number of ways $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ can occur in conditions 1 through 4. For example, let's look at the possible rolls for the first condition:

A	D
(1,1)	(1,1)
(2,2)	(2,2)
(3,3)	(3,3)
(4,4)	(4,4)
(5,5)	(5,5)
(6,6)	(6,6)

There are 15 ways for $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ as seen in Figure 1.

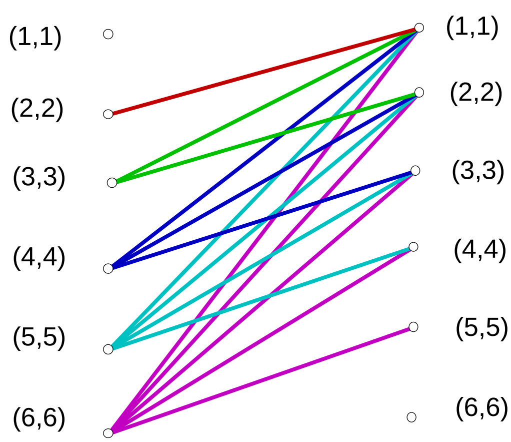

Figure 1

Since condition 1 has a probability of $\frac{1}{1296}$ and there are 15 ways for $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ to occur, the probability of $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ in this condition is $\frac{15}{1296}$. There are 20 ways for $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ in the second condition, 20 ways in condition 3, and 50 ways in condition 4. Therefore, the probability of $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ is $15 \cdot \frac{1}{1296} + 20 \cdot \frac{2}{1296} + 20 \cdot \frac{2}{1296} + 50 \cdot \frac{4}{1296} = 0.2276$. Thus,

$$\begin{aligned} P[X_{n+1} = (a, d-2) | X_n = (a, d)] &= P[(Y^{(1)} > Z^{(1)}) \cap (Y^{(2)} > Z^{(2)})] \\ &= 0.2276. \end{aligned}$$

Using the same procedure, the probability for the second outcome, $Y^{(1)} \leq Z^{(1)}$ and $Y^{(2)} \leq Z^{(2)}$, of Case 5 is found to be

$$\begin{aligned} P[X_{n+1} = (a-2, d) | X_n = (a, d)] &= P[(Y^{(1)} \leq Z^{(1)}) \cap (Y^{(2)} \leq Z^{(2)})] \\ &= 0.4483. \end{aligned}$$

Since the total probability of the 3 outcomes must equal 1, the probability of the third outcome where both the attacker and defender lose an army is found by

$$\begin{aligned}P[X_{n+1}=(a-1,d-1)|X_n=(a,d)]=&1-P[X_{n+1}=(a,d-2)|X_n=(a,d)]-\\&P[X_{n+1}=(a-2,d)|X_n=(a,d)]\\=&1-0.2276-0.4483\\=&0.3241.\end{aligned}$$

Case 6 is different from all the other cases because the attacker rolls three dice. A new variable, Y_3, must be included to represent the value rolled on the third die. Let Y_1, Y_2, and Y_3 represent the outcomes of the attacker's dice. Let $Y^{(1)}$ be the maximum, $Y^{(2)}$ be the second largest, and $Y^{(3)}$ be the minimum of Y_1, Y_2, and Y_3. The probability that $Y^{(1)}$, $Y^{(2)}$, and $Y^{(3)}$ are equal is $\frac{1}{6^3}$. If $Y^{(1)}=Y^{(2)}>Y^{(3)}$, there are three possible ways to achieve this result. For example, $Y^{(1)}=6$, $Y^{(2)}=6$, $Y^{(3)}=5$. This event can be written as (6,6,5), (6,5,6), and (5,6,6). Therefore, the probability of this event occurring is $\frac{3}{6^3}$. Using the same process for the other two conditions, the probability distribution is

$$f\left(Y^{(1)},Y^{(2)},Y^{(3)}\right)=\begin{cases}\frac{1}{216} & Y^{(1)}=Y^{(2)}=Y^{(3)},\\ \frac{3}{216} & Y^{(1)}=Y^{(2)}>Y^{(3)},\\ \frac{3}{216} & Y^{(1)}>Y^{(2)}=Y^{(3)},\\ \frac{6}{216} & Y^{(1)}>Y^{(2)}>Y^{(3)}.\end{cases}$$

The next step is to find the marginal distribution since the values of $Y^{(1)}$ and $Y^{(2)}$ are being compared to the numbers on the defender's dice. To find the marginal distribution, the probability distribution for $Y^{(1)}$, $Y^{(2)}$, and $Y^{(3)}$ needs to be summed over all possible values for $Y^{(3)}$. For example, if $Y^{(1)}=5$ and $Y^{(2)}=5$, then

$$\begin{aligned}P\left(Y^{(1)}=5\text{ and }Y^{(2)}=5\right)=&P\left(Y^{(1)}=5,Y^{(2)}=5,Y^{(3)}=1\right)\\&+P\left(Y^{(1)}=5,Y^{(2)}=5,Y^{(3)}=2\right)\\&+\ldots+P\left(Y^{(1)}=5,Y^{(2)}=5,Y^{(3)}=5\right)\\=&\frac{3}{216}+\frac{3}{216}+\frac{3}{216}+\frac{3}{216}+\frac{1}{216}.\end{aligned}$$

In general,

$$f(Y^{(1)},Y^{(2)})=\begin{cases}\frac{1}{216}+\frac{3}{216}\cdot(Y^{(2)}-1) & Y^{(1)}=Y^{(2)},\\ \frac{3}{216}+\frac{6}{216}\cdot(Y^{(2)}-1) & Y^{(1)}>Y^{(2)}.\end{cases}$$

In Case 6, the distribution for the defender's dice, $Z^{(1)}$ and $Z^{(2)}$, is the same as in Case 5:

$$f\left(Z^{(1)},Z^{(2)}\right)=\begin{cases}\frac{1}{36} & Z^{(1)}=Z^{(2)},\\ \frac{2}{36} & Z^{(1)}>Z^{(2)}.\end{cases}$$

The joint distribution for $Y^{(1)}$, $Y^{(2)}$, $Z^{(1)}$, and $Z^{(2)}$ will be found the same way as described in Case 5 above and has the form

$$f\left(Y^{(1)}, Y^{(2)}, Z^{(1)}, Z^{(2)}\right) = \begin{cases} \frac{1}{36} \cdot (\frac{1}{216} + \frac{3}{216} \cdot (Y^{(2)} - 1)) & Y^{(1)} = Y^{(2)} \text{ and } Z^{(1)} = Z^{(2)}, \\ \frac{2}{36} \cdot (\frac{1}{216} + \frac{3}{216} \cdot (Y^{(2)} - 1)) & Y^{(1)} = Y^{(2)} \text{ and } Z^{(1)} > Z^{(2)}, \\ \frac{1}{36} \cdot (\frac{3}{216} + \frac{6}{216} \cdot (Y^{(2)} - 1)) & Y^{(1)} > Y^{(2)} \text{ and } Z^{(1)} = Z^{(2)}, \\ \frac{2}{36} \cdot (\frac{3}{216} + \frac{6}{216} \cdot (Y^{(2)} - 1)) & Y^{(1)} > Y^{(2)} \text{ and } Z^{(1)} > Z^{(2)}. \end{cases}$$

As with Case 5, there are three possible outcomes when one player attacks another player. Again, let's look at the probability that $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ for condition 1. Since the joint distribution function depends on the value of $Y^{(2)}$, the probability for this part of the distribution needs to be summed using all possible values for $Y^{(2)}$. As in Case 5, the number of ways $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ for condition 1 needs to be found. The results for $Y^{(2)} = 3$, 4, 5, and 6 are found in the following table which shows the number of ways this condition is satisfied given any $Y^{(2)}$:

$Y^{(2)}$	number of ways $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$
1	0
2	1
3	2
4	3
5	4
6	5

Therefore, the probability of $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ occurring in condition 1 using the joint distribution is $(0 \cdot \frac{1}{36} \cdot (\frac{1}{216} + \frac{3}{216} \cdot (1-1)) + (1 \cdot \frac{1}{36} \cdot (\frac{1}{216} + \frac{3}{216} \cdot (2-1)) + (2 \cdot \frac{1}{36} \cdot (\frac{1}{216} + \frac{3}{216} \cdot (3-1)) + (3 \cdot \frac{1}{36} \cdot (\frac{1}{216} + \frac{3}{216} \cdot (4-1)) + (4 \cdot \frac{1}{36} \cdot (\frac{1}{216} + \frac{3}{216} \cdot (5-1)) + (5 \cdot \frac{1}{36} \cdot (\frac{1}{216} + \frac{3}{216} \cdot (6-1)) = 0.0231$. Using the same counting argument for conditions 2 through 4, a probability was found for each condition. The probability of $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ is found by summing up these four probabilities.

$$\begin{aligned} P[X_{n+1} = (a, d-2) | X_n = (a, d)] &= P[(Y^{(1)} > Z^{(1)}) \cap (Y^{(2)} > Z^{(2)})] \\ &= 0.0231 + 0.0707 + 0.0463 + 0.2316 \\ &= 0.3717. \end{aligned}$$

To find the probability of the attacker losing both rolls, use the same procedure as above .

$$\begin{aligned} P[X_{n+1} = (a-2, d) | X_n = (a, d)] &= P[(Y^{(1)} \leq Z^{(1)}) \cap (Y^{(2)} \leq Z^{(2)})] \\ &= 0.2926. \end{aligned}$$

The probability of the last outcome where both the attacker and defender lose an army is

$$\begin{aligned} P[X_{n+1} = (a-1, d-1) | X_n = (a, d)] =& 1 - P[X_{n+1} = (a, d-2) | X_n = (a, d)] \\ & - P[X_{n+1} = (a-2, d) | X_n = (a, d)] \\ =& 1 - 0.3717 - 0.2926 \\ =& 0.3357. \end{aligned}$$

The transition probabilities for Cases 1 through 4 can be found in a similar manner. The results are listed in Table 1. Included in this table are the general formula for finding the probabilities using dice with s faces.

6 sided dice					Transition	General Formula
Case	a	d	From state	To state	Probability	(s = # of faces on die)
I	2	1	(2,1)	(2,0)	0.4166	$\frac{s-1}{2s}$
				(0,1)	0.5834	$\frac{s+1}{2s}$
II	3	1	(3,1)	(3,0)	0.5787	$\frac{(s-1)(4s+1)}{6s^2}$
				(2,1)	0.4213	$\frac{(s+1)(2s+1)}{6s^2}$
III	≥ 4	1	(a,1)	(a,0)	0.6597	$\frac{(s-1)(3s+1)}{4s^2}$
				(a-1,1)	0.3403	$\frac{(s+1)^2}{4s^2}$
IV	2	≥ 2	(2,d)	(2,d-1)	0.2546	$\frac{(s-1)(2s-1)}{6s^2}$
				(1,d)	0.7454	$\frac{(s+1)(4s-1)}{6s^2}$
V	3	≥ 2	(3,d)	(3,d-2)	0.2276	$\frac{(s-1)(2s^2-2s-1)}{6s^3}$
				(1,d)	0.4483	$\frac{(s+1)(2s^2+2s-1)}{6s^3}$
				(2,d-1)	0.3241	$\frac{(s-1)(s+1)}{3s^2}$
VI	≥ 4	≥ 2	(a,d)	(a,d-2)	0.3717	$\frac{(s-1)(6s^3-3s^2-5s-2)}{12s^4}$
				(a-2,d)	0.2926	$\frac{(s+1)(2s+1)(3s^2+3s-1)}{30s^4}$
				(a-1,d-1)	0.3357	$\frac{(s+1)(s-1)(18s^2+15s+8)}{60s^4}$

Table 1

The general formulas provided show what happens when rolling dice with s sides. One can find these formulas by using similar counting techiques to those presented for 6-sided dice.

For example, in Case 1, the attacker and the defender both roll one die. The probability of going from state $(2,1)$ to state $(2,0)$ is the probability that $Y^{(1)} > Z^{(1)}$. If both the attacker and the defender are rolling a die with s sides then the number of ways that $Y^{(1)} > Z^{(1)}$ is $\sum_{i=1}^{s-1} i = \dfrac{s(s-1)}{2}$. Since the total number of ways to roll these two dice is s^2,

$$P[X_{n+1} = (2,0)|X_n = (2,1)] = P(Y^{(1)} > Z^{(1)}) = \frac{s(s-1)}{2s^2} = \frac{s-1}{2s}.$$

In Case 2, the attacker rolls two dice and the defender rolls one die. In order to determine the probability of going from state $(3,1)$ to state $(3,0)$, one must again determine the probability that $Y^{(1)} > Z^{(1)}$. In this case, the number of ways that $Y^{(1)} > Z^{(1)}$ is

$$\sum_{i=1}^{s-1} i + 2i^2 = \frac{s(s-1)(4s+1)}{6}.$$

This is due to the fact that each roll ($Y_1 = i,\ Y_2 = j$) is equivalent to the roll ($Y_1 = j,\ Y_2 = i$) for any $1 \le i, j \le s$, where $i \ne j$. Since the total number of ways to roll three dice is s^3,

$$P[X_{n+1} = (3,0) | X_n = (3,1)] = P(Y^{(1)} > Z^{(1)}) = \frac{s(s-1)(4s+1)}{6s^3} = \frac{(s-1)(4s+1)}{6s^2}.$$

Similar ideas can be used to calculate the general formulas for Cases 3 through 6.

From these general results, it can be concluded that when s is large, the probability of winning increases and the expected loss for the attacker decreases. Therefore, if the attacker was allowed to choose a set of dice, it would be in his/her best interest to choose the set with the largest number of sides.

5 Capturing a Territory and Expected Loss

There is now enough information to answer the two main questions of interest in this paper. The first question asks about the probability of capturing a territory. Now that the entries of matrices Q and R can be calculated, these numbers are used to find the matrix S formed from the formula $(I - Q)^{-1}R$. Since the elements of S consist of the probabilities of a turn ending in an absorbing state, the probability of the attacker winning is found by looking at the intersection of the bottom row (A,D) and the columns where the attacker wins and summing these probabilities. For instance, let $A = 4$ and $D = 1$. The matrix S would look similar to this:

$S =$

	(2,0)	(3,0)	(4,0)	(1,1)
(2,1)	a	b	c	d
(3,1)	e	f	g	h
(4,1)	i	j	k	l

To find the probability of the attacker winning from an initial state (A,D), look at row (A,D), in this case row (4,1). Now, sum up the probabilities found at the intersection of that row and the columns where the attacker wins, in this example the first three columns. In this example, the winning probability would be i + j + k.

The next question asks about the expected loss for the attacker. As done above when calculating the probability of winning, the only row that is looked at is the row with the initial state (A,D). As stated before, the columns of S are the possible absorbing states. For each column, determine the number of armies lost for the attacker. Using the same example as above with $A = 4$ and $D = 1$ and looking only at row (4,1), the attacker went from state (4,1) to (2,0) in the first column and so the attacker lost two armies. In the second column, the attacker lost 1 army and in column three, zero armies were lost. In the last column, three armies were lost. To find the expected loss, sum up the product of the number of armies lost for each column multiplied by the probability in that column for each column in row (A,D). In this example, the expected loss would be equal to 2 · i + 1 · j + 0 · k + 3 · l.

Now that the process of finding the probability of capturing a territory and expected loss is known, let's see an example of a battle.

6 Example

Let $A = 4$ and $D = 3$. The matrices Q and R for this example are shown below.

	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)	(4,1)	(4,2)	(4,3)
(2,1)	0	0	0	0	0	0	0	0	0
(2,2)	0.2546	0	0	0	0	0	0	0	0
(2,3)	0	0.2546	0	0	0	0	0	0	0
(3,1)	0.4213	0	0	0	0	0	0	0	0
(3,2)	0.3241	0	0	0	0	0	0	0	0
(3,3)	0	0.3241	0	0.2276	0	0	0	0	0
(4,1)	0	0	0	0.3403	0	0	0	0	0
(4,2)	0	0.2926	0	0.3357	0	0	0	0	0
(4,3)	0	0	0.2926	0	0.3357	0	0.3717	0	0

Matrix Q

	(2,0)	(3,0)	(4,0)	(1,1)	(1,2)	(1,3)
(2,1)	0.4166	0	0	0.5834	0	0
(2,2)	0	0	0	0	0.7454	0
(2,3)	0	0	0	0	0	0.7454
(3,1)	0	0.5787	0	0	0	0
(3,2)	0	0.2276	0	0	0.4483	0
(3,3)	0	0	0	0	0	0.4483
(4,1)	0	0	0.6597	0	0	0
(4,2)	0	0	0.3717	0	0	0
(4,3)	0	0	0	0	0	0

Matrix R

Using the formula $(I-Q)^{-1}R$, the resultant matrix, S, is calculated and is given below.

	(2,0)	(3,0)	(4,0)	(1,1)	(1,2)	(1,3)
(2,1)	0.4166	0	0	0.5834	0	0
(2,2)	0.1061	0	0	0.1485	0.7454	0
(2,3)	0.0270	0	0	0.0378	0.1898	0.7454
(3,1)	0.1755	0.5787	0	0.2458	0	0
(3,2)	0.1350	0.2276	0	0.1891	0.4483	0
(3,3)	0.0743	0.1317	0	0.1041	0.2416	0.4483
(4,1)	0.0597	0.1969	0.6597	0.0836	0	0
(4,2)	0.0899	0.1943	0.3717	0.1260	0.2181	0
(4,3)	0.0754	0.1496	0.2452	0.1056	0.2060	0.2181

Matrix S

To find the probability that the attacker wins, the probabilities from the row that is the initial state $X_n = (A,D)$, in this case the row (4,3), and the columns where the attacker is victorious, in this case the first three columns, need to be summed. In this example, the probability that the attacker conquers the territory is $0.0754 + 0.1496 + 0.2452 = 0.4702$. This means that there is a 47.02 % chance that if a player attacks a territory, he/she will conquer that territory.

The expected loss for the attacker is found using the entries in matrix S. The columns of this matrix correspond to the headings (1,0), (2,0), (3,0), (4,0), (0,1), (0,2), and (0,3). For each column, determine the number of armies that the attacker lost from the initial state. For example, in column one the attacker goes from four armies to two armies, resulting in a loss of two armies. Once all of these values have been found, multiply the probability in the bottom row of each column by the number of armies lost for each column and add all of these new values together. This will give you the expected number of armies lost by the attacker in this battle. In this example, the expected loss of armies for the attacker is 1.8895.

7 Conclusion

Each individual player has their own strategy in mind when playing the game of *RISK*. Hopefully, the information produced in this article will be of assistance to players in adapting their strategy or creating an entirely new strategy. The use of Markov chains to create a state-space model was key in finding the probabilities of conquering a territory and approximating expected lost. From this model, further research was done to determine if the number of sides on the dice would have an effect on the probabilites calculated to answer the first two questions.

Further research that would be of interest to a *RISK* player would be the decision to conquer an entire continent. One would have to look at the expected loss of capturing the countries on a continent that are currently not in that player's possession. Also, one would have to keep in mind that in order to be awarded extra armies due to the possession of an entire continent, a player has to fortify its armies in countries that border territories in other continents in order to defend against possible attacks from opponents. One possible aspect to consider when trying to decide on a strategy is comparing the expected loss to the expected gain (i.e. ruling a continent) and deciding if it's worth the risk to attack a territory. One other area of possible further research involves the use of the cards. In the game of *RISK*, a player receives a card at the end of his/her turn if at least one territory is conquered during the turn. On this card is an icon and a territory. There are three possible icons and once a player has a matching set of cards (either three of the same icon or one of each icon), the player can turn in the cards at the beginning of their next turn and receive extra armies. A player must turn in a set of cards after earning five or six cards. If the player occupies any of the territories shown on the set of cards turned in, extra armies are earned. It would be interesting to see how an attacker's strategy would change when he/she turns in a set of cards and receives extra armies.

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