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Hamilton Cycles in Addition Graphs

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Abstract

If A is a square-free subset of an abelian group G, then the addition graph of A on G is the graph with vertex set G and distinct vertices x and y forming an edge if and only if $x + y \in A$. We prove that every connected cubic addition graph on an abelian group G whose order is divisible by 8 is Hamiltonian as well as every connected bipartite cubic addition graph on an abelian group G whose order is divisible by 4. We show that connected bipartite addition graphs are Cayley graphs and prove that every connected cubic Cayley graph on a group of dihedral type whose order is divisible by 4 is Hamiltonian.

1 Introduction

For basics of graph theory and group theory, we refer to [2] and [5], respectively.

Let S be a subset of a finite group G such that $1 \notin S$ and, for every $x \in G$, $x \in S$ if and only if $x^{-1} \in S$. The graph $Cay(S, G) = (G, E)$, where $E = \{\{x, xs\}: s \in S\}$ is called the *Cayley graph* of S on G. It is connected if and only if $\langle S \rangle = G$. (Here and throughout the paper, if S is a subset of a group G, then $\langle S \rangle$ denotes the subgroup of G generated by S.) It was conjectured by many people that every connected Cayley graph

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is Hamiltonian. This conjecture was proven for all abelian groups by several people (see, for instance, [3]). It was also proven for some classes of non-abelian groups. In particular, Alspach and Zhang [1] proved the conjecture for cubic Cayley graphs on dihedral groups. The paper [4] is the most recent survey on Hamilton cycles in Cayley graphs.

In this paper, we introduce *addition graphs*. Let A be a square-free subset of an abelian group G, that is, there is not $x \in G$ such that $x + x \in A$. The *addition graph of A on G* is the graph $Add(A, G) = (G, E)$, where $E = \{\{x, y\} \colon x + y \in A\}$. The graph $Add(A, G)$ is regular of degree $|A|$. We show in Section 2 that the graph $Add(A, G)$ is connected if and only if the set A generates G and the set of all differences $a - b$ of elements of A generates either the entire group G or a subgroup of index 2. In the latter case, the graph $Add(A, G)$ is bipartite.

We make the following conjecture:

Every connected addition graph on a finite abelian group is Hamiltonian.

In Section 3 we prove this conjecture for cubic addition graphs on a group G whose order is divisible by 8 and for bipartite cubic addition graphs on a group G whose order is divisible by 4. (Note that if $Add(A, G)$ is connected and not complete, then the order of G has to be even and, if this graph is not bipartite, the order of G has to be divisible by 4.)

In Section 4 we show that if $Add(A, G)$ is a connected bipartite addition graph, then it is a Cayley graph on a group G^* of *dihedral type*, which means that G^* has an abelian subgroup H of index 2 such that every element of $G^* \setminus H$ is of order 2. We use this observation to prove that every cubic connected Cayley graph on a finite group of dihedral type whose order is divisible by 4 is Hamiltonian.

In Section 5, we apply Witte's result [6] on Cayley digraphs to show that every connected bipartite addition graph on an abelian group of order $2q$, where q is a prime power, is Hamiltonian.

2 Preliminaries

All graphs in this paper are simple graphs, that is, without loops or multiple edges. For any graph Γ, $V(\Gamma)$ and $E(\Gamma)$ denote its vertex set and edge set, respectively. We will call a sequence of vertices $(x_0, x_1, \ldots, x_n)$ a *walk* of length n from vertex x_0 to vertex x_n if $\{x_{i-1}, x_i\}$ is an edge for $i = 1, 2, \ldots, n$. For walks $\alpha = (x_0, x_1, \ldots, x_m)$ and $\beta = (x_m, x_{m+1}, \ldots, x_{m+n})$, $\alpha * \beta$ denotes the walk $(x_0, x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$. If the vertices of a walk are distinct, we will call it a *path*. If the vertices of a walk $(x_0, x_1, \ldots, x_n)$ are distinct, except $x_0 = x_n$, we will call it a *cycle*. A graph is *connected* if there is a walk from any vertex to any other vertex. A cycle that contains all vertices of the graph is called a *Hamilton cycle*. A graph with a Hamilton cycle is said to be *Hamiltonian*.

A subset A of an (additively written) abelian group G is said to be *square-free* if $x + x \notin A$ for all $x \in G$.

Definition 2.1 *Let G be a finite abelian group and let A be a square-free subset of G. An* addition graph *$\Gamma = Add(A, G)$ is the graph with $V(\Gamma) = G$ and $E(\Gamma) = \{\{x, y\} \colon x + y \in A\}$.*

Figure 1 shows an addition graph $Add(A, G)$, where $G = \mathbf{Z}_2 \times \mathbf{Z}_6$ is the direct product of a cyclic group of order 6 and a cyclic group of order 2. The elements of G, that is, the vertices of the graph are represented by ordered pairs ab with $a = 0, 1$ and $b = 0, 1, 2, 3, 4, 5$. The operation in the group is coordinate-wise addition with the first coordinates added modulo 2 and the second coordinates added modulo 6. The set A consists of elements $a = 10$, $b = 01$, and $c = 13$. Two vertices of the graph are connected with a thin edge if their sum is a, with a thick edhe if their sum is b, and with a dashed edge if their sum is c.

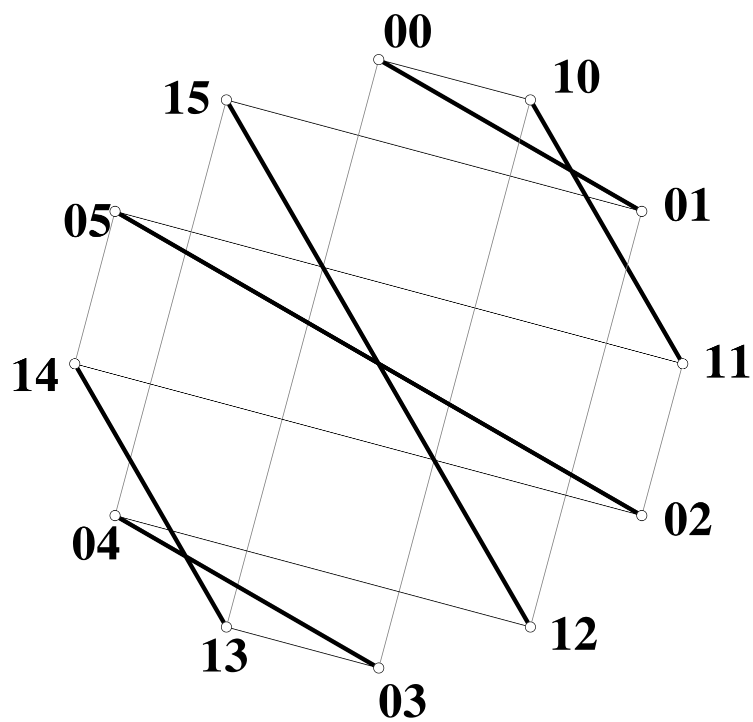

Figure 1

Notation. Let G be a finite abelian group and let A be a nonempty subset of G. Let W be the free semigroup of words over the alphabet A. For $a_1, a_2, \ldots, a_n \in A$, $w = a_1 a_2 \cdots a_n \in W$, and $x \in G$, $(x; w)$ denotes the walk $(x_0, x_1, \ldots, x_n)$ in $Add(A, G)$

with $x_0 = x$ and $x_i = a_i - x_{i-1}$ for $i = 1, 2, \ldots, n$.

For instance, in the graph shown in Figure 1, $(02; a^2b^3c)$ denotes the following walk: $(02, 14, 02, 15, 02, 15, 04)$.

Notation. For any subset A of an abelian group G, $A' = \{a - b \colon a, b \in A, a \neq b\}$.

For the graph $Add(A, G)$ in Figure 1, we have $A' = \{15, 03, 10, 14, 12\}$. Figure 2 shows the Cayley graph $Cay(A', G)$.

Remark 2.2 *Observe that* $0 \notin A'$ *and* $-x \in A'$ *whenever* $x \in A'$. *Therefore, for any subset* A *of an abelian group* G, *there is a Cayley graph* $Cay(A', G)$. *If* $\{x, y\}$ *is an edge of this graph, then* $y = x + a - b$ *with* $a, b \in A$, $a \neq b$. *Therefore,* $(x, b - x, y)$ *is a path of length* 2 *from* x *to* y *in* $Add(A, G)$. *Conversely, if there is a path of length* 2 *from* x *to* y *in* $Add(A, G)$, *then* $\{x, y\}$ *is an edge of* $Cay(A', G)$.

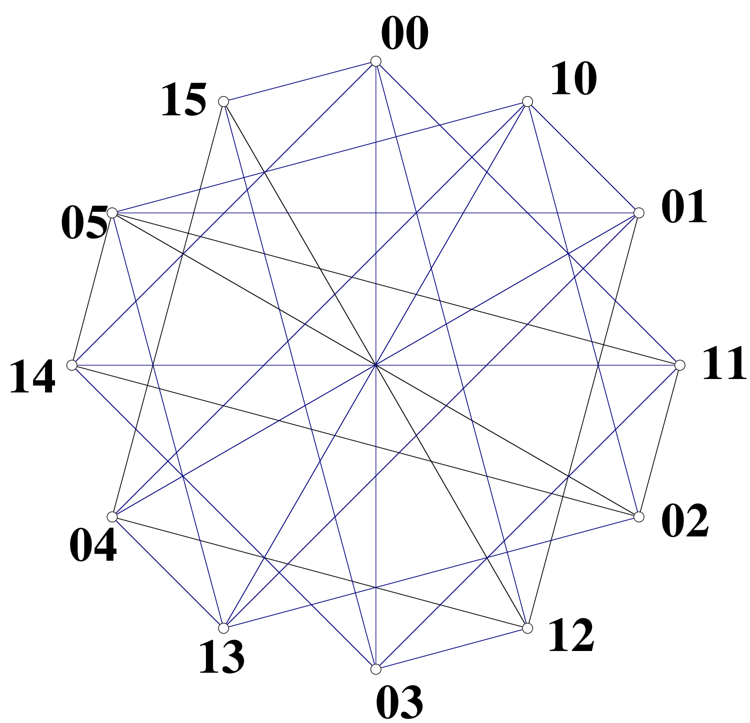

Figure 2

The following proposition gives a necessary and sufficient condition for a graph $Add(A, G)$ to be connected.

Proposition 2.3 *Let G be an abelian group and let A be a square-free subset of G. The graph $Add(A, G)$ is connected if and only if $\langle A \rangle = G$ and $|\langle A' \rangle| \geq |G|/2$.*

Proof. 1. Suppose $\langle A \rangle = G$ and $\langle A' \rangle = G$. Then $Cay(A', G)$ is connected and, by Remark 2.2, the graph $Add(A, G)$ is connected.

2. Suppose $\langle A \rangle = G$ and $|\langle A' \rangle| = \frac{1}{2}|G|$. Since $\Gamma_0 = Cay(A', \langle A' \rangle)$ is connected, for each $x \in \langle A' \rangle$ there is a walk from 0 to x in $Add(A, G)$. Since $\langle A \rangle = G$, the set A is not contained in $\langle A' \rangle$, so we fix $a \in A \setminus \langle A' \rangle$. Let $z \in G \setminus \langle A' \rangle$. Since $|\langle A' \rangle| = \frac{1}{2}|G|$, $a - z \in \langle A' \rangle$. Therefore, there is a walk from 0 to $a - z$ in $Add(A, G)$. Since $\{a - z, z\}$ is an edge in $Add(A, G)$, there is a walk from 0 to z in $Add(A, G)$, so $Add(A, G)$ is connected.

3. Suppose the graph $Add(A, G)$ is connected. Let $x \in G$, $x \neq 0$, and let $(0, x_1, x_2, \ldots, x_n = x)$ be a walk from 0 to x. We prove by induction on n that $x \in \langle A \rangle$. If $n = 1$, then $x \in A$. Let $n \geq 2$ and let $x_{n-1} \in \langle A \rangle$. Since $x_{n-1} + x \in A$, we have $x \in \langle A \rangle$. Thus, $\langle A \rangle = G$.

In order to prove that $|\langle A' \rangle| \geq \frac{1}{2}|G|$, we define an equivalence relation on G by declaring x and y equivalent if there is a walk of even length from x to y in $Add(A, G)$. Observe that if there is a walk of odd length from x to z and a walk of odd length from z to y, then there is a walk of even length from x to y. Thus, with respect to this equivalence relation, the group G consists of at most two equivalence classes. Let O be the equivalence class that contains 0. It suffices to prove the following two claims: (i) $O \subseteq \langle A' \rangle$ and (ii) $|O| \geq |G|/2$.

(i) Let $x \in O$. Since there is a walk of even length from 0 to x in $Add(A, G)$, there is a walk from 0 to x in $Cay(A', G)$. Therefore, $x \in \langle A' \rangle$.

(ii) Let $x \in G \setminus O$ and let $(0, x_1, \ldots, x_n = x)$ be a walk of odd length from 0 to x in $Add(A, G)$. Let $a \in A$. Then $(0, x_1, \ldots, x_n, a - x)$ is a walk of even length from 0 to $a - x$ in $Add(A, G)$, so $a - x \in O$. Therefore, $|O| \geq |G \setminus O|$, which implies that $|O| \geq |G|/2$. □

The following result was obtained in the course of the above proof.

Proposition 2.4 *If $Add(A, G)$ is a connected graph, then, for any $x \in \langle A' \rangle$, there is a walk of even length from 0 to x.*

A graph is bipartite if and only if it has no cycle of odd length. Applying this criterion to addition graphs yields the following result.

Proposition 2.5 *Let A be a square-free subset of a finite abelian group G. The graph $\Gamma = Add(A, G)$ is not bipartite if and only if there exist integers k_a, $a \in A$, such that $\sum_{a \in A} k_a a$ is a square and $\sum_{a \in A} k_a$ is odd.*

Proof. Suppose Γ has a cycle $(x_0, x_1, \ldots, x_{k-1}, x_k = x_0)$ of odd length. For each $a \in A$, let k_a be the number of indices $j \in \{0, 1, \ldots, k-1\}$ such that $x_j + x_{j+1} = a$. Then $\sum_{a \in A} k_a a = 2 \sum_{i=0}^{k-1} x_i$ is a square and $\sum_{a \in A} k_a = k$ is odd.

Conversely, suppose $\sum_{a\in A} k_a a$ is a square and $\sum_{a\in A} k_a$ is odd. For each $a \in A$, let $l_a \in \{0,1\}$ be such that $l_a \equiv k_a \pmod 2$. Then $\sum_{a\in A} l_a a = 2u$ is a square and $\sum_{a\in A} l_a = k$ is odd. Let $B = \{a \in A\colon l_a = 1\} = \{a_0, a_1, \ldots, a_{k-1}\}$. For $j = 0, 1, \ldots, k-1$, let

$$x_j = u - \sum_{i=1}^{(k-1)/2} a_{2i+j-1}$$

(with indices taken modulo k). Then $(x_0, x_1, \ldots, x_{k-1}, x_k)$ is a cycle of odd length in Γ. $\square$

The next result characterizes connected bipartite addition graphs.

Proposition 2.6 *Let A be a square-free subset of a finite abelian group G. Suppose that the graph $Add(A, G)$ is connected. Let $A' = \{a - b\colon a, b \in A, a \neq b\}$. Then the following statements are equivalent:*

(i) *$Add(A, G)$ is a bipartite graph;*

(ii) *$|\langle A' \rangle| = \frac{1}{2}|G|$;*

(iii) *$A \cap \langle A' \rangle = \varnothing$.*

Furthermore, if the graph $Add(A, G)$ is bipartite, then $\langle A' \rangle$ and $G \setminus \langle A' \rangle$ are the bipartition sets.

Proof. (i) $\Rightarrow$ (ii).

Suppose $Add(A, G)$ is a connected bipartite graph. Proposition 2.4 implies that $\langle A' \rangle$ is contained in a bipartition set of $Add(A, G)$. Therefore, $\langle A' \rangle \neq G$ and Proposition 2.3 implies that $|\langle A' \rangle| = \frac{1}{2}|G|$.

(ii) $\Rightarrow$ (iii).

Suppose $|\langle A' \rangle| = \frac{1}{2}|G|$ but $A \cap \langle A' \rangle \neq \varnothing$. Let $a \in A \cap \langle A' \rangle$. Then, for any $b \in A$, $b = a + (b - a) \in \langle A' \rangle$, i.e., $A \subseteq \langle A' \rangle$. Therefore $\langle A \rangle \neq G$. This contradicts connectedness of $Add(A, G)$.

(iii) $\Rightarrow$ (i).

Suppose $A \cap \langle A' \rangle = \varnothing$. Then $\langle A' \rangle \neq G$ and Proposition 2.3 implies that $|\langle A' \rangle| = \frac{1}{2}|G|$. Let $\{x, y\}$ be an edge of $Add(A, G)$. Then $y = a - x$ with $a \in A$. Since $a \notin \langle A' \rangle$, $\langle A' \rangle$ and $a + \langle A' \rangle$ are the two cosets of $\langle A' \rangle$ in G. Therefore, if $x \in \langle A' \rangle$, then $y \notin \langle A' \rangle$, and, if $x \notin \langle A' \rangle$, then $y \in \langle A' \rangle$. Thus, $Add(A, G)$ is a bipartite graph with bipartition sets $\langle A' \rangle$ and $G \setminus \langle A' \rangle$. $\square$

Three distinct vertices x, y, and z of a graph form a *triangle* if $\{x, y\}$, $\{y, z\}$, and $\{z, x\}$ are edges of the graph. A graph that has no triangle is said to be *triangle-free.*

Proposition 2.7 *Let G be a finite abelian group and let A be a square-free subset of G. The graph $Add(A, G)$ contains a triangle if and only if there exist distinct $a, b, c \in A$ such that $a+b+c$ is a square. If $G = \mathbf{Z}_{q_1} \times \mathbf{Z}_{q_2} \times \cdots \times \mathbf{Z}_{q_s}$, where $q_1, q_2, \ldots, q_s$ are prime powers, then $Add(A, G)$ has exactly $2^k t$ triangles, where t is the number of 3-subsets $\{a, b, c\}$ of A, such that $a + b + c$ is a square, and k is the number of indices $i \in \{1, 2, \ldots, s\}$ such that q_i is even.*

Proof. If $Add(A,G)$ has a triangle with vertices x, y, and z, then $x+y=a$, $y+z=b$, and $z+x=c$ are three distinct elements of A and $a+b+c=2(x+y+z)$ is a square. Conversely, if $a+b+c=2u$ for distinct $a,b,c\in A$, then vertices $x=u-b$, $y=u-c$, and $z=u-a$ form a triangle.

Let $\{a,b,c\}$ be a 3-subset of A such that $a+b+c=2u$ for some $u\in G$. Then $a+b+c=2v$ if and only if $u-v=0$ or $ord(u-v)=2$. Since $G=\mathbf{Z}_{q_1}\times\mathbf{Z}_{q_2}\times\cdots\times\mathbf{Z}_{q_s}$ has exactly 2^k-1 elements of order 2, the graph $Add(A,G)$ has exactly 2^k triangles $\{x,y,z\}$ such that $\{x+y,y+z,z+x\}=\{a,b,c\}$. Therefore, the total number of triangles in $Add(A,G)$ is 2^kt. □

Remark 2.8 *Recall that a graph Γ is* vertex-transitive *if for any two vertices u and v there is an automorphism σ of Γ such that $\sigma(u)=v$. Proposition 2.7 implies that an addition graph does not have to be vertex-transitive. Indeed, let $\Gamma=Add(A,G)$, where $G=\mathbf{Z}_{q_1}\times\mathbf{Z}_{q_2}\times\cdots\times\mathbf{Z}_{q_s}$ with prime powers $q_1,q_2,\ldots,q_s$ and $A=\{a,b,c\}$ is a square-free 3-subset of G such that $a+b+c$ is a square. By Proposition 2.7, Γ has exactly 2^k triangles, where k is the number of indices $i\in\{1,2,\ldots,s\}$ such that q_i is even. On the other hand, if Γ is vertex-transitive, then each vertex of Γ is contained in the same number of triangle, and therefore $|G|$ must divide $3\cdot 2^k$. This is possible only if $G=\mathbf{Z}_2^k$ or $G=\mathbf{Z}_2^k\times\mathbf{Z}_3$.*

Notation. Let A be a square-free subset of a finite abelian group G and let a and b be distinct elements of A. Let $G'=G/\langle a-b\rangle$ and let π be the natural homomorphism from G to G'. If $\Gamma=Add(A,G)$, we will denote by $\Gamma_{a,b}$ the graph $\Gamma_{a,b}=(G',E')$, where $E'=\{\{\pi(x),\pi(y)\}\colon \pi(x)+\pi(y)\in A, \pi(x)\neq\pi(y)\}$.

If Γ is connected, then so is $\Gamma_{a,b}$. Note that if the set $\pi(A)$ is square-free, then $\Gamma_{a,b}=Add(\pi(A),G')$.

Proposition 2.9 *If $\Gamma=Add(A,G)$ is a bipartite graph and a and b are distinct elements of A, then $\Gamma_{a,b}$ is a bipartite graph.*

Proof. Let $(\pi(x_1),\pi(x_2),\ldots,\pi(x_n))$ be a cycle in $\Gamma_{a,b}$. Then $x_1+x_2=a_1+k_1(a-b)$, $x_2+x_3=a_2+k_2(a-b)$,...,$x_n+x_1=a_n+k_n(a-b)$, where $a_1,a_2,\ldots,a_n\in A$ and $k_1,k_2,\ldots,k_n$ are integers. Let $z=a_1+a_2+\cdots+a_n-\sum_{i=1}^n k_i(a-b)$. Then $z=2(x_1+x_2+\cdots+x_n)$ is a square and $z=\sum_{a\in A}k_aa$ with $\sum_{a\in A}k_a=n$. Since Γ is bipartite, Proposition 2.5 implies that n is even. Therefore, $\Gamma_{a,b}$ is bipartite. □

For an element x of a finite abelian group G, we denote by $ind(x)$ the order of the factor-group $G/\langle x\rangle$. The following technical lemma will be used in the sequel.

Lemma 2.10 *Let G_0 be a finite abelian group of odd order and let G_1 be a nontrivial abelian 2-group. Let $G=G_0\times G_1$ and let A be a square-free 3-subset of G. If* (i) *G_1 is not isomorphic to $\mathbf{Z}_2$ and is not isomorphic to $\mathbf{Z}_2^2$ or* (ii) *G_1 is isomorphic to $\mathbf{Z}_2^2$ and the sum of the three elements of A is not a square, then there exist distinct $a,b\in A$ such that $ind(a-b)\equiv 0 \pmod 4$.*

Proof. Let $G_1 = H_1 \times \cdots \times H_s$, where H_i is a cyclic group of order 2^{k_i} with $1 \le k_1 \le k_2 \le \cdots \le k_s$. For any $x \in G$, $ord(x) = 2^k \cdot m$, where $0 \le k \le k_s$ and m is odd. Therefore, if $s \ge 3$ or $s = 2$ and $k_1 \ge 2$, then $ind(x) \equiv 0 \pmod 4$ for all $x \in G$. There are three special cases to consider.

Case 1: $s = 2$, $k_1 = 1$, and $k_2 \ge 2$.

We will represent each $x \in G$ as $x = (x_0, x_1, x_2)$ with $x_0 \in G_0$, $x_1 \in \{0, 1\}$, and $x_2 \in \{0, 1, \ldots, 2^{k_2} - 1\}$. Among any three elements of G, one can choose two whose last components are of the same parity. If a and b are such elements, then $ord(a - b)$ divides $2^{k_2 - 1} \cdot |G_0|$, and therefore, $ind(a - b) \equiv 0 \pmod 4$.

Case 2: $s = 2$, $k_1 = k_2 = 1$, and the sum of the three elements of A is not a square.

We will represent each $x \in G$ as $x = (x_0, x_1, x_2)$ with $x_0 \in G_0$ and $x_1, x_2 \in \{0, 1\}$. Let $A = \{a, b, c\}$. Since a, b, c, and $a + b + c$ are not squares, each of these elements has at least one of the last two components equal to 1. Without loss of generality, we assume that $a_1 = b_1 = c_1 = 1$ and either (1) $a_2 = b_2 = c_2 = 1$ or (2) $a_2 = b_2 = 0$ and $c_2 = 1$. In either case, $ord(a - b)$ is odd and therefore $ind(a - b) \equiv 0 \pmod 4$.

Case 3: $s = 1$ and $k_1 \ge 2$.

We will represent each $x \in G$ as $x = (x_0, x_1)$ with $x_0 \in G_0$, $x_1 \in \{0, 1, \ldots, 2^{k_1} - 1\}$. Since the three elements of A are not squares, their last components are odd. Therefore, one can choose two distinct elements of A, say a and b, whose last components are congruent modulo 4. Then $ord(a-b)$ divides $2^{k_1-2} \cdot |G_0|$ and therefore $ind(a-b) \equiv 0 \pmod 4$. $\square$

3 Cubic addition graphs

Throughout this section, G is a finite abelian group and $A = \{a, b, c\}$ is a square-free 3-subset of G such that the graph $\Gamma = Add(A, G)$ is connected. If $\{x, y\}$ is an edge of Γ, we will call it an a-edge, a b-edge, or a c-edge if $x + y = a$, b, or c, respectively.

Let $H = \langle a - b, a - c \rangle$. Then either $|H| = \frac{1}{2}|G|$ and Γ is bipartite or $H = G$. The bipartite case has the following simple characterization.

Proposition 3.1 *Graph Γ is bipartite if and only if $a + b + c$ is not a square.*

Proof. If $a + b + c$ is a square, then, by Proposition 2.7, Γ contains a triangle and therefore it is not bipartite.

Suppose Γ is not bipartite. Then it has a cycle $(x_0, x_1, \ldots, x_n = x_0)$ of odd length. For $i = 1, 2, \ldots, n$, let $a_i = x_{i-1} + x_i$. Then $a_1 + a_2 + \cdots + a_n$ is a square, i.e., there are integers k, l, and m such that $k + l + m = n$ is odd and $ka + lb + mc = 2u$ is a square. If, say, k and l are even and m is odd, then $c = 2u - ka - lb - (m - 1)c$ is a square. Therefore, k, l, and m are odd, and then $a+b+c = 2u-(k-1)a-(l-1)b-(m-1)c$ is a square. $\square$

Let π be the natural homomorphism from G to $G/\langle a - b \rangle$. Since $\pi(a) = \pi(b)$, each vertex of graph $\Gamma_{a,b}$ is of degree at most 2.

Suppose $\pi(a)$ is a square. Then $a = 2u + k(a-b)$, where $u \in G$ and $k \in \mathbb{Z}$. Since a is not a square, k is odd. But then $b = 2u + (k-1)a - (k-1)b$ is a square. Thus $\pi(a)$ is not a square, and therefore, every vertex of $\Gamma_{a,b}$ is incident with a $\pi(a)$-edge. Since $\Gamma_{a,b}$ is a connected graph, it is either a path or a cycle.

Suppose now that $\pi(c)$ is a square, i.e., $c = 2u + k(a-b)$, where $u \in G$ and $k \in \mathbb{Z}$. Since c is not a square, k is odd. But then $a+b+c = 2u+(k+1)a-(k-1)b$ is a square. Conversely, if $a+b+c = 2u$ is a square, then $\pi(c) = 2\pi(u) - 2\pi(a)$ is a square.

Thus, if Γ is bipartite and connected, then $\Gamma_{a,b}$ is regular of degree 2 and connected, that is, a cycle. If Γ is connected but not bipartite, then $\Gamma_{a,b}$ is a path.

If $\Gamma_{a,b}$ is a path of length 1, then $G = \langle a-b \rangle \cup (a + \langle a-b \rangle)$ and Γ has a Hamilton cycle

$$(0, a, b-a, 2a-b, 2b-2a, 3a-2b, \ldots, (g-1)b-(g-1)a, ga-(g-1)b, 0),$$

where $g = \frac{1}{2}|G|$.

From now on, we assume that $ind(a-b) = 2n \equiv 0 \pmod 4$ (cf. Lemma 2.10), so $|G| = 2mn$, where $m = ord(a-b)$. Then $\Gamma_{a,b}$ is a cycle or a path of length $2n$. Let $X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n$ be consecutive vertices of $\Gamma_{a,b}$. If $\Gamma_{a,b}$ is a path, then, since $\pi(a)$ is not a square, we have $X_1 + Y_1 = \pi(a)$. If $\Gamma_{a,b}$ is a cycle, we assume without loss of generality that $X_1 + Y_1 = \pi(a)$. Since the edges of $\Gamma_{a,b}$ are alternating $\pi(a)$- and $\pi(c)$-edges, we obtain that $X_j + Y_j = \pi(a)$ for $j = 1, 2, \ldots, n$, $Y_j + X_{j+1} = \pi(c)$ for $j = 1, 2, \ldots, n-1$, and, if $\Gamma_{a,b}$ is a cycle, $Y_n + X_1 = \pi(a)$.

Therefore, if $x \in X_j$, then $a-x, b-x \in Y_j$ and, for $j \geq 2$, $c-x \in Y_{j-1}$; if $y \in Y_j$, then $a-y, b-y \in X_j$ and, for $j \leq n-1$, $c-y \in X_{j+1}$. If $\Gamma_{a,b}$ is a cycle and $x \in X_1$, $y \in Y_n$, then $c-x \in Y_n$ and $c-y \in X_1$. If $\Gamma_{a,b}$ is a path and $x \in X_1$, $y \in Y_n$, then $c-x \in X_1$ and $c-y \in Y_n$.

In the next two theorems we construct a Hamilton cycle on Γ for the following two cases: (i) $|G| \equiv 0 \pmod 4$ and Γ is bipartite and (ii) $|G| \equiv 0 \pmod 8$ and Γ is not bipartite.

Theorem 3.2 *Let A be a square-free 3-subset of a finite abelian group G with $|G| \equiv 0 \pmod 4$. If $\Gamma = Add(A,G)$ is a connected bipartite graph, then it is Hamiltonian.*

Proof. Let $A = \{a,b,c\}$ and let $\Gamma = Add(A,G)$ be a connected bipartite graph. Group G and set A satisfy the condition of Lemma 2.10, so we assume that $ind(a-b) = 2n$, where n is even. Let $m = ord(a-b)$ and let $X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n$ be consecutive vertices of $\Gamma_{a,b}$.

Let $H = \langle a-b, a-c \rangle$. Then $|H| = mn$, so $n(c-a) \in \langle b-a \rangle$. Since $\pi(2c-a-b) = 2\pi(c-a)$, we obtain that $\frac{n}{2}(2c-a-b) \in \langle a-b \rangle$. Let $\frac{n}{2}(2c-a-b) = q(b-a)$ with $0 \leq q \leq m-1$. Let d be the greatest common divisor of q and m and let $m = sd$.

For each $z \in X_1$, let $P(z) = (z; ((ba)^{d-1}bc)^{n/2}((ab)^{d-1}ac)^{n/2})$. For $j = 1, 2, \ldots, n$, let $u_j(z)$ be the first vertex of $P(z)$ that lies in X_j. Then $u_1(z) = z$ and, for $j = 2, 3, \ldots, n$,

$$u_j(z) = \begin{cases} u_{j-1}(z) + (d-1)(a-b) + c - b & \text{if } j \leq \frac{n}{2}+1, \\ u_{j-1}(z) + (d-1)(b-a) + c - a & \text{if } j > \frac{n}{2}+1. \end{cases}$$

Therefore, for $z_1, z_2 \in X_1$ and, for $j = 1, 2, \ldots, n$,

$$u_j(z_1) - u_j(z_2) = z_1 - z_2. \tag{1}$$

By a straightforward calculation, we obtain that the last vertex $v(z)$ of $P(z)$ is $v(z) = z + \frac{n}{2}(2c - b - a) = z + q(b - a)$. Note that $v(z) \in X_1$. Let $V(z)$ be the set of all vertices of $P(z)$, except the last one. Then, for $j = 1, 2, \ldots, n$,

$$V(z) \cap X_j = \begin{cases} \{u_j + i(a-b) \colon 0 \le i \le d-1\} & \text{if } j \le \frac{n}{2}, \\ \{u_j - i(a-b) \colon 0 \le i \le d-1\} & \text{if } j > \frac{n}{2} \end{cases} \tag{2}$$

and

$$V(z) \cap Y_j = \begin{cases} b - (V(z) \cap X_j) & \text{if } j \le \frac{n}{2}, \\ a - (V(z) \cap X_j) & \text{if } j > \frac{n}{2}. \end{cases} \tag{3}$$

Equations (1), (2), and (3) imply that if $z_1, z_2 \in X_1$ and $z_1 - z_2 = kd(a - b)$, where $k \not\equiv 0 \pmod{s}$, then $V(z_1) \cap V(z_2) = \varnothing$.

Fix $x \in X_1$ and let $z_k = x + kq(b-a)$ for $k = 0, 1, \ldots, s-1$. Then sets $V(z_0), V(z_1), \ldots, V(z_{s-1})$ are pairwise disjoint. Since $\sum_{k=0}^{s-1} |V(z_k)| = s|V(z_0)| = s \cdot 2nd = 2mn = |G|$, the union of the sets $V(z_k)$, $k = 0, 1, \ldots, s-1$, is the group G. Observe that the $v(z_{k-1}) = z_k$ for $k = 1, 2, \ldots, s-1$ and $v(z_{s-1}) = z_0$. Therefore, $P(z_0) * P(z_1) * \ldots * P(z_{s-1})$ is a Hamilton cycle in Γ. $\square$

Figure 3 demonstrates a bipartite addition graph $Add(A, G)$, where G is the direct product of cyclic groups ogf order 6 and of order 12. The vertices of the graph are ordered pairs ab with $a = 0, 1, 2, 3, 4, 5$ and $b = 0, 1, 2, \ldots, 11$. For convenience, we use E instead of 11 and T instead of 12. Theset $A = \{a, b, c\}$ with $a = 10$, $b = 12$, and $c = 21$. The thick edges of the graph form a Hamilton cycle obtained according to the algorithm described in Theorem 3.2

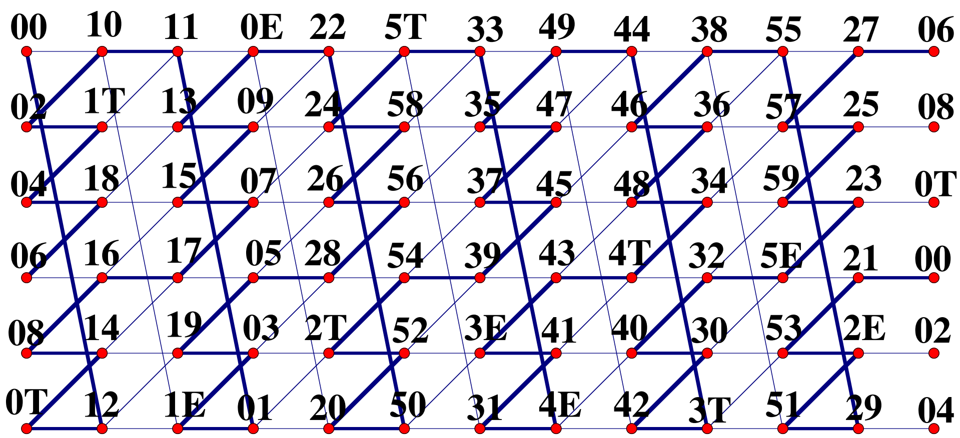

Figure 3

Remark 3.3 *The only case of connected cubic bipartite addition graphs that is not covered by Theorem 3.2 is* $Add(A, \mathbf{Z}_2 \times H)$*, where* H *is an abelian group of odd order and* $A = \{(1,a),(1,b),(1,c)\}$ *with* $a, b, c \in H$ *such that* $\langle a-b, a-c\rangle = H$*. Theorems 4.6 and 5.2 cover this case if* H *is a cyclic group or* $|H|$ *is a prime power.*

Remark 3.4 *Observe that the vertex* x *in the last paragraph of the proof of Theorem 3.2 could be any vertex of* $Add(A, G)$*. One of the two edges of the Hamilton cycle that are incident with* x *is a* c*-edge and the other is either an* a*-edge or a* b*-edge. Since* a *and* b *are interchangeable in the proof, we conclude that for any edge of* $Add(A, G)$ *there is a Hamilton cycle that contains this edge.*

We will now turn our attention to non-bipartite connected cubic addition graphs. For such a graph Γ, every element of X_1 is connected by a c-edge to another element of X_1 and therefore $|X_1| = m$ is even. This implies that $|G| = 2mn$ is divisible by 4.

Theorem 3.5 *Let* A *be a square-free* 3*-subset of a finite abelian group* G *with* $|G| \equiv 0 \pmod 8$*. If* $\Gamma = Add(A, G)$ *is a connected graph, then it is Hamiltonian.*

Proof. Let $A = \{a, b, c\}$ and let $\Gamma = Add(A, G)$ be a connected graph. Group G and set A satisfy the condition of Lemma 2.10, so we assume that $ind(a - b) = 2n$, where n is even. Let $m = ord(a - b)$. Due to Theorem 3.2, we may assume that Γ is not bipartite, i.e., $a + b + c$ is a square. Let $(X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$ be the path formed by the vertices of $\Gamma_{a,b}$. Since every element of X_1 is connected by a c-edge to another element of X_1, m is even. Observe that if $z \in X_1$, then, for $j = 0, 1, \ldots, n-1$,

$z+j(c-a) \in X_{j+1}$, $z+(n+j)(c-a) \in Y_{n-j}$, and $z+2n(c-a) \in X_1$. Therefore, $ord(\pi(c-a)) = 2n$. Since $\pi(2c-a-b) = 2\pi(c-a)$, we have $n(2c-a-b) \in \langle a-b \rangle$. Let $n(2c-a-b) = q(b-a)$ with $0 \leq q \leq m-1$. Then $(n-q)a+(n+q)b-2nc = 0$. If q is odd, then $c = (a+b+c)-(n-q+1)a-(n+q+1)b+2nc$ is a square. Therefore, q is even. Let d be the greatest common divisor of q and m. Since m is even, we have $d \geq 2$. Let $s = md$.

For each $z \in X_1$, let $P(z) = (z; w_1 w_2 \cdots w_{2n})$, where

$$w_j = \begin{cases} bc & \text{if } j \leq n \text{ and } j \text{ is odd,} \\ ac & \text{if } j \leq n \text{ and } j \text{ is even,} \\ (ab)^{d-2}ac & \text{if } j > n \text{ and } j \text{ is odd,} \\ (ba)^{d-2}bc & \text{if } j > n \text{ and } j \text{ is even.} \end{cases}$$

The last vertex of $P(z)$ is $v(z) = z+n(2c-a-b) = z+q(b-a)$. Note that $v(z) \in X_1$

Let $V(z)$ be the set of all vertices of $P(z)$, except the last one. For $j = 1, 2, \ldots, n$, let $V_j(z) = V(z) \cap X_j$. Then

$$V_1(z) = \{z\} \cup \{c - v(z) + i(a-b) \colon 0 \leq i \leq d-2\} \tag{4}$$

and, for $j = 2, \ldots, n$,

$$V_j(z) = \begin{cases} V_{j-1}(z) + c - b & \text{if } j \text{ is even,} \\ V_{j-1}(z) + c - a & \text{if } j \text{ is odd.} \end{cases} \tag{5}$$

For $j = 1, 2, \ldots, n$, we have

$$V(z) \cap Y_j = \begin{cases} b - V_j(z) & \text{if } j \text{ is odd,} \\ a - V_j(z) & \text{if } j \text{ is even.} \end{cases} \tag{6}$$

The order of group $G' = G/\langle d(a-b) \rangle$ is d. Let ρ be the natural homomorphism from G to G'.

Claim. There exists $z_0 \in X_1$ such that $\rho(V_1(z_0)) = G'$.

Since $\rho(v(z_0)) = \rho(z_0)$, (4) implies that this claim is equivalent to

$$\rho(z_0) \notin \{\rho(c) - \rho(z_0) + i(\rho(a) - \rho(b)) \colon 0 \leq i \leq d-2\},$$

which is equivalent to

$$2\rho(z_0) = \rho(c) - \rho(a) + \rho(b).$$

Let $x \in X_1$. Then $c - x \in X_1$, so $c - 2x \in \langle a-b \rangle$. let $c - 2x = k(a-b)$. Since c is not a square, k is odd. Let $z_0 = x + \frac{k-1}{2}(a-b)$. Then $z_0 \in X_1$, $2z_0 = c-a+b$, and the claim is proven.

Let $z \in X_1$ be such that $\rho(V_1(z)) = G'$. Then $\rho(V_j(z)) = G'$ for $j = 1, 2, \ldots, n$. Let $z' = z + kd(a-b)$, where $k \not\equiv 0 \pmod{s}$. Then $z' \in X_1$ and $\rho(V_1(z')) = G'$. Moreover, since $\rho(z') = \rho(z)$ and $\rho(V_1(z') \setminus \{z'\}) = \rho(V_1(z) \setminus \{z\})$, the sets $V_1(z)$ and $V_1(z')$ are disjoint. Eqs. (5) and (6) then imply that $V(z) \cap V(z') = \varnothing$.

Let $z_0 \in X_1$ be provided by the above claim and let $z_k = z_0 + kq(b-a)$ for $k = 0, 1, \ldots, s-1$. Then sets $V(z_0), V(z_1), \ldots, V(z_{s-1})$ are pairwise disjoint. Since $\sum_{k=0}^{s-1} |V(z_k)| = s|V(z_0| = |G|$, the union of the sets $V(z_k)$, $k = 0, 1, \ldots, s-1$, is G. Since $v(z_{k-1}) = z_k$, for $k = 1, 2, \ldots, s-1$, and $v(z_{s-1}) = z_0$, we obtain a Hamilton cycle $P(z_0) * P(z_1) * \ldots * P(z_{s-1})$ in Γ. $\square$

Remark 3.6 *The only case of connected cubic non-bipartite addition graphs that is not covered by Theorem 3.5 is* $Add(A, \mathbf{Z}_2^2 \times H)$*, where* H *is an abelian group of odd order and* $A = \{(1,1,a), (1,0,b), (0,1,c)\}$ *with* $a, b, c \in H$ *such that* $\langle a-b, a-c \rangle = H$*.*

4 Bipartite addition graphs and Cayley graphs

In this section we will show that every connected bipartite addition graph is in fact a Cayley graph. We begin with two group-theoretic constructions.

Let G be an additively written abelian group with a fixed subgroup H of index 2. Define a multiplication on G as follows:

$$xy = \begin{cases} y + x & \text{if } y \in H, \\ y - x & \text{if } y \notin H. \end{cases} \tag{7}$$

This multiplication is associative: for $y, z \in H$, $(xy)z = x(yz) = z + y + x$; for $y \in H, z \notin H$, $(xy)z = x(yz) = z - y - x$; for $y \notin H, z \in H$, $(xy)z = x(yz) = z + y - x$; for $y, z \notin H$, $(xy)z = x(yz) = z - y + x$. For the neutral element 0 of G and for all $x \in G$, we have $x0 = 0x = x$. For $x \in H$, $x(-x) = 0$; for $x \notin H$, $xx = 0$. Thus, the elements of G form a group with respect to this multiplication. We will denote this group by G^*. Observe that H is an abelian subgroup of G^* (of index 2) and every element of $G^* \setminus H$ is of order 2.

Remark 4.1 *The group* G^* *is a semidirect product* $\mathbf{Z}_2 \ltimes H$*.*

We follow Curran and Gallian [4] and call G^* a group of *dihedral type.*

Definition 4.2 *A group* G *is said to be of* dihedral type *if it has a subgroup* H *of index 2 such that every element of* $G \setminus H$ *is of order 2.*

Remark 4.3 *A nonabelian group* G *of dihedral type is a dihedral group if and only if the subgroup* H *is cyclic.*

Let G be a multiplicatively written group of dihedral type with respect to an abelian subgroup H of index 2. Fix $a \in G \setminus H$ and define an addition on G as follows:

$$x + y = \begin{cases} xy & \text{if } y \in H, \\ axay & \text{if } y \notin H. \end{cases} \tag{8}$$

This addition is associative: for $y, z \in H$, $(x+y)+z = x+(y+z) = xyz$; for $y \in H, z \notin H$, $(x+y)+z = x+(y+z) = axyaz$; for $y \notin H, z \in H$, $(x+y)+z = x+(y+z) = axayz$; for $y, z \notin H$, $(x+y)+z = x+(y+z) = xayaz$. This addition is commutative: for $x, y \in H$, $x+y = y+x = xy$; for $x \in H, y \notin H$, $x+y = axay = (ax)^{-1}ay = x^{-1}y = (x^{-1}y)^{-1} = yx = y+x$; for $x \notin H, y \in H$, $y+x = ayax = (ay)^{-1}ax = y^{-1}x = (y^{-1}x)^{-1} = xy = x+y$; for $x, y \notin H$, $x+y = (ax)(ay) = (ay)(ax) = y+x$. If e is the neutral element of G, then $x+e = x$ for all $x \in G$. For $x \in H$, $x + x^{-1} = e$; for $x \notin H$, $x + axa = e$.

Thus, with respect to this addition, the elements of G form an abelian group. We will denote this group by G^+.

Proposition 4.4 *Let A be a square-free subset of a finite abelian group G and let $\Gamma = Add(A, G)$ be a connected bipartite graph. Let $H = \langle A' \rangle$ and let G^* be the group defined by* (7)*. Then $\Gamma = Cay(A, G^*)$.*

Proof. If $\Gamma = Add(A, G)$ is connected and bipartite, then, by Proposition 2.6, $|H| = \frac{1}{2}|G|$ and $A \cap H = \varnothing$. Therefore, for all $x, y \in G$ and for any $a \in A$, $y = xa$ in the group G^* if and only if $y = a - x$ in the group G. Thus, the graphs $Add(A, G)$ and $Cay(A, G^*)$ have the same edge set (and the same vertex set). $\square$

Since all Cayley graphs are vertex-transitive, we obtain that all connected bipartite addition graphs are vertex-transitive. It is possible to show that all regular bipartite addition graphs are vertex-transitive. As we have shown in Remark 2.8, not all regular connected addition graphs are vertex-transitive.

Proposition 4.5 *Let G be a finite group of dihedral type with an abelian subgroup H of index 2 and a fixed element $a \in G \setminus H$. Let $A \subseteq G \setminus H$ and let $\Gamma = Cay(A, G)$. Let G^+ be the group defined by* (8)*. If Γ is connected, then $\Gamma = Add(A, G^+)$ and Γ is bipartite.*

Proof. Let $\Gamma = Cay(A, G)$ and $\Delta = Add(A, G^+)$.

If $\{x, y\} \in E(\Gamma)$, then $x = yc$ with $c \in A$. Without loss of generality, we assume that $x \notin H$ and $y \in H$. Then $x + y = xy = ycy = (yc)^{-1}y = c$, so $\{x, y\} \in E(\Delta)$. Since Γ is connected, so is Δ. If $b, c \in A$, then $b - c = b + aca = (ab)(ca) \in H$. Therefore, the subgroup generated by A' in G^+ is contained in H. Since Δ is connected and $|H| = \frac{1}{2}|G|$, Propositions 2.3 and 2.6 imply that the subgroup generated by A' in G^+ is H, and therefore Δ is bipartite. If $\{x, y\} \in E(\Delta)$, then $x + y = c \in A$, and we assume without loss of generality that $x \notin H$ and $y \in H$. Then $xy = c$ and $yc = yxy = (yx)^{-1}y = x$, so $\{x, y\} \in E(\Gamma)$. Therefore, $\Gamma = \Delta$. $\square$

We will apply Propositions 4.4 and 4.5 to obtain new results for both addition graphs and Cayley graphs.

In the paper [1], Alspach and Zhang proved that every connected cubic Cayley graph on a dihedral group is Hamiltonian. Combining this result with Proposition 4.5 yields the following theorem.

Theorem 4.6 *Let A be a square-free 3-subset of a group $G = \mathbf{Z}_2 \times H$, where H is a cyclic group of odd order. If $\Gamma = Add(A, G)$ is a connected bipartite graph, then it is a Hamiltonian graph.*

Proof. Let $\Gamma = Add(A, G)$ be a connected bipartite graph. Since A is square-free, we can represent its three elements as $(1, a)$, $(1, b)$, and $(1, c)$ with $a, b, c \in H$. Therefore $\langle A' \rangle \subseteq H$ and then Proposition 2.3 implies that $\langle A' \rangle = H$. Let G^* be the group defined by (7). Then H is a cyclic subgroup of G^*, and therefore G^* is either abelian or dihedral. In either case, $Cay(A, G^*)$ is Hamiltonian. By Proposition 4.4, $\Gamma = Cay(A, G^*)$ and therefore Γ is Hamiltonian. $\square$

Theorem 3.2 is instrumental in obtaining a previously unknown family of groups, on which all connected cubic Cayley graphs are Hamiltonian.

Theorem 4.7 *If G is a finite group of dihedral type and $|G| \equiv 0 \pmod 4$, then every connected cubic Cayley graph on G is Hamiltonian.*

Proof. Let G be a finite group of dihedral type, that is, G has an abelian subgroup H of index 2 such that every element of $G \setminus H$ is of order 2. Let $|G| \equiv 0 \pmod 4$. Let $A = \{a, b, c\}$ be a 3-subset of G such that the neutral element e of G is not in A and, for any $x \in G$, $x \in A$ if and only if $x^{-1} \in A$. Let $\Gamma = Cay(A, G)$ be a connected Cayley graph. Then $\langle A \rangle = G$ and therefore $0 \leq |A \cap H| \leq 2$. Let $a \notin H$.

Case 1: $A \cap H = \varnothing$.

Consider the group G^+ defined by (8). By Proposition 4.5, $\Gamma = Add(A, G^+)$. By Theorem 3.2, Γ is Hamiltonian.

Case 2: $|A \cap H| = 1$.

Let $c \in A \cap H$. Then $ord(c) = 2$.

Let $ord(ab) = m$. We have $ac = (ac)^{-1} = ca$, $bc = cb$, and $ba = (ab)^{-1}$. Therefore, $H = \{(ab)^k c^l \colon 0 \leq k \leq m-1, 0 \leq l \leq 1\}$. If $c \in \langle ab \rangle$, then H is a cyclic group. Therefore, either G is a dihedral group and Γ is Hamiltonian by Alspach and Zhang [1], or G is abelian, and again Γ is Hamiltonian. Suppose $c \notin \langle ab \rangle$. Then $|G| = 4m$ and G can be decomposed into four cosets of cardinality m: $X = \{x_k = (ab)^k \colon k \in \mathbb{Z}\}$, $Y = \{y_k = (ab)^k a \colon k \in \mathbb{Z}\}$, $Z = \{z_k = (ab)^k c \colon k \in \mathbb{Z}\}$, and $W = \{w_k = (ab)^k ac \colon k \in \mathbb{Z}\}$.

Observe that

$$E(\Gamma) = \{\{x_k, y_k\}, \{x_k, y_{k-1}\}, \{x_k, z_k\}, \{y_k, w_k\}, \{z_k, w_k\}, \{z_k, w_{k-1}\} \colon k \in \mathbb{Z}\}.$$

Therefore, Γ admits the following Hamilton cycle:

$$(x_0, y_0, x_1, y_1, \ldots, x_{m-1}, y_{m-1}, w_{m-1}, z_{m-1}, w_{m-2}, z_{m-2}, \ldots, w_0, z_0, x_0).$$

Case 3: $|A \cap H| = 2$.

Then $c = b^{-1}$ and therefore $H = \langle b \rangle$ is a cyclic group. Then G is abelian or dihedral and therefore Γ is Hamiltonian. $\square$

5 Cayley digraphs and bipartite addition graphs

Let S be a subset of a group H that does not contain the neutral element. The *Cayley digraph* of S on G is the digraph $Cay(S,H) = (H,E)$ with the arc set $E = \{(x,xs)\colon s \in S\}$.

An addition graph $\Gamma = Add(A,G)$ on an abelian group G induces a Cayley digraph $\Delta = Cay(A',H)$, where $H = \langle A' \rangle$. If Γ is bipartite and has a Hamilton cycle $(0, y_0, x_1, y_1, \dots, x_{n-1}, y_{n-1}, 0)$, then $(0, x_1, x_2, \dots, x_{n-1})$ is a Hamilton cycle on Δ. However, Cayley digraphs that can be obtained in this manner are rather special, and the construction is not reversible. We introduce another relation between bipartite addition graphs and Cayley digraphs which seems to be more promising.

For a subset A of an abelian group G and an element a of A, let $A_a = \{b - a\colon b \in A, b \neq a\}$. For $b, c \in A$, $b - c = (b-a) - (c-a)$, so $\langle A_a \rangle = \langle A' \rangle$. Let $\Gamma = Add(A,G)$ and let $a \in A$. We denote by Γ_a the digraph $Cay(A_a, H)$, where $H = \langle A_a \rangle$.

Proposition 5.1 *Let A be a square-free subset of a finite abelian group G and let $\Gamma = Add(A,G)$ be a connected bipartite graph. Let $a \in A$ and let $H = \langle A_a \rangle$. Then the following two statements are equivalent.*

(i) *The digraph $Cay(A_a, H)$ is Hamiltonian.*

(ii) Γ *admits a Hamilton cycle that contains all a-edges of* Γ.

Proof. Let $H = \{x_0, x_1, \dots, x_{n-1}\}$. Since Γ is bipartite, $|G| = 2n$ and $G = H \cup (H+a)$. If $(x_0, x_1, \dots, x_{n-1}, x_n = x_0)$ is a Hamilton cycle in the digraph $Cay(A_a, H)$, then, for $i = 1, 2, \dots, n$, $x_i = x_{i-1} + b_i - a$ with $b_i \in A$. Then $(x_0, a - x_0, x_1, a - x_1, \dots, x_{n-1}, a - x_{n-1}, x_0)$ is a Hamilton cycle in G. It contains a-edges $\{x, a - x\}$ for each $x \in H$, that is, all a-edges of Γ.

Conversely, if $(0 = x_0, y_0, x_1, y_1, \dots, x_{n-1}, y_{n-1}, x_n = 0)$ is a Hamilton cycle in G that contains all a-edges, then either $x_0 + y_0 = a$ or $x_n + y_{n-1} = a$. Without loss of generality we assume that $x_0 + y_0 = a$ and then $x_i + y_i = a$ for $i = 0, 1, \dots, n-1$. For $i = 1, 2, \dots, n$, let $b_i = x_i + y_{i-1}$. Then $x_i - x_{i-1} = b_i - a$ and therefore $(x_0, x_1, \dots, x_n)$ is a Hamilton cycle in the digraph $Cay(A_a, H)$. □

In the paper [6], Witte proved that every connected Cayley digraph on a group of prime-power order greater than 2 is Hamiltonian. This result immediately implies

Theorem 5.2 *Let A be a square-free subset of a finite abelian group G such that $|G| = 2q$, where q is a prime power. If the graph $Add(A,G)$ is connected and bipartite, then it is Hamiltonian.*

Remark 5.3 *The definition of addition graphs can be generalized in the following way: if A is a subset of an abelian group G, then $Add(A,G) = (G,E)$, where $E = \{\{x,y\}\colon x+y \in A, x \neq y\}$. This generalization does not assume that the set A is square-free. Such graph $Add(A,G)$ is regular of degree $|A|$ if and only if A is square-free and it is regular of degree $|A|-1$ if and only if $x + x \in A$ for all $x \in G$. In the latter case, it would be interesting to* *know whether all connected cubic addition graphs are Hamiltonian (the set A has to be of cardinality 4). Note that if $Add(A, G)$ is connected but not regular, it does not have to be Hamiltonian. For instance, let $G = \mathbf{Z}_7$ and let $A = \{1, 2, 4\}$. Then $Add(A, G)$ has three verices of degree 2, namely, 1, 2, and 4, and four vertices of degree 3. Since the graph has an edge from 0 to each vertex of degree 2, it is not Hamiltonian.*

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