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1. Introduction

The old hats problem goes by many names (originally described by Montmort in 1713) but is generally described as:

> A group of n men enter a restaurant and check their hats. The hat-checker is absent minded, and upon leaving, she redistributes the hats back to the men at random. What is the probability P_n that no man gets his correct hat, and how does P_n behave as n approaches infinity?

This problem is just a standard question about derangements[i]. Three solutions are presented in this paper. The first two apply standard approaches:

- Using the inclusion-exclusion principle, and
- Using the recurrence relation $P_n = P_{n-1} - 1/n\,(P_{n-1} - P_{n-2})$[ii].

Finally, the main point of interest in this paper is a relatively different approach using the technique of binomial inversion. In this presentation, we will also solve the related question: What is the expected value of the number of men who receive their correct hats? We conclude the paper with a derivation of the binomial inversion formula itself.

2. Solution Using the Inclusion-Exclusion Principle

Let N denote the total number of permutations of n hats. To calculate the number of derangements, D_n, we want to exclude all permutations possessing any of the attributes $a_1, a_2, \ldots, a_n$ where a_i is the attribute that man i gets his correct hat for $1 \le i \le n$. Let $N(i)$ denote the number of permutations possessing attribute a_i (and possibly others), $N(i,j)$ the number of permutations possessing attributes a_i and a_j (and possibly others), etc. Then the inclusion-exclusion principle states that

$$D_n = N - \sum_{1\le i\le n} N(i) + \sum_{1\le i<j\le n} N(i,j) - \sum_{1\le i<j<k\le n} N(i,j,k) + \cdots + (-1)^n N(1,2,\ldots,n)$$

By symmetry, $N(1) = N(2) = \ldots = N(i)$, $N(1,2) = N(1,3) = \ldots = N(i,j)$, and so on.

So we have

$$D_n = N - \binom{n}{1}N(1) + \binom{n}{2}N(1,2) - \binom{n}{3}N(1,2,3) + \cdots + (-1)^n \binom{n}{n}N(1,2,3,\ldots n)$$

Now, N(1), the number of permutations where man 1 gets his correct hat, is simply (n-1)!, since the remaining hats can be distributed in any order. Similarly, N(1,2)=(n-2)!, N(1,2,3)=(n-3)!, and so forth. Therefore

$$D_n = N - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n \binom{n}{n}(n-n)!$$

Replacing N, the total permutations for n hats by n!, and simplifying the above expression, we obtain:

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!})$$

To generate P_n, the probability of a derangement occurring for n hats, we simply divide the total number of derangements D_n by the total permutations of the n hats, n!, to get:

$$P_n = (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!})$$

We easily recognize this series as approaching 1/e as n approaches infinity.

3. Solution Using a Recurrence Relation

In this section we derive the formula for D_n using a recurrence relation, as follows. If there is a derangement, then man #1 will not have his correct hat. We begin by looking at the case where man #1 gets hat #2. Note that this case can be broken down into two subcases:

a) man #2 gets hat #1, or
b) man #2 does not get hat #1.

In case (a), for a derangement to occur, we need the remaining n-2 men to get the wrong hats. Therefore, the total number of derangements in this subcase is simply D_{n-2}.

In case (b), for a derangement to occur, man #2 cannot get hat #1 (that's case a), man #3 cannot get hat #3, man #i cannot get hat #i, etc. In this subcase, the number of derangements is D_{n-1}. The fact that in this subcase man #2 cannot get hat #1 rather than hat #2 is inconsequential.

We can treat the cases where man #1 receives hat #3, or hat #4, or hat #i, in exactly the same way. Therefore, to account for all possible derangements, there are n-1 such possibilities for all the different incorrect hats which man #1 can get. So,

$$D_n = (n-1)\,(D_{n-1} + D_{n-2})$$

The probability of a derangement is again the number of derangements, D_n, divided by all possible outcomes, $n!$. So,

$$P_n = \frac{D_n}{n!} = \frac{(n-1)}{n!}(D_{n-1} + D_{n-2})$$

$$P_n = (n-1)\left[\frac{1}{n}\frac{D_{n-1}}{(n-1)!} + \frac{1}{n(n-1)}\frac{D_{n-2}}{(n-2)!}\right]$$

$$= (1-\frac{1}{n})P_{n-1} + \frac{1}{n}P_{n-2}$$
$$= P_{n-1} - \frac{1}{n}(P_{n-1} - P_{n-2})$$

Now, we know that $P_1 = 0$, and $P_2 = 1/2$. So, using our formula, we can calculate successively

$$P_3 = P_2 - \frac{1}{3}(P_2 - P_1) = \frac{1}{2} - \frac{1}{3}.\frac{1}{2} = \frac{1}{2} - \frac{1}{2\cdot 3}$$
$$P_4 = P_3 - \frac{1}{4}(P_3 - P_2) = \frac{1}{2} - \frac{1}{2\cdot 3} - \frac{1}{4}(-\frac{1}{2\cdot 3}) = \frac{1}{2} - \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4}$$

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$$P_n = P_{n-1} - \frac{1}{n}(P_{n-1} - P_{n-2})$$

$$= \frac{1}{2} - \frac{1}{2\cdot 3} + \ldots + \frac{(-1)^{n-1}}{2\cdot 3\cdot 4\cdots(n-1)} - \frac{1}{n}\frac{(-1)^{n-1}}{2\cdot 3\cdot 4\cdots(n-1)}$$
$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\ldots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}$$

Once again, we recognize this as approaching $1/e$ as n approaches infinity.

More formally, we can write

$$P_n = P_{n-1} - \frac{1}{n}(P_{n-1} - P_{n-2})$$

as

$$P_n - P_{n-1} = -\frac{1}{n}(P_{n-1} - P_{n-2})$$

If we let $Q_n = P_n - P_{n-1}$, then

$$\begin{aligned} Q_n &= -\frac{1}{n} Q_{n-1} \\ &= -\frac{1}{n}\left(\frac{-1}{n-1}\right) Q_{n-2} \\ &= -\frac{1}{n}\left(\frac{-1}{n-1}\right)\left(\frac{-1}{n-2}\right) Q_{n-3}; \\ &= \ldots = (-1)^{n-2} \frac{1}{n}\left(\frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)\cdots\left(\frac{1}{3}\right) Q_2 \end{aligned}$$

Now, our base case is Q_2, which equals P_2-P_1= 1/2-0 = 1/2. Thus,

$$Q_n = (-1)^n \frac{1}{n!}.$$

Now, we can set up a telescoping sum to calculate P_n:

$$\begin{aligned} Q_n &= P_n - P_{n-1} \\ Q_{n-1} &= P_{n-1} - P_{n-2} \\ &\vdots \\ Q_2 &= P_2 - P_1 \end{aligned}$$

Therefore, summing both sides, we get

$P_n - P_1 = \sum_{i=2}^{n} Q_i$, and since P_1=0, it follows that $P_n = \sum_{i=2}^{n} (-1)^i \frac{1}{i!}$.

4. Solution by Binomial Inversion

The old hats problem can also be solved by using binomial inversion. Specifically, given that

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i$$

we can retrieve the a_i coefficients using the formula

$$a_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} b_i$$

This is the binomial inversion formula. For now, we assume it to be true, and use it to solve our problem. We will prove the formula using exponential generating functions in Section 6.

The set of permutations of the hats can be expressed as the disjoint union of n+1 subsets $A_0, A_1, A_2,\ldots,A_n$, where A_i is the set of permutations where exactly n-i men receive their correct hats. For instance, if only two of the men get their correct hat back, this would correspond to the set of permutations A_{n-2}.

If we take as an example this subset A_{n-2}, we see that if the two men who get the correct hats are man #1 and man #2, then the number of possible arrangements is just D_{n-2}, the number of derangements for the remaining n-2 men. However, there are $\binom{n}{2}$ possible pairings of the two men who get their correct hats. Therefore, the number of permutations in set A_{n-2} is:

$$|A_{n-2}| = \binom{n}{2} D_{n-2}\,.$$

Proceeding with a similar logic, and using the fact that

$\binom{n}{i} = \binom{n}{n-i}$, we may now write the total permutations as:

$$n! = \binom{n}{n} D_n + \binom{n}{n-1} D_{n-1} + \binom{n}{n-2} D_{n-2} + \binom{n}{n-3} D_{n-3} + \cdots + \binom{n}{0} D_0$$

Setting $n! = b_n$, we have $b_n = \sum_{i=0}^{n} \binom{n}{i} D_i$.

We can now use binomial inversion to obtain D_n:

$$D_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} b_i \,,$$

where each $b_i = i!$. Therefore,

$$D_n = \sum_{i=0}^{n} (-1)^{n-i} \frac{n!}{(n-i)!i!} \cdot i!$$

Dividing by *n!*, letting $j = n\text{-}i$, and reversing the order of summation, we get once again:

$$P_n = \sum_{j=0}^{n} (-1)^{j} \frac{1}{j!}$$

5. The Expected Value of Correct Hats

A nice application to show the utility of this "binomial" approach is the calculation of the expected value of the correct number of hats. In other words, if n men walk into the restaurant and check their hats, and then walk out, receiving their hats at random, any number from 0 to n (with the exception of n-1) men may have their correct hat. If this experiment is repeated numerous times, how many men on average would have gotten their correct hat? The answer to this question can be defined to be $E(n)$, the expected number of correct hats for n men. If we let P_j be the probability that when n leave the restaurant, j men have their correct hats, then

$$E(n) = \sum_{j=0}^{n} P_j (j)$$

Let us use P_2 as an example to better understand the form of P_j. This is simply the number of ways in which two men may receive their correct hat, divided by the total number of possible arrangements of the hats, *n!*. From the foregoing, we see that this can be written as

$$P_2 = \frac{\binom{n}{2} D_{n-2}}{n!}$$

Proceeding analogously, we can then write

$$E(n) = \sum_{j=0}^{n} \frac{\binom{n}{j} D_{n-j}(j)}{n!}$$

Now, if we let j=n-i, and recall that $\binom{n}{j} = \binom{n}{n-i} = \binom{n}{i}$, we can write

$$E(n) = \sum_{i=0}^{n} \frac{\binom{n}{i} D_i(n-i)}{n!}$$

When this is simplified, we get

$$E(n) = \sum_{i=0}^{n} \frac{D_i(n-i)}{(n-i)!i!} = \sum_{i=0}^{n} \frac{D_i}{(n-i-1)!i!}.$$

Now, if we multiply both sides by (n-1)!, we get

$$(n-1)!E(n) = \sum_{i=0}^{n} \frac{D_i(n-1)!}{(n-1-i)!i!} = \sum_{i=0}^{n} \binom{n-1}{i} D_i \ .$$

Since the term in the summation will equal 0 for i=n, we can rewrite this equation as

$$(n-1)!E(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} D_i$$

Comparing this to our result from above, we recognize the right hand side of the equation immediately as $(n-1)!$, yielding the amazing result that $E(n)$ = 1, and is independent of n.

6. Derivation of the Binomial Inversion Formula

We now derive the binomial inversion formula used in Section 4.

If we know that

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i \,, \tag{1}$$

we wish to show that this can be inverted to retrieve the a_i coefficients as follows:

$$a_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} b_i \tag{2}$$

This can be done by introducing exponential generating functions. If

$$A(x) = \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \tag{3}$$

$$B(x) = \sum_{k=0}^{\infty} \frac{b_k x^k}{k!} \tag{4}$$

then substituting b_k from (1) into (4), we get

$$B(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{k}{i} a_i \right) \frac{x^k}{k!} \tag{5}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} a_i \frac{x^k}{k!} \tag{6}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{a_i}{i!(k-i)!} x^k \tag{7}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \left(\frac{a_i x^i}{i!} \right) \left(\frac{x^{k-i}}{(k-i)!} \right) \tag{8}$$

We now interchange the order of summation, still summing over the same half-plane.

Hence, (8) can be rewritten as,

$$B(x) = \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \left(\frac{a_i x^i}{i!} \right) \left(\frac{x^{k-i}}{(k-i)!} \right) \qquad (9)$$

Since the terms indexed by *i* can be considered constant with respect to the summation over *k*, the equation can be rearranged as follows:

$$B(x) = \sum_{i=0}^{\infty} \left(\frac{a_i x^i}{i!} \right) \sum_{k=i}^{\infty} \left(\frac{x^{k-i}}{(k-i)!} \right) \qquad (10)$$

Letting $j = k\text{-}i$, we can rewrite (10) as

$$B(x) = \sum_{i=0}^{\infty} \left(\frac{a_i x^i}{i!} \right) \sum_{j=0}^{\infty} \left(\frac{x^j}{j!} \right) \qquad (11)$$

The left-hand summation is recognized as *A(x),* while the right-hand summation is e^x.

Therefore, we can now write

$$B(x) = A(x)\, e^x \qquad (12)$$

from which we get that

$$A(x) = e^{-x} B(x) \qquad (13)$$

Since $B(x) = \sum_{i=0}^{\infty} \frac{b_i x^i}{i!}$, and $e^{-x} = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}$, we have

$$A(x) = \left[\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!} \right] \left[\sum_{i=0}^{\infty} b_i \frac{x^i}{i!} \right] \qquad (14)$$

Becaues the indices are constant with respect to each other, we can write (14) as

$$A(x) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^j x^j}{j!} b_i \frac{x^i}{i!} \qquad (15)$$

Letting $n = i + j$, and once again re-indexing to change the order in which we sum over the i-j plane, we have

$$A(x) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-1)^{n-i} x^{n-i}}{(n-i)!} b_i \frac{x^i}{i!} \tag{16}$$

Now, we simplify, rearrange the variables, and multiply numerator and denominator by *n!* to get

$$A(x) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{n!}{(n-i)!\,i!} (-1)^{n-i} \frac{x^n}{n!} b_i \tag{17}$$

This is again rearranged to give

$$A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b_i \tag{18}$$

Now, comparing (18) with (3),

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

(written with *n* instead of *k* as the summation label), we see that

$$a_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} b_i \ .$$

[i] Mathematics of Choice. Ivan Niven. Mathematical Association of America, 1965, p. 78.

[ii] Challenging Mathematical Problems with Elementary Solutions, Volume I. A.M. Yaglom and I.M. Yaglom. Dover Publications, New York, 1964, p.158.

Interesting related articles that generalize the problem of derangements are:

[iii] Matching, Derangements, Recontres; D. Hanson, K. Seyfforth, J.H. Weston
Mathematics Magazine:Volume 56, Number 4, pages 224-229.

[iv] The Secretary's Packet Problem; Steve Fisk
Mathematics Magazine: Volume 61, Number 2, pages 103-105