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An identity of derangements

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Abstract

In this note, we present a new identity for derangements. As a corrolary, we have a combinatorial proof of the irreducibility of the standard representation of symmetric groups.

1 Introduction

A derangement is the permutation $\sigma$ of $\{1, 2, \ldots, n\}$ that there is no $i$ satisfying $\sigma(i) = i$. It is well-known that the number $d(n)$ of derangements equals:

$$d(n) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

and satisfies the following identity (since both sides are the number of permutations on $n$ letters)

$$\sum_{k=0}^{n} \binom{n}{k} d(k) = n!.$$  \hspace{1cm} (1)

The Stirling set number $S(n, m)$ is the number of ways of partitioning a set of $n$ elements into $m$ nonempty sets. We define $[x]_r = x(x-1)\ldots(x-r+1)$ (by convention $[x]_0 = 1$). Then (see [4])

$$x^n = \sum_{m=0}^{n} S(n, m)[x]_r.$$  \hspace{1cm} (2)

The number of ways a set of $n$ elements can be partitioned into nonempty subsets is called a Bell number and is denoted $B_n$. We use the convention that $B_0 = 1$. The integer $B_n$ can also be defined by the sum (see [3])

$$B_n = \sum_{m=0}^{n} S(n, m)$$  \hspace{1cm} (3)

The main result of this note is the following generalization of (1).
Theorem 1 Suppose that \( n \geq m \) are two natural numbers. Then

\[
\sum_{k=0}^{n} k^m \binom{n}{k} d(n-k) = B_m n!.
\] (4)

We use the convention that \( \binom{n}{m} = 0 \) if \( m < 0 \) or \( n < m \). Also set \( d(k) = 0 \) if \( k < 0 \) and \( d(0) = 1 \). Note that taking \( m = 0 \) in (4) implies (1) since \( B_0 = 1 \). Furthermore, by linearity we have the following corollary.

Corollary 1 Suppose that \( n \geq m \) are two natural numbers. Let \( g(x) = a_m x^m + \ldots + a_0 \) be a polynomial with integer coefficients. Then

\[
\sum_{k=0}^{n} g(k) \binom{n}{k} d(n-k) = \left\{ \sum_{i=0}^{m} a_i B_i \right\} n!.
\] (5)

2 Some Lemmas

We define \( f_n(k) \) to be the number of permutations of \( \{1, \ldots, n\} \) that fix exactly \( k \) positions. By convention, \( f_n(k) = 0 \) if \( k < 0 \) or \( k > n \). We have the following recursion for \( f_n(k) \).

Lemma 1 Suppose that \( n, k \) are positive integers. Then

\[
f_{n+1}(k) = f_n(k-1) + (n-k)f_n(k) + (k+1)f_n(k+1).
\]

Proof Let \( \sigma \) be any permutation of \( \{1, \ldots, n+1\} \) which has exactly \( k \) fixed points. We have two cases.

1. Suppose that \( \sigma(n+1) = n+1 \). Then \( \sigma \) corresponds to a restricted permutation on \( \{1, \ldots, n\} \) which fixes \( k-1 \) points of \( \{1, \ldots, n\} \). This case applies to the first term in the statement of the lemma.

2. Suppose that \( \sigma(n+1) = i \) for some \( i \in \{1, \ldots, n\} \). Then there exists \( j \in \{1, \ldots, n\} \) such that \( \sigma(j) = n+1 \). There are two separate subcases.

   (a) If \( i = j \) then we can obtain a correspondence between \( \sigma \) and a permutation \( \sigma' \) of \( \{1, \ldots, n\} \) from \( \sigma \) as follows: \( \sigma'(i) = i \) and \( \sigma'(t) = \sigma(t) \) for \( t \neq i \). It is clear that \( \sigma' \) has \( k+1 \) fixed points. Conversely, for each permutation of \( \{1, \ldots, n\} \) that has \( k+1 \) fixed points, we can choose \( i \) to be any of its fixed points and then swapping \( i \) and \( n+1 \) to have a permutation of \( \{1, \ldots, n+1\} \) that has \( k \) fixed points. This case applies to the third term in the statement of the lemma.

   (b) If \( i \neq j \) then we can obtain a correspondence between \( \sigma \) and a permutation \( \sigma' \) of \( \{1, \ldots, n\} \) from \( \sigma \) as follows: \( \sigma'(j) = i \) and \( \sigma'(t) = \sigma(t) \) for \( t \neq j \). It is clear that \( \sigma' \) has \( k \) fixed points. Conversely, for each permutation \( \sigma' \) of \( \{1, \ldots, n\} \) that has \( k \) fixed points, we can choose any \( j \) such that \( \sigma'(j) = i \neq j \), and get back a permutation \( \sigma \) of \( \{1, \ldots, n+1\} \) that has \( k \) fixed points by letting \( \sigma(t) = \sigma'(t) \) for \( t \neq j, n+1, \sigma(j) = n+1 \) and \( \sigma(n+1) = \sigma'(j) = i \). This case applies to the second term in the statement of the lemma.
Hence \( f_{n+1}(k) = f_n(k-1) + (n-k)f_n(k) + (k+1)f_n(k+1) \) for all \( n, k \). This concludes the proof. \( \square \)

Lemma 1 can be applied to obtain the following identity for \( f_n(k) \) (Note that \( f_n(k) = 0 \) whenever \( k < 0 \) or \( k > n \)).

**Lemma 2** Suppose that \( n, k, t \) are integers, \( t \geq -1 \). Then

\[
\sum_{k=0}^{n} [k]_{t+1} f_n(k) = \begin{cases} 
  n! & \text{if } n \geq t + 1, \\
  0 & \text{otherwise}
\end{cases}
\]

**Proof** We prove this using a double induction. The outer induction is on \( t \) and the inner one is on \( n \). By convention, \([k]_0 = 1\). Also we have \( \sum_k f_n(k) = n! \) which is trivial from the definition of \( f_n(k) \). Hence the claim holds for \( t = -1 \). Next, suppose that the claim holds for \( t - 1 \). We prove that it holds for \( t \). Define

\[
F(n, t) := \sum_{k=0}^{n} [k]_{t+1} f_n(k) = \sum_{k=0}^{n} k(k-1) \ldots (k-t) f_n(k).
\]

Suppose that \( n \leq t \). If \( f_n(k) \neq 0 \) then \( 0 \leq k \leq n \leq t \). But this implies that \( k(k-1) \ldots (k-t) = 0 \). Hence \( F(n, t) = 0 \) if \( n \leq t \).

Suppose that \( n = t + 1 \). Then

\[
F(n, t) = \sum_{k=0}^{n} k(k-1) \ldots (k-(n-1)) f_n(k) = n! f_n(n) = n!
\]

since all but the last term of the sum equal zero. Hence the claim holds for \( n = t + 1 \). For the inner induction, suppose that \( F(n, t) = n! \) for some \( n \geq t + 1 \). We will show that \( F(n+1, t) = (n+1)! \). From Lemma 1, we have

\[
F(n+1, t) = \sum_{k=0}^{n+1} k(k-1) \ldots (k-t) f_{n+1}(k)
\]

\[
= \sum_{k=0}^{n+1} k(k-1) \ldots (k-t) [f_n(k-1) + (n-k)f_n(k) + (k+1)f_n(k+1)]
\]

\[
= nF(n, t) + \sum_{k=0}^{n+1} k(k-1) \ldots (k-t) [f_n(k-1) - kf_n(k) + (k+1)f_n(k+1)]
\]

Since \( f_n(-1) = 0 \), we have

\[
\sum_{k=0}^{n+1} k(k-1) \ldots (k-t) f_n(k-1) = \sum_{k=0}^{n} (k+1)k \ldots (k-t+1) f_n(k).
\]

3
Similarly \( f_n(n + 1) = 0 \) implies that
\[
\sum_{k=0}^{n+1} k(k - 1) \ldots (k - t) k f_n(k) = \sum_{k=0}^{n} k(k - 1) \ldots (k - t) k f_n(k).
\]

And \( f_n(n + 1) = f_n(n + 2) = 0 \) implies that
\[
\sum_{k=0}^{n+1} k(k - 1) \ldots (k - t) (k + 1) f_n(k + 1) = \sum_{k=0}^{n} (k - 1) \ldots (k - t - 1) k f_n(k).
\]

Therefore, we have
\[
F(n + 1, t) = nF(n, t)
\]
\[
+ \sum_{k=0}^{n} k(k - 1) \ldots (k - t + 1)((k + 1) - (k - t)k + (k - t)(k - t - 1)) f_n(k)
\]
\[
= nF(n, t) + \sum_{k=0}^{n} k(k - 1) \ldots (k - t + 1)((k + 1) - (k - t)(t + 1)) f_n(k)
\]
\[
= nF(n, t) + \sum_{k=0}^{n} k(k - 1) \ldots (k - t + 1)(t + 1) - (k - t)t f_n(k)
\]
\[
= nF(n, t) + (t + 1)F(n, t - 1) - tF(n, t)
\]
\[
= nn! + (t + 1)n! - tn!
\]
\[
= (n + 1)!.
\]

To see (6), note that the claim is true for \( t - 1 \) by the outer induction. So \( F(n, t - 1) = n! \). Also \( F(n, t) = n! \) by the inner inductive hypothesis. Hence the claim holds for \( n + 1 \). Therefore, it holds for every \( n, t \). This concludes the proof of the lemma. \( \square \)

3 Proof of Theorem 1

Suppose that \( n \geq m \) are two natural numbers. From (2), we have
\[
\sum_{k=0}^{n} k^m f_n(k) = \sum_{k=0}^{n} \sum_{j=0}^{m} S(m, j)[k]_j f_n(k)
\]
\[
= \sum_{j=0}^{m} S(m, j) \left( \sum_{k=0}^{n} [k]_j f_n(k) \right)
\]
\[
= \sum_{j=0}^{m} S(m, j) F(n, j - 1).
\]
From Lemma 2, \( F(n, j - 1) = n! \) for all \( 0 \leq j \leq n \). Also, from (3) \( B_m = \sum_{j=0}^{m} S(m, j) \).

Thus, (7) implies that
\[
\sum_{k=0}^{n} k^m f_n(k) = \sum_{j=0}^{m} S(m, j)n!
= B_m n!.
\]  

(8)

To have a permutation with exactly \( k \) fixed points, we can first choose \( k \) fixed points in \( \binom{n}{k} \) ways. Then for each set of \( k \) fixed points, we have \( d(n - k) \) ways to arrange the \( n - k \) remaining numbers such that we have no more fixed points. Hence
\[
f_n(k) = \binom{n}{k} d(n - k).
\]  

(9)

Substituting (9) into (8), we obtain (4). This concludes the proof of the theorem.

### 4 An application

In this section, we will apply Theorem 1 to prove the irreducibility of the standard representation of symmetric groups. Let \( G = S_n \) be the symmetric group on \( X = \{1, \ldots, n\} \).

Let \( \mathbb{C} \) denote the complex numbers. Let \( GL(d) \) stand for the group of all \( d \times d \) complex matrices that are invertible with respect to multiplication.

**Definition 1** A matrix representation of a group \( G \) is a group homomorphism
\[
\rho : G \to GL(d).
\]

Equivalently, to each \( g \in G \) is assigned \( \rho(g) \in GL(d) \) such that
1. \( \rho(1) = I \), the identity matrix,
2. \( \rho(gh) = \rho(g)\rho(h) \) for all \( g, h \in G \).

The parameter \( d \) is called the degree or dimension of the representation and is denoted by \( \text{deg}(\rho) \). All groups have the trivial representation of degree 1 which sends every \( g \in G \) to the matrix (1). We denote the trivial representation by 1. An important representation of the symmetric group \( S_n \) is the permutation representation \( \pi \), which is of degree \( n \). If \( \delta \in S_n \) then we let \( \pi(\delta) = (r_{i,j})_{n \times n} \) where
\[
r_{i,j} = \begin{cases} 
1 & \text{if } \delta(j) = i, \\
0 & \text{otherwise}.
\end{cases}
\]

**Definition 2** Let \( G \) be a finite group and let \( \rho \) be a matrix representation of \( G \). Then the character of \( \rho \) is
\[
\chi_\rho(g) = \text{tr} \rho(g),
\]
where \( \text{tr} \) denotes the trace of a matrix.
It is clear from Definition 2 that if $\delta \in S_n$ then
\[
\chi_1(\delta) = 1,
\chi_\pi(\delta) = \text{number of fixed points of } \delta.
\]

**Definition 3** Let $\chi$ and $\phi$ be characters of a finite group $G$. Then
\[
\langle \chi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\phi(g^{-1}).
\]

A matrix representation $\rho$ of a group is called irreducible if $\langle \chi_\rho, \chi_\rho \rangle = 1$. Maschke’s Theorem (see [2, 5]) states that every representation of a finite group having positive dimension can be written as a direct sum of irreducible representations. The permutation representation $\pi$ can be written as a direct sum of the trivial representation $1$ and another representation $\sigma$. The representation $\sigma$ is called the standard representation of $S_n$. We have $\chi_\pi = \chi_1 + \chi_\sigma$ since for any $\delta \in S_n$ then $\pi(\delta) = 1(\delta) \oplus \sigma(\delta)$. Thus, for all $\delta \in S_n$ then
\[
\chi_\sigma(\delta) = (\text{number of fixed points of } \delta) - 1.
\]

Now we want to prove that $\sigma$ is irreducible. In other words, we need to show $\langle \chi_\sigma, \chi_\sigma \rangle = 1$, which is equivalent to
\[
\sum_{k=0}^{n} (k - 1)^2 f_n(k) = n!
\]

Identity (10) can be obtained easily from Corollary 1 as follows.
\[
\sum_{k=0}^{n} (k - 1)^2 f_n(k) = \sum_{k=0}^{n} (k^2 - 2k + 1) f_n(k) \\
= (2 - 2 + 1)n! = n!
\]
since $B_0 = B_1 = 1$ and $B_2 = 2$. This implies the irreducibility of standard representation of symmetric groups.

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References


    
    http://mathworld.wolfram.com/BellNumber.html

    