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Domains That Do Not Have a Nice Complementary Set of Rays

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ABSTRACT. We will introduce conditions which are sufficient to guarantee that a closed connected domain in $\mathbb{R}^2$ that has a smooth boundary does not have a nice complementary set of rays.

1. INTRODUCTION

In this paper we will introduce and prove conditions that will guarantee that a given closed connected domain with smooth boundary in $\mathbb{R}^2$ will not have a nice complementary set of rays defined on it. Informally, a closed domain D with smooth boundary in $\mathbb{R}^2$ has a nice complementary set of rays if you can fill the complement of the domain with noncrossing rays each of which emanates from the boundary of D. A formal definition will be given later. A few examples of domains that have and do not have a nice complementary set of rays are given in Figure 1.1. The top two figures show a few rays of the nice complementary set for each. You can imagine filling the complement with noncrossing rays each of which emanates on the boundary. The bottom figures do not possibly have a nice complementary set of rays defined on them.

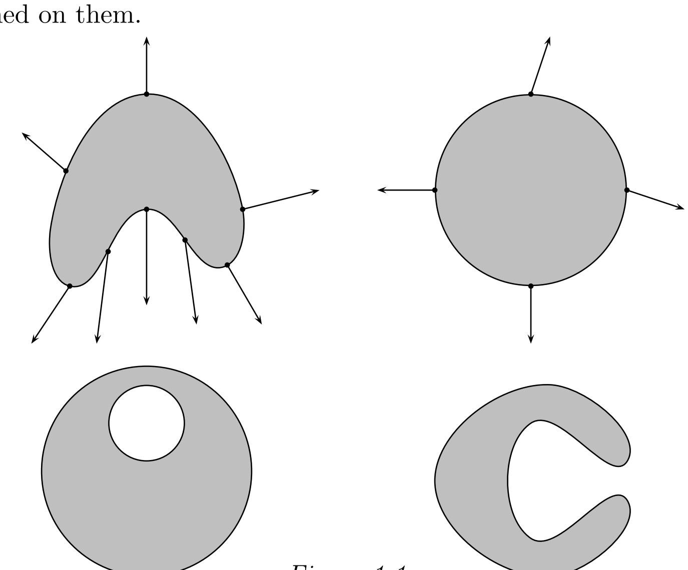

Figure 1.1

Domains with nice complementary sets of rays have applications in the field of area-minimizing surfaces.In the paper *Area-Minimizing Minimal Graphs Over Nonconvex Domains* by Michael Dorff, Denise Halverson and Gary Lawlor sufficient conditions were introduced "for which a minimal graph over a nonconvex domain is area-minimizing. The conditions are shown to hold for subsurfaces of Enneper's surface, the singly periodic Scherk surface, and the associated surfaces of the doubly periodic Scherk surface which previously were unknown to be area-minimizing." Since a domain with a nice complementary set of rays will have a set of rays originating from the boundary, filling the complement then we can project the complement along these rays onto the boundary of the domain in a continuous way.

The conditions given in this paper will prove that certain connected closed domains with smooth boundary will not have a nice complementary set of rays and therefore certain minimal graphs on that domain will not be area-minimizing (see [1]).

2. Preliminaries

For the purposes of this paper, we will define a domain to be any connected bounded subregion of $\mathbb{R}^2$ and a closed domain as a domain together with its boundary points. In this paper we consider only closed domains that are connected. A domain D in $\mathbb{R}^2$ is nonconvex if there exists a line segment whose endpoints are in D and the line segment intersects $\mathbb{R}^2 \setminus D$. Figure 2.1 below is an example of a nonconvex domain. Also an area-minimizing surface is a surface that has the least surface area of any surface having that same boundary and a minimal surface in $\mathbb{R}^3$ is a regular surface that has a mean curvature of zero at every point of the surface.

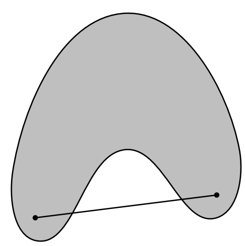

Figure 2.1

Definition 2.1. Let D be a closed domain in $\mathbb{R}^2$ with smooth boundary. If there exists a set of rays Υ with the properties given below then we say that D has a nice complementary set of rays.

(1) For every point $p \in \partial D$ there is at least one $\mathcal{R} \in \Upsilon$ such that $\partial\mathcal{R} = p$,
(2) $\mathcal{R} \cap D = \partial\mathcal{R}$ for every $\mathcal{R} \in \Upsilon$,
(3) $\mathcal{R} \cap \mathcal{R}' \subset \partial D$ for every distinct pair of rays $\mathcal{R}, \mathcal{R}' \in \Upsilon$,
(4) $\overline{D^C} = \bigcup_{\mathcal{R}\in\Upsilon} \mathcal{R}$, and
(5) There is a $\delta > 0$ such that for all $p \in \partial D$, the angle between ∂D and any ray $\mathcal{R} \in \Upsilon$ which emanates from p is defined and is at least δ.

Lemma 2.2. *Suppose $M = \{(x, y) \in \mathbb{R}^2 \ : \ |x| < \rho, |y| < \rho\}$ for some $0 < \rho < \infty$ and Γ is the graph of a function $y = f(x)$ where $f(x)$ is continuous for all $|x| < \rho$,*

$f(0)=0$ and $f'(0)=0$. Also suppose that l is the line $y=mx$ where $m\neq 0$. Then Γ separates M into two disjoint open sets, A and B with $A\cup B\cup\Gamma=M$ and there exists two distinct points $p,q\in l$ such that $p\in A$ and $q\in B$.

Proof of Lemma 2.2. Note that $\Gamma=\{(x,f(x)):|x|<\rho\}$ and let $A=\{(x,y):y>f(x)\}$ and $B=\{(x,y):y<f(x)\}$. It is easily verified that A and B are open, disjoint and that $A\cup B\cup\Gamma=M$. Now let l be the line $y=mx$ where $m\neq 0$.

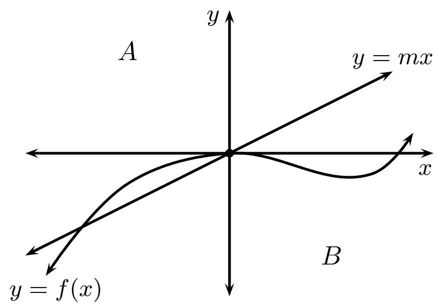

Figure 2.2

Case I. Suppose $m>0$. Pick two points $r=(x_1,y_1)$ and $s=(x_2,y_2)$ of l such that $x_1<0$ and $x_2>0$. Note that we can choose x_1 sufficiently close to 0 to guarantee that $m>\frac{f(x_1)-f(0)}{x_1-0}$ and likewise we can choose x_2 sufficiently close to 0 to also guarantee that $m>\frac{f(x_2)-f(0)}{x_2-0}$. We know that we can choose x_1 and x_2 in this way because

$$\lim_{x_1\to 0}\frac{f(x_1)-f(0)}{x_1-0}=f'(0)=0$$

and

$$\lim_{x_2\to 0}\frac{f(x_2)-f(0)}{x_2-0}=f'(0)=0$$

and $m>0$.

Note that $m=\frac{y_1-f(0)}{x_1-0}=\frac{y_1}{x_1}$ means that $y_1=mx_1$. Also we know that $m>\frac{f(x_1)-f(0)}{x_1-0}=\frac{f(x_1)}{x_1}$ implies that $mx_1<f(x_1)$ (because $x_1<0$). Therefore $mx_1=y_1<f(x_1)$. Now since $y_1<f(x_1)$ then $r=(x_1,y_1)\in B$. Similarly $m=\frac{y_2-f(0)}{x_2-0}=\frac{y_2}{x_2}$ implies that $y_2=mx_2$ and $m>\frac{f(x_2)-f(0)}{x_2-0}=\frac{f(x_2)}{x_2}$ means that $mx_2>f(x_2)$ (because $x_2>0$). Therefore $mx_2=y_2>f(x_2)$. Now since $y_2>f(x_2)$ then $s=(x_2,y_2)\in A$. So for Case I we have shown that $r\in B$ and $s\in A$.

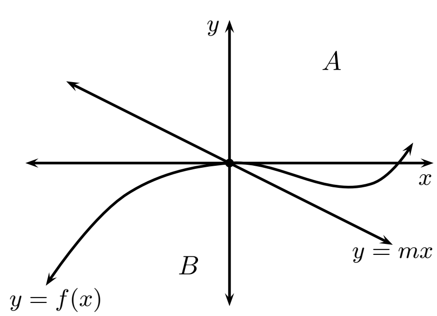

Figure 2.3

Case II. Suppose $m < 0$. Pick two points $r = (x_1, y_1)$ and $s = (x_2, y_2)$ of l such that $x_1 < 0$ and $x_2 > 0$. Note that we can choose x_1 sufficiently close to 0 to guarantee that $m < \frac{f(x_1)-f(0)}{x_1-0}$ and likewise we can choose x_2 sufficiently close to 0 to also guarantee that $m < \frac{f(x_2)-f(0)}{x_2-0}$. Again we know that we can choose x_1 and x_2 in this way because

$$\lim_{x_1 \to 0} \frac{f(x_1) - f(0)}{x_1 - 0} = f'(0) = 0$$

and

$$\lim_{x_2 \to 0} \frac{f(x_2) - f(0)}{x_2 - 0} = f'(0) = 0$$

and $m < 0$.

Note that $m = \frac{y_1 - f(0)}{x_1 - 0} = \frac{y_1}{x_1}$ means that $y_1 = mx_1$. Also we know that $m < \frac{f(x_1)-f(0)}{x_1-0} = \frac{f(x_1)}{x_1}$ implies that $mx_1 > f(x_1)$ (because $x_1 < 0$). Therefore $mx_1 = y_1 > f(x_1)$. Now since $y_1 > f(x_1)$ then $r = (x_1, y_1) \in A$. Similarly $m = \frac{y_2 - f(0)}{x_2 - 0} = \frac{y_2}{x_2}$ implies that $y_2 = mx_2$ and $m < \frac{f(x_2)-f(0)}{x_2-0} = \frac{f(x_2)}{x_2}$ means that $mx_2 < f(x_2)$ (because $x_2 > 0$). Therefore $mx_2 = y_2 < f(x_2)$. Now since $y_2 < f(x_2)$ then $s = (x_2, y_2) \in B$. So for Case II we have shown that $r \in A$ and $s \in B$.

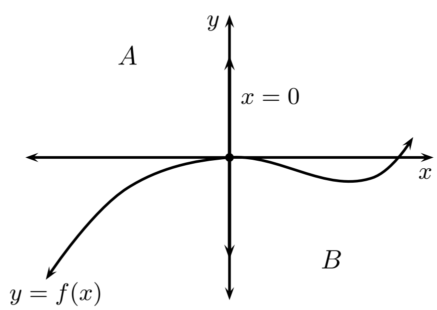

Figure 2.4

Case III. Suppose l is the line $x = 0$. Since l is the vertical line containing the origin then all the points $(0, y_1) \in l$ with $y_1 < 0$ are contained in B because $f(0) = 0$ and $B = \{(x, y) : y < f(x)\}$. Similarly all the points $(0, y_2) \in l$ with $y_2 > 0$ are contained in A because $A = \{(x, y) : y > f(x)\}$. Since all the points of $l \setminus (0, 0)$ are in A or B then there exists two points of $r, s \in \{l \setminus (0, 0)\}$ such that $r \in A$ and $s \in B$.

$\square$

The following lemma is a classical result in Euclidean Geometry.

Lemma 2.3. *Suppose a line l meets two other lines l_1 and l_2 so that the sum of the interior angles, α and β, on one side of l is less than π radians. Then the lines l_1 and l_2 meet at a point on the same side of l as α and β.*

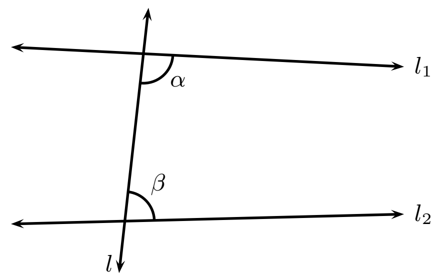

Figure 2.5

3. Proof of the Main Theorem

Theorem 3.1. *Suppose D is a domain with smooth boundary, ∂D, and there are two distinct points $P, Q \in \partial D$ such that:*

(1) $\overline{PQ} \subset \mathbb{R}^2 \backslash int\ D$,
(2) $t_P \cap t_Q = \varnothing$ *where t_P and t_Q are the tangent lines to ∂D at P and Q respectively, and*
(3) $t_P \parallel t_Q$.

Then D does not have a nice complementary set of rays.

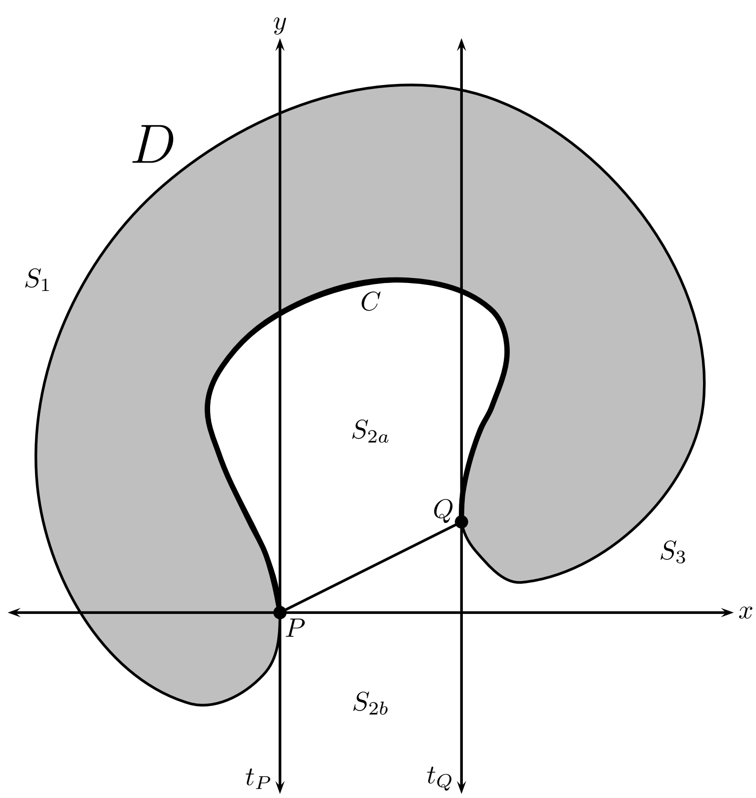

Figure 3.1

Proof of Main Theorem. First we will define regions of the plane to divide the domain into disjoint regions.

Let R_P and R_Q be two rays emanating from point P and Q respectively. Orient the Domain D on the coordinate axes so that $t_P \subset y - axis$, $P = (0,0)$ and so that $Q = (x_1, y_1)$ such that $x_1 > 0$. Define S_1, S_{2a}, S_{2b} and S_3 in the following way. Let $S_1 = \{(x,y) : x < 0\}$, $S_{2a} = \{(x,y) : 0 < x < x_1 \text{ and } y > (\frac{y_1}{x_1})x\}$, $S_{2b} = \{(x,y) : 0 < x < x_1 \text{ and } y < (\frac{y_1}{x_1})x\}$ and $S_3 = \{(x,y) : x > x_1\}$. Note that $t_P \parallel t_Q$. See Figure 3.1 for the representation of the domain and regions as defined above. It is easily verified that S_1, S_{2a}, S_{2b} and S_3 exist because $t_P \parallel t_Q$ and $t_P \neq t_Q$.

Note that if $R_P \subset t_P$ or if $R_Q \subset t_Q$, then we have a contradiction to the definition part (5). Therefore the rays cannot coincide with the tangent lines at the points P and Q. Also note that if $\overline{PQ} \subset R_P$ then $P, Q \in R_P$ which is a contradiction to Definition 2.1 part (2). Similarly if $\overline{PQ} \subset R_P$ then $P, Q \in R_Q$ which is again a contradiction to Definition 2.1 part (2). Therefore we need to consider only six cases.

Before we consider these six cases, let's consider two things. First, if D separates $\mathbb{R}^2$ into two disjoint sets, then it is clear that D cannot have a nice complementary set of rays because since D is bounded then one of the disjoint sets of $\mathbb{R}^2 \setminus D$ is also bounded and therefore cannot contain an infinite ray. Second, let's consider the points P and Q. Locally we will have two cases for each of the two points. Keeping the orientation as constructed above for the domain, consider a small neighborhood of P and consider the intersection of D with this neighborhood to be N. In this neighborhood, either N is "mostly" contained in the region $U = \{(x,y) : x \leq 0\}$ or $V = \{(x,y) : x \geq 0\}$. If N is "mostly" contained in V then Lemma 2.3 is seen to give that $\overline{PQ}$ will contain a point other than P that is contained in N, contradicting our assumption that $\overline{PQ} \subset \mathbb{R}^2 \setminus \text{int } D$. A parallel argument made locally with Q will show that N must be contained "mostly" to the right of Q. Figure 3.2 shows the different possible configurations that can occur with regards to the domain N and clearly shows that only case (a) fulfills the assumptions of Theorem 3.1. Also note that vertical placement of P and Q is irrelevant to the argument. The gray region represents N.

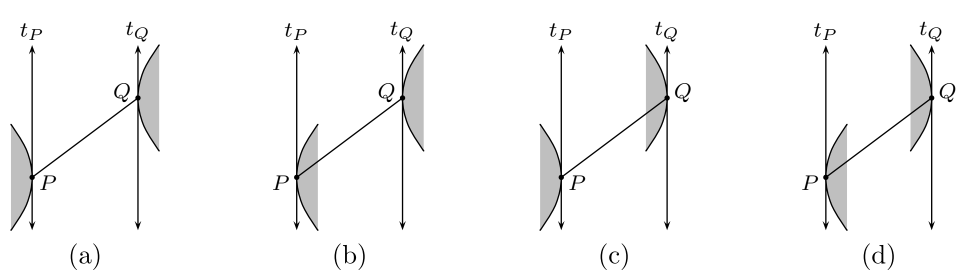

Figure 3.2

Although Figure 3.2 only represents a particular concavity of ∂D, the case arguments work regardless of concavity or if ∂D is locally a straight line at either P or Q.

Case I. $R_P \subseteq S_1$. Note that since ∂D is smooth then by the Implicit Function Theorem, there exists an open Euclidean neighborhood of P, call it M, such that ∂D is the graph of a function $y = f(x)$ where the x-axis in the coordinate patch of M corresponds to t_P and P corresponds to the origin. Let Γ be the graph of $y = f(x)$ in M. Note that Γ is the boundary of D in M. Now by Lemma 2.3, Γ separates M into two disjoint regions, A and B where $A \cup B \cup \Gamma = M$. Let the regions A and B be defined as in the proof of Lemma 2.3. Let $\Omega = \{(x, y) \in M : y < 0\}$ and let $\Omega \subset S_1$. Therefore $B \subset D$. Now for any $R_P \subset S_1$, let l be the line $y = mx$ such that $R_P \subset l$. Now by Lemma 2.4 we have that there exists a point $p \in R_P \subset l$ such that $p \in B$. Since $B \subset D$, we have a point p of R_P where $p \in D$ and $p \notin \Gamma$ therefore we have a contradiction to Definition 2.1 part (2) of a nice complementary set of rays. Therefore Case I cannot happen and $R_P \nsubseteq S_1$.

Case II. $R_Q \subseteq S_3$. Note that since ∂D is smooth then by the Implicit Function Theorem, there exists an open Euclidean neighborhood of Q, call it N, such that ∂D is the graph of a function $y = f(x)$ where the x-axis in the coordinate patch of N corresponds to t_Q and Q corresponds to the origin. Let Γ be the graph of $y = f(x)$ in N. Note that Γ is the boundary of D in N. Now by Lemma 2.3, Γ separates N into two disjoint regions, A and B where $A \cup B \cup \Gamma = N$. Let the regions A and B be defined as in the proof of Lemma 2.2. Let $\Omega = \{(x, y) \in N : y < 0\}$ and let $\Omega \subset S_3$. It can be verified by Lemma 2.2 that if $A \subset D$ then $\overline{PQ} \cap \mathit{int}\ D \neq \varnothing$ and therefore does not satisfy the conditions of Theorem 3.1. Therefore we then have that $B \subset D$ and Case II is now parallel to Case I. So there exists a point $q \in R_Q$ such that $q \notin \Gamma$ and $q \in B \subset D$. Thus we again have a contradiction to Definition 2.1 part (2) which means that Case II cannot happen and $R_Q \nsubseteq S_3$.

Case III. $R_P \cap S_{2a} \neq \varnothing$ and $R_Q \cap S_{2a} \neq \varnothing$. Let α be the angle between R_P and t_P, let β be the angle between R_Q and t_Q and let γ be the sum of the two angles made by R_P and $\overline{PQ}$ and made by R_Q and $\overline{PQ}$. Since $t_P \parallel t_Q$ then $\alpha + \beta + \gamma = \pi$ and since $R_P \nsubseteq t_P$, $R_Q \nsubseteq t_Q$, $R_P \cap S_{2a} \neq \varnothing$ and $R_Q \cap S_{2a} \neq \varnothing$ then $\alpha + \beta > 0$. Therefore $\gamma < \pi$ which means that by Lemma 2.5, we have that R_P intersects R_Q at some point in S_{2a} which is a contradiction to Definition 2.1 part (3) and thus Case III can not happen.

Case IV. $R_P \cap S_{2b} \neq \varnothing$ and $R_Q \cap S_{2b} \neq \varnothing$. Case IV is parallel in proof to Case III. Therefore we have that R_P intersects R_Q at some point in S_{2b} which is a contradiction to Definition 2.1 part (3) and thus Case IV can not happen.

Case V. $R_P \cap S_{2a} \neq \varnothing$ and $R_Q \cap S_{2b} \neq \varnothing$. Let C be the arc of ∂D from P to Q. Note that $C \cup \overline{PQ}$ bounds two regions of the complement of D, a bounded one and an unbounded one. Let $\overline{D_1^c}$ be the bounded region with boundary $C \cup \overline{PQ}$. Since $\overline{D_1^c}$ is bounded and R_P is an infinite ray that enters $\overline{D_1^c}$ then R_P must exit $\overline{D_1^c}$ at some point $p \in R_P$ on $C \cup \overline{PQ}$. By our hypothesis, $R_P \cap \overline{PQ} = \varnothing$ and therefore since $p \in C \cup \overline{PQ}$ then $p \in \partial D$ which is a contradiction to Definition 2.1 part (2). Therefore Case V can not happen.

Case VI. $R_P \cap S_{2b} \neq \varnothing$ and $R_Q \cap S_{2a} \neq \varnothing$. Case VI is parallel in proof to Case V. Therefore there exists a point $q \neq Q$ of R_Q such that $q \in \partial D$ which is a contradiction to Definition 2.1 part (2). Thus Case VI can not happen.

Therefore under the conditions of the hypothesis of Theorem 3.1 we have shown that there is no possibility for R_P or R_Q to exist and not contradict Definition 2.1. Thus such a domain will not have a nice complementary set of rays. □

4. A Brief Explanation of Condition (2) of Theorem 3.1

Condition (2) of Theorem 3.1 states essentially that $t_P \neq t_Q$. From Figure 4.1, it can easily be verified that even though D' satisfies all the other conditions of Theorem 3.1, a nice complementary set of rays does exist. Thus condition (2) needs to be fulfilled as well as the other stated conditions.

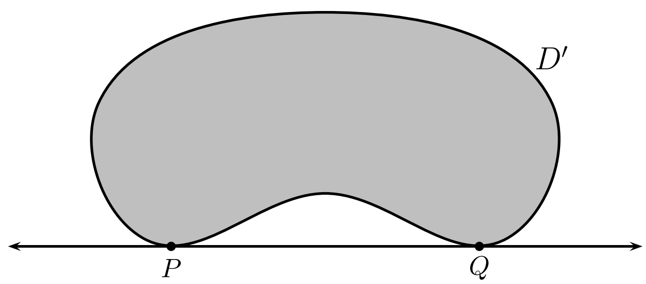

Figure 4.1

References

[1] Dorff, Michael, Denise Halverson and Gary Lawlor, Area-Minimizing Minimal Graphs Over Nonconvex Domains, *Pacific Journal of Mathematics.* **1 (2002).**