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Safiya Moran
Columbia College, South Carolina, safiya.moran@gmail.com

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Abstract

A model of the pretzel knot is described. A method for predicting the ropelength of pretzel knots is given. An upper bound for the minimum ropelength of a pretzel knot is determined and shown to improve on existing upper bounds.

1 Introduction

A (p, q, r) pretzel knot is composed of 3 twists which are connected at the tops and bottoms of each twist. The p,q, and r are integers representing the number of crossings in the given twist. However, a general pretzel knot can contain any number n of twists, as long as $n \geq 3$, where the knot is denoted by $(p_1, p_2, p_3, ..., p_n)$ and is illustrated by a diagram of the form shown in figure 1.

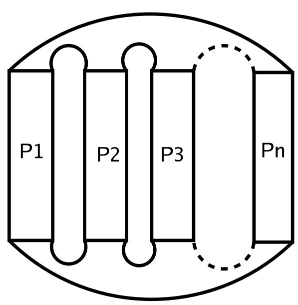

Fig. 1: A 2-dimensional representation of a pretzel knot.

Here, the p_i's denote integers, some of which may be negative, designating the left- or right-handedness of all the crossings in each of the various twists. Note that such a "knot" might turn out to not be knot, which has just one component, but instead a link of more than one component. For example, the pretzel knot $(p, q, r) = (2, 2, 2)$ is a chained link of three simple components. Nevertheless, here we will use the accepted reference of "pretzel knot," even in such cases.

*Columbia College, Columbia, SC

The goal of this paper is to find a model where the knot is tightest and the ropelength is minimized. The ropelength of a knot is determined by the ratio of the length of the rope to its radius.

$$Rop(K) = \frac{L}{r}$$

It follows from corollary 1 of [1] that the ropelength of the unknot is 2π . The lowest known provable upper bound for the trefoil knot is 32.7433864 [2], while a lower bound for the ropelength of the trefoil knot is also known to be 31.32 [3]. An upper bound for ropelength for knots in general is given in a paper by Cantarella, Faber, and Mullikin [4]. This upper bound is a function of crossing number, $c(L)$, and is given by the following equation for prime links:

$$Rop(L) \leq 1.64c(L)^2 + 7.69c(L) + 6.74$$

The main theorem of this paper is an upper bound on ropelength for any $(p_1, p_2, p_3, ..., p_n)$ pretzel knot.

Theorem 1. If K is a pretzel knot defined by $(p_1, p_2, p_3, ..., p_n)$, then

$$Rop(K) \leq \pi\ (2\sqrt{2}(\sum_{i=1}^{n} p_i) + 3n + (\sum_{i=1}^{n-1} |p_i - p_{i+1}|) + |p_n - p_1|) + 1.014263831(2n)$$

A direct corollary of this theorem is an equation for an upper bound on ropelength of the (p, q, r) pretzel knot.

Corollary 1.1 If k is a pretzel knot defined by (p, q, r), then

$$Rop(k) \leq \pi\ (2\sqrt{2}(p + q + r) + 9 + |p - q| + |q - r| + |r - p|) + 6.085582986$$

It should be noted that one can use Theorem 1 to give an upper bound in terms of the crossing number as well. One naive approach follows. For $(p_1, p_2, p_3, ..., p_n)$ where all p_i's are positive, the crossing number is given by $c = \sum_{i=1}^{n} p_i$. If all p_i's are negative, then $c = -\sum_{i=1}^{n} p_i$. Since

$\sum_{i=1}^{n-1}(|p_i - p_{i+1}|) + |p_n - p_1| \leq \sum_{i=1}^{n-1}(|p_i| + |p_{i+1}|) + |p_n| + |p_1|$, and $\sum_{i=1}^{n-1} |p_i| + |p_{i+1}|) + |p_n| + |p_1| = 2c$,
we can simplify the upper bound to

$$Rop(K) \leq 15.1689\ c\ +\ 10.4390\ n$$

As mentioned above, the previously known upper bound is a quadratic function of the crossing number, while ours is linear. Of course, the bound given in Theorem 1 is much better than that of equation 1.

2 Constructing the Twists

In order to prove Theorem 1, we first analyze the construction of the twists. For the purpose of this paper, we let the radius of the rope be equal to 1. Two different models were compared for constructing the twists of the pretzel knot. The first model was a double helix constructed with rope of radius 1. The second model was a single helix constructed of rope of radius 1 wrapping around a straight rope also of radius 1. The second model was considered due to its successful application to the (59,2) torus knot in the paper by Baranska, Pieranski, Przybyl, and Rawden [2]. The construction of one helix wrapping around a straight rope appeared in the torus knot and seemed to perhaps minimize the ropelength necessary. In the first model, the pitch of the helices must remain 1 to prevent the helices from overlapping. In the second model, since we did not have to worry about one helix running into another, the pitch could be reduced to at most 0.3225. The length of a twist with a single crossing for this model is about 9.67. However, when compared with the length of the double helix model for a twist with a single crossing, 8.89, this was not enough to provide a shorter length than the first model so the second model was rejected.
Therefore the twists are modeled by the double helix and given by the parametric equations: $(cos(t), sin(t), t)$ *and* $(cos(t+\pi), sin(t+\pi), t)$. A few concerns about overlapping ropes when minimizing ropelength were dealt with in both [2] and [4] and lead us to impose the following requirements on the construction of the double helix:

1. The distance between the two helices is at least twice the radius.
2. The distance between doubly-critical points must be at least twice the radius.
3. The radius of curvature must be at least the radius of the rope.

Lemma 1: The double helix with pitch 1 and radius 1 satisfies all three of the above requirements.

Proof. To satisfy requirement 1 we find the minimal distance between one helix to the other helix. Let the first helix, $(cos(t), sin(t), t)$, be H_1 and the second helix $(cos(t+\pi), sin(t+\pi), t)$, H_2. Let $t = 0$ in H_2, so that $H_2(0) = (-1, 0, 0)$. The distance between H_1 and $H_2(0)$ is $d(t) = \sqrt{(cos(t)+1)^2 + (sin(t))^2 + t^2}$. When this distance is minimized, $t = 0$ and $d(0) = 2$. Since the radius of the rope is one, condition 1 is satisfied.

Doubly-critical points occur when $(x, y) \in K$ and $\overrightarrow{(x-y)}$ is orthogonal to both the vector tangent to x and the vector tangent to y. To show that requirement 2 is true when radius is 1, we must find the vector on H_1 orthogonal to the tangent vector of $H_1(0)$. Thus we solve $(\vec{c}(t) - \vec{c}(0)) - \vec{c}'(0) = 0$.
In our case:

$\vec{c}(t) = (cos(t), sin(t), t)$
$\vec{c}(0) = (1, 0, 0)$
$\vec{c'}(t) = (-sin(t), cos(t), 1)$
$\vec{c'}(0) = (0, 1, 1)$.
The vector between $\vec{c}(t)$ *and* $\vec{c}(0)$ is $(cos(t) - 1, sin(t), t)$. If we take the dot product of this vector and the vector tangent to the curve at $t = \vec{c'}(0)$ and set this value equal to 0 we will find a vector orthogonal to both. In this case the dot product gives us $sin(t) + t = 0$ which gives us $t = 0$. Since this is the same as our original point, this tells us that there are no doubly-critical points when the pitch is equal to 1.

To satisfy requirement 3 the radius of curvature of a single helix must be at least 1. The curvature formula is

$$\kappa = \frac{\|\vec{c}\,'\,(t_0) \times \vec{c}\,''\,(t_0)\|}{\|\vec{c}\,'\,(t_0)\|^{\,3}}$$

and the radius of curvature is $\frac{1}{\kappa}$ For the double helix the curvature is $\kappa(\vec{c}) = \frac{1}{2}$ and the radius of curvature equals 2. So the curvature of the double helix will not be a problem. □

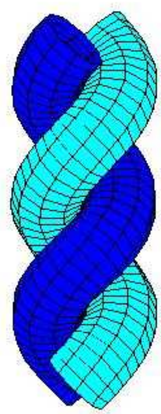

Fig. 2: A double-helix with radius of 1 and pitch of 1.

The length of one twist of a single helix is found by

$$l = \int_0^{\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1}\, dt = \sqrt{2}\pi$$

The length of a double helix with one crossing is equal to twice the above value. Therefore the length of a double helix with n crossings is $L = n2\sqrt{2}\pi$ and the length of the double helices in any pretzel knot is

$$l_{helix} = (p + q + r)\; 2\,\sqrt{2}\,\pi$$

3 Proof of Theorem 1

The following section explains the construction of a (p, q, r) pretzel knot, or a pretzel knot with three twists. The pretzel knot was constructed by using partial circles and straight ropes to connect the helices. The length of each individual part is denoted by l_{helix} for the helices, l_{circ} for the partial circles, l_{vert} for vertical straight ropes, and l_{hor} for horizontal straight ropes.

3.1 Placing the Helices

The issue now is to place the 3 separate twists/double helices in such a way that minimizes the ropelength needed to connect the helices. Each double helix is contained within a cylinder of radius 2. By placing the three double helices in such a way that the cylinders of radius 2 do not intersect, the double helices are guaranteed not to overlap. Changing the rotation of the double helices will minimize the distance between the helices to be connected later on. Figure 3 shows the placement of the helices which will reduce the ropelength necessary to connect the helices while ensuring that no ropes will overlap.

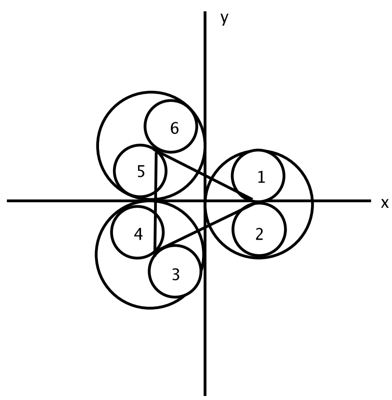

Fig. 3: Placement of the helices as seen looking down the z-axis.

The equations for the helix 1 and helix 2 in these locations are

$$\text{Helix 1: } (\cos(t + \tfrac{\pi}{2}) + \tfrac{4}{\sqrt{3}}, \sin(t + \tfrac{\pi}{2}), t)$$
$$\text{Helix 2: } (\cos(t + \tfrac{3\pi}{2}) + \tfrac{4}{\sqrt{3}}, \sin(t + \tfrac{3\pi}{2}), t)$$

The equations for helix 3 through 6 were obtained by starting with helix 1 and helix 2 and rotating them by $\frac{2\pi}{3}$ around the z-axis.

3.2 Connecting the Helices

Following the numbering used in figure 3, helix 1 and helix 6, helix 2 and helix 3, and helix 4 and helix 5 must be connected on the top and the bottom of each

helix. The first step to be able to connect the helices is to flatten the tops and bottoms of each helix. At this point each helix begins and ends at a $45°$ angle with the horizontal. In order to flatten these, $\frac{1}{8}$th of a circle with radius 1 will be added to the top and bottom of each helix. These partial circles must satisfy similar conditions as stated above for the helices. First, the distance between the centers of the circles must be at least twice the radius. Since the centers of the circles will be located at the same location on their helix and it has already been shown that the helices have a distance of 2, the centers of the circle must be at least 2 units apart. In this case there will be no doubly-critical points. Finally, the radius of curvature must be at least 1. The radius of curvature for a circle with radius 1 is 1. So these partial circles satisfy all conditions for the construction of this knot.
The length of these $\frac{1}{8}$th circles is $l = \frac{1}{8}(2\pi)$. Since 12 of these $\frac{1}{8}$th circles will be needed the length that they contribute to the total ropelength is 3π, which will contribute to l_{circ} later.

Once the helices have been flattened, it is necessary to make the helices which are to be connected the same height. So helix 1 must be the same height as helix 6, helix 2 the same height as helix 3, and helix 4 the same height as helix 5. This will be accomplished by simply connecting a straight rope to the top of the shorter of the two helices. This straight rope will not intersect itself or any other section because it will extend vertically from the attached $\frac{1}{8}$th circles which are of distance of at least 2 apart. The radius of curvature and doubly-critical points will not be an issue since this is a straight rope.

The length of this vertical straight rope will be equal to the difference in height between the two helices. Since each twist has height of π the length contributed to the total ropelength by these vertical straight ropes is

$$l_{vert} = (|p - q| + |q - r| + |r - p|)\pi.$$

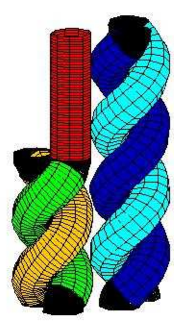

Fig. 4: Construction of two double-helices.

Now that the helices to be connected are at the same height, the helices can be connected. The connections will be constructed by a horizontal straight rope with a $\frac{1}{4}$ circle of radius 1 on either end. It must be shown that both the horizontal straight rope and quarter circles satisfy the conditions given above for the construction of a knot. The quarter circle will be similar to the $\frac{1}{8}$th circle described above. The centers of the circles will be on their respective helices, which are at least distance 2 away from each other, in the direction of the helix that they are to connect to. Because of the placement of the helices no two quarter circles will overlap. Again, there are no doubly-critical points and the radius of curvature is 1. The horizontal straight ropes will be similar to the vertical straight ropes described above.

The $\frac{1}{4}$ circle connects the top or bottom of the twist with the horizontal connecting rope. The $\frac{1}{4}$ circles will have length of $\frac{1}{4}(2\pi)$. Once again 12 of these $\frac{1}{4}$ circles are needed. So the total length that the $\frac{1}{4}$ circles along with the $\frac{1}{8}$th circles contribute to the total ropelength is

$$l_{circ} = 9\pi$$

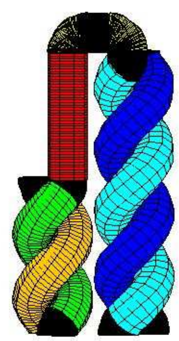

Fig. 5: Connection at the bottom of two helices created by a straight horizontal rope and two $\frac{1}{4}$ circles.

The length of each horizontal rope is the distance between the two $\frac{1}{4}$ circles to be connected. In our construction this distance is 1.014263831 for a (p, q, r) pretzel knot. Since 6 of these horizontal straight ropes are needed, the length these contribute to the total ropelength is

$$l_{hor} = 6.085582986$$

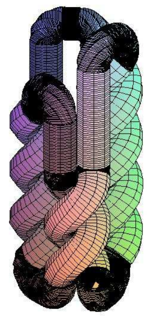

Fig. 6: Final construction of a (2,4,3) pretzel knot.

3.3 Total Ropelength

Using this construction an upperbound for the ropelength of (p, q, r) pretzel knots can be found. When all of the individual parts are added an upper bound for ropelength is given.

$$Rop(L) \leq \pi\ (2\ \sqrt{2}(p+q+r) + 9 + |p-q| + |q-r| + |r-p|) + 6.085582986$$

It was found that this construction can be applied to all $(p_1, p_2, p_3, ..., p_n)$ pretzel knots. We can generalize the above analysis to give an upper bound on ropelength for all pretzel knots with n twists. The helices are found by rotating helix 1 and helix 2 by $\theta = \frac{2\pi}{n}$ around the z-axis where helix 1 and helix 2 are placed a distance of $\frac{2}{\sin\frac{\pi}{n}}$ along the x-axis. The distance between the two quarter circles to be connected for a $(p_1, p_2, p_3, ..., p_n)$ pretzel knot is

$$\sqrt{|\frac{2}{\sin\frac{\pi}{n}} - 1 + \sqrt{\frac{1}{2}} - \cos\frac{2\pi}{n}(\frac{2}{\sin\frac{\pi}{n}} + 1 - \sqrt{\frac{1}{2}}) - \sin\frac{2\pi}{n}|^2 + |1 - \sin\frac{2\pi}{n}(\frac{2}{\sin\frac{\pi}{n}} + 1 - \sqrt{\frac{1}{2}}) + \cos\frac{2\pi}{n}|^2}$$

So the length of the horizontal straight pieces is 2 less than the above distance. However, since this length decreases as n increases, we can simply use the value found for the pretzel knot with three twists, 1.014263831, in the upper bound. The following are the general equations for all parts of the ropelength:

$$L_{helix} = 2\ \sqrt{2}\ (\sum_{i=1}^{n} p_i)\ \pi$$
$$L_{circ} = 3\ n\ \pi$$
$$L_{vert} = ((\sum_{i=1}^{n-1} |p_i - p_{i+1}|) + |p_n - p_1|)\ \pi$$
$$L_{hor} = (1.01426383)\ 2n$$

Summing these four quantities yields the desired inequality of Theorem 1.

4 Conclusion

When this upperbound is applied to the (2,4,3) pretzel knot we are given an upper bound of 126.8981804. The previously known upper bound for knots gives an upper bound of 208.79. Compared to the previous upper bound, this upper bound greatly improves upon the upper bound of ropelength for (p,q,r) pretzel knots. This can also be seen in the equations by the fact that both are functions of crossing numbers. However, as mentioned before the previously known upper bound is a quadratic function while the upperbound given in this paper is a linear function, resulting in a lower value. However, it should be noted that there is room for improvement on this upper bound. It may be possible to bring the helices even closer together, which would slightly reduce the upper bound on ropelength. There could also be a better construction that would minimize the open space in the middle of the knot, which would also hopefully reduce the upper bound on ropelength.

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