Two Questions on Continuous Mappings

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TWO QUESTIONS ON CONTINUOUS MAPPINGS

XUN GE

Abstract. In this paper, it is shown that a mapping from a sequential space is continuous iff it is sequentially continuous, which improves a result by relaxing first-countability of domains to sequentiality. An example is also given to show that open mappings do not imply Darboux-mappings, which answers a question posed by Wang and Yang.

A mapping \( f : X \rightarrow Y \) is continuous if \( f^{-1}(U) \) is open in \( X \) for every open subset \( U \) of \( Y \). In [5], Wang and Yang give some interesting generalizations of continuous mappings.

**Definition 1.** Let \( f : X \rightarrow Y \) be a mapping.

1. \( f \) is called a sequentially continuous mapping if for every sequence \( \{x_n\} \) converging to \( x \) in \( X \), \( \{f(x_n)\} \) is a sequence converging to \( f(x) \) in \( Y \).
2. \( f \) is called a Darboux-mapping if \( f(F) \) is connected in \( Y \) for every connected subset \( F \) of \( X \).

It is a standard result that every continuous mapping is both a sequentially continuous mapping and a Darboux-mapping, but neither sequentially continuous mappings nor Darboux-mappings need to be continuous[5, 1]. However, the following result is well known (see [1], for example).

**Theorem 2.** Let \( f : X \rightarrow Y \) be a mapping, where \( X \) is first countable. If \( f \) is sequentially continuous, then \( f \) is continuous.

Take the above theorem into account, the following question naturally arises.

**Question 3.** Can first-countability of \( X \) in Theorem 2 be relaxed?

On the other hand, Wang and Yang posed the following question in [5].

**Question 4.** Does there exist an open mapping \( f : X \rightarrow Y \) such that \( f \) is not a Darboux-mapping?

In this paper, we investigate the above Questions. We show that we can relax first-countability of \( X \) in Theorem 2 to sequentiality, which gives an affirmative answer for Question 3. We also give an example to answer Question 4 affirmatively.

Throughout this paper, all spaces are assumed to be \( T_1 \). The set of all natural numbers is denoted by \( \mathbb{N} \). A sequence is denoted by \( \{x_n\} \), where the \( n \)-th term is \( x_n \). Let \( X \) be a space and \( P \subset X \).

**Definition 5.** Let \( X \) be a space.

1. A sequence \( \{x_n\} \) converging to \( x \) in \( X \) is eventually in \( P \) if \( \{x_n : n > k\} \cup \{x\} \subset P \) for some \( k \in \mathbb{N} \).

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(2) Let $x \in X$. A subset $P$ of $X$ is called a sequential neighborhood of $x$ if every sequence $\{x_n\}$ converging to $x$ is eventually in $P$, and a subset $U$ of $X$ is called sequentially open if $U$ is a sequential neighborhood of each of its points.

(3) $X$ is called a Fréchet-space if for every $P \subset X$ and for every $x \in \overline{P}$, there exists a sequence $\{x_n\}$ in $P$ converging to the point $x$.

(4) $X$ is called a sequential space if for every $A \subset X$, $A$ is closed in $X$ if and only if $A \cap S$ is closed in $S$ for every convergent sequence $S$ (containing its limit point) in $X$.

(5) $X$ is called a k-space if for every $A \subset X$, $A$ is closed in $X$ if $A \cap K$ is closed in $K$ for every compact subset $K$ of $X$.

**Remark 6.** It is well known that first countable spaces $\implies$ Fréchet-spaces $\implies$ sequential spaces $\implies$ k-spaces (see [4], for example).

**Lemma 7.** Let $X$ be a space. The following are equivalent.

1. $X$ is a sequential space.
2. For every non-closed subset $F$ of $X$, there exists a sequence $\{x_n\}$ in $F$ converging to $x$ for some $x \in X - F$.
3. Every sequentially open subset of $X$ is open in $X$.

**Proof.** (1) $\implies$ (2): Let $F$ be a non-closed subset of $X$. Since $X$ is a sequential space, there exists a sequence $S$ converging to a point $x \in X$ such that $F \cap S$ is not closed in $S$. It is clear that $F \cap S$ is infinite. So there exists a subsequence $\{x_n\}$ of $S$ such that $x_n \in F$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to $x$. Put $L = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. If $x \in F$, then $x \in F \cap S$, thus $F \cap S$ is closed in $S$, a contradiction. So $x \in X - F$.

(2) $\implies$ (3): Let $U$ be a sequentially open subset of $X$. If $U$ is not open in $X$, that is, $X - U$ is not closed in $X$, then there exists a sequence $\{x_n\}$ in $X - U$ converging to $x$ for some $x \in U$. Thus $U$ is not a sequentially open subset of $X$, a contradiction.

(3) $\implies$ (1): If $X$ is not a sequential space, then there exists a non-closed subset $F$ of $X$ such that $F \cap S$ is closed in $S$ for every convergent sequence $S$ in $X$, where $S$ containing its limit point. Since $X - F$ is not open in $X$, $X - F$ is not a sequentially open subset of $X$, so there exist a point $x \in X - F$ and a sequence $\{x_n\}$ converging to $x$ such that $\{x_n\}$ is not eventually in $X - F$. Thus there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \notin X - F$ for all $n \in \mathbb{N}$, that is, $y_n \in F$ for all $n \in \mathbb{N}$. Put $S = \{y_n : n \in \mathbb{N}\} \cup \{x\}$, then $x \notin F \cap S$. Note that $x$ is a cluster point of $F \cap S$, $F \cap S$ is not closed in $S$. This is a contradiction. $\square$

**Theorem 8.** Let $f : X \rightarrow Y$ be a mapping, where $X$ is sequential. If $f$ is sequentially continuous, then $f$ is continuous.

**Proof.** Let $f : X \rightarrow Y$ be sequentially continuous and let $U$ be an open subset of $Y$. Since $X$ is a sequential space, it suffices to prove that $f^{-1}(U)$ is a sequentially open subset of $X$ from Lemma 7.

Let $x \in f^{-1}(U)$ and $\{x_n\}$ be a sequence converging to $x$. Since $f : X \rightarrow Y$ is sequentially continuous, $\{f(x_n)\}$ is a sequence converging to $f(x) \in U$. Note that $U$ is an open neighborhood of $f(x)$, there exists $k \in \mathbb{N}$ such that $f(x_n) \in U$ for all $n > k$. So $x_n \in f^{-1}(U)$ for all $n > k$, thus $\{x_n\}$ is eventually in $f^{-1}(U)$. This proves that $f^{-1}(U)$ is a sequentially open subset of $X$. $\square$

The above theorem improves Theorem 2 and gives an affirmative answer for Question 3. However, the following question is still open.
Question 9. Let \( f : X \rightarrow Y \) be a mapping, where \( X \) is a \( k \)-space. If \( f \) is sequentially continuous, is \( f \) continuous?

The following example answers Question 4 affirmatively.

Example 10. There exists an open mapping \( f : X \rightarrow Y \) such that \( f \) is not a Darboux-mapping.

Proof. Let \( X = \mathbb{R} \) with the Euclidean topology and \( Y = \mathbb{R} \) with the discrete topology, where \( \mathbb{R} \) is the set of all real numbers. Let \( f : X \rightarrow Y \) be the identity mapping. Then \( f \) is an open mapping because every subset of discrete space \( Y \) is open in \( Y \). Notice that \( X \) is a connected space and \( Y = f(X) \) is a discrete space, thus \( Y \) is not connected. So \( f \) is not a Darboux-mapping. \( \square \)

References


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