Two Questions on Continuous Mappings

Xun Ge
Suzhou University, zhugexun@163.com

Follow this and additional works at: https://scholar.rose-hulman.edu/rhumj

Recommended Citation
Available at: https://scholar.rose-hulman.edu/rhumj/vol7/iss2/3
TWO QUESTIONS ON CONTINUOUS MAPPINGS

XUN GE

Abstract. In this paper, it is shown that a mapping from a sequential space is continuous iff it is sequentially continuous, which improves a result by relaxing first-countability of domains to sequentiality. An example is also given to show that open mappings do not imply Darboux-mappings, which answers a question posed by Wang and Yang.

A mapping $f : X \longrightarrow Y$ is continuous if $f^{-1}(U)$ is open in $X$ for every open subset $U$ of $Y$. In [5], Wang and Yang give some interesting generalizations of continuous mappings.

Definition 1. Let $f : X \longrightarrow Y$ be a mapping.

(1) $f$ is called a sequentially continuous mapping if for every sequence \( \{x_n\} \) converging to $x$ in $X$, \( \{f(x_n)\} \) is a sequence converging to $f(x)$ in $Y$.

(2) $f$ is called a Darboux-mapping if $f(F)$ is connected in $Y$ for every connected subset $F$ of $X$.

It is a standard result that every continuous mapping is both a sequentially continuous mapping and a Darboux-mapping, but neither sequentially continuous mappings nor Darboux-mappings need to be continuous[5, 1]. However, the following result is well known (see [1], for example).

Theorem 2. Let $f : X \longrightarrow Y$ be a mapping, where $X$ is first countable. If $f$ is sequentially continuous, then $f$ is continuous.

Take the above theorem into account, the following question naturally arises.

Question 3. Can first-countability of $X$ in Theorem 2 be relaxed?

On the other hand, Wang and Yang posed the following question in [5].

Question 4. Does there exist an open mapping $f : X \longrightarrow Y$ such that $f$ is not a Darboux-mapping?

In this paper, we investigate the above Questions. We show that we can relax first-countability of $X$ in Theorem 2 to sequentiality, which gives an affirmative answer for Question 3. We also give an example to answer Question 4 affirmatively.

Throughout this paper, all spaces are assumed to be $T_1$. The set of all natural numbers is denoted by $\mathbb{N}$. A sequence is denoted by $\{x_n\}$, where the $n$-th term is $x_n$. Let $X$ be a space and $P \subset X$.

Definition 5. Let $X$ be a space.

(1) A sequence $\{x_n\}$ converging to $x$ in $X$ is eventually in $P$ if \( \{x_n : n > k\} \cup \{x\} \subset P \) for some $k \in \mathbb{N}$.

2000 Mathematics Subject Classification. 54C05, 54C10, 54D55.

Key words and phrases. Continuous mapping, sequentially continuous mapping, Darboux-mapping, sequential space.
(2) Let \( x \in X \). A subset \( P \) of \( X \) is called a sequential neighborhood of \( x \) if every sequence \( \{x_n\} \) converging to \( x \) is eventually in \( P \), and a subset \( U \) of \( X \) is called sequentially open if \( U \) is a sequential neighborhood of each of its points.

(3) \( X \) is called a Fréchet-space if for every \( P \subset X \) and every \( x \in \overline{P} \), there exists a sequence \( \{x_n\} \) in \( P \) converging to the point \( x \).

(4) \( X \) is called a sequential space if for every \( A \subset X \), \( A \) is closed in \( X \) if and only if \( A \cap S \) is closed in \( S \) for every convergent sequence \( S \) (containing its limit point) in \( X \).

(5) \( X \) is called a k-space if for every \( A \subset X \), \( A \) is closed in \( X \) if and only if \( A \cap K \) is closed in \( K \) for every compact subset \( K \) of \( X \).

**Remark 6.** It is well known that first countable spaces \( \Rightarrow \) Fréchet-spaces \( \Rightarrow \) sequential spaces \( \Rightarrow \) k-spaces (see [4], for example).

**Lemma 7.** Let \( X \) be a space. The following are equivalent.

1. \( X \) is a sequential space.
2. For every non-closed subset \( F \) of \( X \), there exists a sequence \( \{x_n\} \) in \( F \) converging to \( x \) for some \( x \in X - F \).
3. Every sequentially open subset of \( X \) is open in \( X \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( F \) be a non-closed subset of \( X \). Since \( X \) is a sequential space, there exists a sequence \( S \) converging to a point \( x \in X \) such that \( F \cap S \) is not closed in \( S \). It is clear that \( F \cap S \) is infinite. So there exists a subsequence \( \{x_n\} \) of \( S \) such that \( x_n \in F \) for all \( n \in \mathbb{N} \) and \( \{x_n\} \) converges to \( x \). Put \( L = \{x_n : n \in \mathbb{N}\} \cup \{x\} \). If \( x \in F \), then \( x \in F \cap S \), thus \( F \cap S \) is closed in \( S \), a contradiction. So \( x \in X - F \).

(2) \( \Rightarrow \) (3): Let \( U \) be a sequentially open subset of \( X \). If \( U \) is not open in \( X \), that is, \( X - U \) is not closed in \( X \), then there exists a sequence \( \{x_n\} \) in \( X - U \) converging to \( x \) for some \( x \in U \). Thus \( U \) is not a sequentially open subset of \( X \), a contradiction.

(3) \( \Rightarrow \) (1): If \( X \) is not a sequential space, then there exists a non-closed subset \( F \) of \( X \) such that \( F \cap S \) is closed in \( S \) for every convergent sequence \( S \) in \( X \), where \( S \) containing its limit point. Since \( X - F \) is not open in \( X \), \( X - F \) is not a sequentially open subset of \( X \), so there exist a point \( x \in X - F \) and a sequence \( \{x_n\} \) converging to \( x \) such that \( \{x_n\} \) is not eventually in \( X - F \). Thus there exists a subsequence \( \{y_n\} \) of \( \{x_n\} \) such that \( y_n \notin X - F \) for all \( n \in \mathbb{N} \), that is, \( y_n \in F \) for all \( n \in \mathbb{N} \). Put \( S = \{y_n : n \in \mathbb{N}\} \cup \{x\} \), then \( x \notin F \cap S \). Note that \( x \) is a cluster point of \( F \cap S \), \( F \cap S \) is not closed in \( S \). This is a contradiction.

**Theorem 8.** Let \( f : X \rightarrow Y \) be a mapping, where \( X \) is sequential. If \( f \) is sequentially continuous, then \( f \) is continuous.

**Proof.** Let \( f : X \rightarrow Y \) be sequentially continuous and let \( U \) be an open subset of \( Y \). Since \( X \) is a sequential space, it suffices to prove that \( f^{-1}(U) \) is sequentially open subset of \( X \) from Lemma 7.

Let \( x \in f^{-1}(U) \) and \( \{x_n\} \) be a sequence converging to \( x \). Since \( f : X \rightarrow Y \) is sequentially continuous, \( \{f(x_n)\} \) is a sequence converging to \( f(x) \in U \). Note that \( U \) is an open neighborhood of \( f(x) \), there exists \( k \in \mathbb{N} \) such that \( f(x_n) \in U \) for all \( n > k \). So \( x_n \in f^{-1}(U) \) for all \( n > k \), thus \( \{x_n\} \) is eventually in \( f^{-1}(U) \). This proves that \( f^{-1}(U) \) is a sequentially open subset of \( X \).

The above theorem improves Theorem 2 and gives an affirmative answer for Question 3. However, the following question is still open.
Question 9. Let \( f : X \rightarrow Y \) be a mapping, where \( X \) is a \( k \)-space. If \( f \) is sequentially continuous, is \( f \) continuous?

The following example answers Question 4 affirmatively.

Example 10. There exists an open mapping \( f : X \rightarrow Y \) such that \( f \) is not a Darboux-mapping.

Proof. Let \( X = \mathbb{R} \) with the Euclidean topology and \( Y = \mathbb{R} \) with the discrete topology, where \( \mathbb{R} \) is the set of all real numbers. Let \( f : X \rightarrow Y \) be the identity mapping. Then \( f \) is an open mapping because every subset of discrete space \( Y \) is open in \( Y \). Notice that \( X \) is a connected space and \( Y = f(X) \) is a discrete space, thus \( Y \) is not connected. So \( f \) is not a Darboux-mapping.

References


Department of Mathematics, Suzhou University, Suzhou, 215006, P.R.China
E-mail address: zhugexun@163.com