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1 Introduction

In 1579 Francois Vieta (1540-1603) derived the following formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots \tag{1}$$

to approximate the value of π. He obtained this formula, which subsequently became known as Vieta's formula using a geometric approach [3]. As it is often the case, things can be achieved in many different ways in mathematics. Vieta's formula can be derived from the identity

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos \frac{x}{2^k} \tag{2}$$

by setting $x = \frac{\pi}{2}$, and identity (2) can be obtained by using the elementary trigonometric identity

$$\sin 2x = 2 \sin x \cos x$$

and elementary calculus.

In his beautiful monograph [1] Mark Kac (1914 - 1984) began with a proof of Vieta's formula using the Rademacher functions and their independence property, and in the first chapter, a generalization of Vieta's formula was suggested as an exercise. In this paper we provide a proof following Kac's idea of using the independence property of the Rademacher functions. To the best of our knowledge, this generalization has only been achieved by Kent E. Morrison using the Fourier transform and delta distributions (see [2]).

In the following section we introduce some facts about the expansion of numbers in the closed interval $[0, 1]$ to any base $g \geq 2$. In section 3 we briefly present the Rademacher functions and their independence property. In section 4 we give the generalization and its proof using the Rademacher functions and their independence property. In section 5 we use the general formulas to approximate the value of π and give a comparison of different formulas.

2 Preliminaries

Real numbers in the interval $[0,1]$ can be represented using the binary expansion. In fact, each real number $0 \leq t \leq 1$ can be written as

$$t = \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} + \frac{\varepsilon_3}{2^3} + \cdots$$

where each value of ε_i is either 0 or 1. For example

$$\frac{3}{5} = \frac{1}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} + \frac{0}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots.$$

To ensure uniqueness we also impose a terminating expansion in which all the digits from a certain point on are all equal to 0. For example, we write

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{0}{2^4} + \frac{0}{2^5} + \cdots$$

instead of

$$\frac{3}{4} = \frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$$

The digits ε_i depend on t, so they are functions of t. Hence we write

$$t = \frac{\varepsilon_1(t)}{2} + \frac{\varepsilon_2(t)}{2^2} + \frac{\varepsilon_3(t)}{2^3} + \cdots$$

This idea can of course be generalized to any integer base $g \geq 2$, so in general we can write

$$t = \frac{\omega_1(t)}{g} + \frac{\omega_2(t)}{g^2} + \frac{\omega_3(t)}{g^3} + \cdots \qquad (3)$$

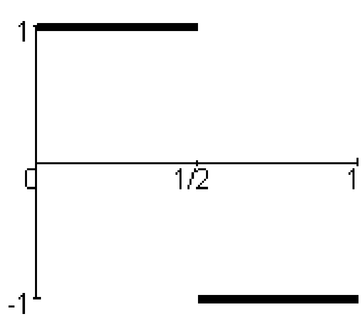
where $t \in [0,1]$ and $\omega_k \in \{0, 1, \cdots, g-1\}$.

3 Rademacher functions

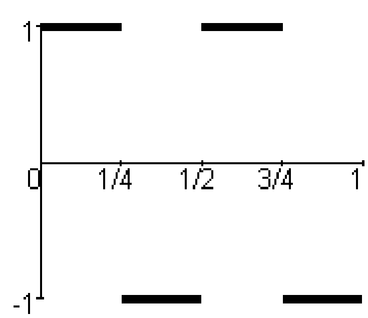
In this section we introduce the functions $r_k(t)$ which are defined in terms of the binary digits $\varepsilon_k(t)$ as follows

$$r_k(t) = 1 - 2\varepsilon_k(t) \qquad k = 1, 2, 3, \cdots. \qquad (4)$$

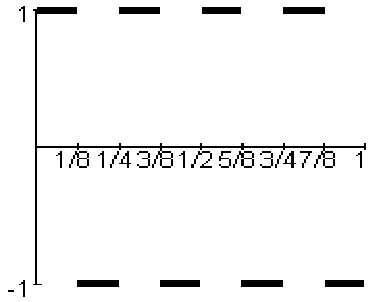
These functions were first introduced by Hans Rademacher (1892 - 1969) as a system of orthogonal functions. He introduced them in a paper which was published in 1922 (more information can be found in [4]), and these functions are now known as the Rademacher functions. For illustrative purposes, the graphs of the first four functions $r_1(t)$, $r_2(t)$, $r_3(t)$, $r_4(t)$ are given below.



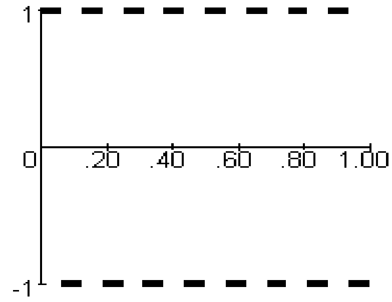
$r_1(t)$



$r_2(t)$



$r_3\,(t)$



$r_4\,(t)$

Notice from the graphs above that $r_n\,(t)$, takes the values $+1$ or -1 alternatively over the intervals $(\frac{s}{2^n}, \frac{s+1}{2^n})$, $\;s = 0, \cdots, 2^n - 1$, and so the length of each small interval is $\frac{1}{2^n}$. Also we deduce that the function $r_n\,(t)$ has 2^n intervals, half of them have the value $+1$ and the other half have the value -1.

The Rademacher functions satisfy the independence property

$$\mu\,\{r_1\,(t) = \delta_1, r_2\,(t) = \delta_2, \cdots, r_n\,(t) = \delta_n\} = \prod_{k=1}^{n} \mu\{r_k\,(t) = \delta_k\} \tag{5}$$

where μ stands for the measure (length) of the set defined inside the braces and $\delta_1, \cdots, \delta_n$ is a sequence of $+1$ and -1. Thus the product in (5) is $\left(\frac{1}{2}\right)^n$.

3.1 Generalization of the Rademacher functions

As we can have an expansion to any base for the numbers in the interval $[0, 1]$, we can also define the Rademacher functions as

$$r_k(t) = 1 - \frac{2\omega_k(t)}{g-1} \qquad k = 1, 2, 3, \cdots \tag{6}$$

where $\omega_k \in \{0, 1, \cdots, g-1\}$ and g is a positive integer greater than 1.

Moreover, the functions defined above can also be shown to be independent. Therefore, we have

$$\mu\,\{r_1\,(t) = \delta_1, r_2\,(t) = \delta_2, \cdots\cdot, r_n\,(t) = \delta_n\} = \prod_{k=1}^{n} \mu\{r_k\,(t) = \delta_k\} = \left(\frac{1}{g}\right)^n,$$

where $\delta_1, \cdots, \delta_n \in \{1, \frac{g-3}{g-1}, \frac{g-5}{g-1}, \cdots, \frac{1}{g-1}, \frac{-1}{g-1}, \cdots, \frac{-(g-5)}{g-1}, \frac{-(g-3)}{g-1}, -1\}$ when g is an even number and $\delta_1, \cdots, \delta_n \in \{1, \frac{g-3}{g-1}, \frac{g-5}{g-1}, \cdots, \frac{2}{g-1}, 0, \frac{-2}{g-1}, \cdots, \frac{-(g-5)}{g-1}, \frac{-(g-3)}{g-1}, -1\}$ when g is an odd number.

4 The general formulas

The following theorem generalizes formula (2).

Theorem 4.1 *Let g be a positive integer ≥ 2. If g is even, then*

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \frac{2}{g} \left(\sum_{m=1}^{\frac{g}{2}} \cos \left[\frac{(2m-1)x}{g^k} \right] \right). \tag{7}$$

If g is odd, then

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \frac{1}{g} \left(2 \left(\sum_{m=1}^{\frac{g-1}{2}} \cos \left[\frac{2mx}{g^k} \right] \right) + 1 \right). \tag{8}$$

Proof. Equality (8) can be proved by establishing the next five identities:

$$\frac{1-2t}{g-1} = \sum_{k=1}^{\infty} \frac{r_k(t)}{g^k} \tag{9}$$

$$\int_0^1 e^{(g-1)ix(\frac{1-2t}{g-1})} dt = \int_0^1 e^{ix(1-2t)} dt = \frac{\sin x}{x} \tag{10}$$

$$\int_0^1 \exp\left[(g-1)ix \frac{r_k(t)}{g^k} \right] dt = \frac{1}{g} \left(2 \sum_{m=1}^{\frac{g-1}{2}} \cos \left[\frac{2mx}{g^k} \right] + 1 \right) \tag{11}$$

$$\int_0^1 e^{(g-1)ix(\frac{1-2t}{g-1})} dt = \int_0^1 \exp\left[(g-1)ix \lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k(t)}{g^k} \right] dt \tag{12}$$

$$\begin{aligned} \int_0^1 \exp\left[(g-1)ix \lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k(t)}{g^k} \right] dt &= \lim_{n\to\infty} \int_0^1 \prod_{k=1}^{n} \exp\left[(g-1)ix \frac{r_k(t)}{g^k} \right] dt \\ &= \lim_{n\to\infty} \prod_{k=1}^{n} \int_0^1 \exp\left[(g-1)ix \frac{r_k(t)}{g^k} \right] dt. \end{aligned} \tag{13}$$

First, equation (9) can be proved as follows. From (3) and (6) we have

$$\begin{aligned} t &= \sum_{k=1}^{\infty} \frac{\omega_k(t)}{g^k} \\ &= \sum_{k=1}^{\infty} \frac{(g-1)(1-r_k(t))}{2.g^k}. \end{aligned}$$

Thus,

$$\frac{2t}{g-1} = \sum_{k=1}^{\infty} \frac{1}{g^k} - \sum_{k=1}^{\infty} \frac{r_k(t)}{g^k}.$$

But $\sum_{k=1}^{\infty} \frac{1}{g^k}$ is a geometric series and converges to $\frac{1}{g-1}$. Hence

$$\frac{2t}{g-1} = \frac{1}{g-1} - \sum_{k=1}^{\infty} \frac{r_k(t)}{g^k}$$

which proves equation (9). Then equation (10) can be obtained by a simple integration and the identity

$$\frac{e^{ix} - e^{-ix}}{2i} = \sin x.$$

Similarly, equation (11) can be derived by a simple integration and noticing that the function $r_k(t)$ is integrable over the intervals $(\frac{s}{g^k}, \frac{s+1}{g^k}), \quad s = 0, \cdots, g^k - 1$ and has the values $1, \frac{g-3}{g-1}, \frac{g-5}{g-1}, \cdots, \frac{2}{g-1}, 0, \frac{-2}{g-1}, \cdots, \frac{-(g-5)}{g-1}, \frac{-(g-3)}{g-1}, -1$ alternatively over these intervals. Therefore we have g^k possible values and the set of points t satisfying each value has a total length of $\frac{1}{g}$. Notice

$$\int_0^1 \exp\left[(g-1)ix\frac{r_k(t)}{g^k}\right] dt = g^{k-1} \int_0^{\frac{1}{g^{k-1}}} \exp\left[(g-1)ix\frac{r_k(t)}{g^k}\right] dt.$$

and

$$\begin{aligned}
& \int_0^{\frac{1}{g^{k-1}}} \exp\left[(g-1)ix\frac{r_k(t)}{g^k}\right] dt \\
= \; & \int_0^{\frac{1}{g^k}} \exp\left[ix\frac{(g-1)}{g^k}\right] dt + \int_{\frac{g-1}{g^k}}^{\frac{g}{g^k}} \exp\left[ix\frac{-(g-1)}{g^k}\right] dt \\
& + \int_{\frac{1}{g^k}}^{\frac{2}{g^k}} \exp\left[ix\frac{(g-3)}{g^k}\right] dt + \int_{\frac{g-2}{g^k}}^{\frac{g-1}{g^k}} \exp\left[ix\frac{-(g-3)}{g^k}\right] dt \\
& + \int_{\frac{2}{g^k}}^{\frac{3}{g^k}} \exp\left[ix\frac{(g-5)}{g^k}\right] dt + \int_{\frac{g-3}{g^k}}^{\frac{g-2}{g^k}} \exp\left[ix\frac{-(g-5)}{g^k}\right] dt \\
& + \cdots \\
& + \int_{\frac{\frac{g-3}{2}}{g^k}}^{\frac{\frac{g-1}{2}}{g^k}} \exp\left[ix\frac{2(g-1)}{g^k}\right] dt + \int_{\frac{\frac{g+1}{2}}{g^k}}^{\frac{\frac{g+3}{2}}{g^k}} \exp\left[ix\frac{-2(g-1)}{g^k}\right] dt \\
& + \int_{\frac{\frac{g-1}{2}}{g^k}}^{\frac{\frac{g+1}{2}}{g^k}} \exp\left[ix\frac{0(g-1)}{g^k}\right] dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^1 \exp\left[(g-1)ix\frac{r_k(t)}{g^k}\right]dt \\
&= g^{k-1}\left[\frac{1}{g^k}\left(\begin{array}{c} e^{\frac{ix(g-1)}{g^k}} + e^{\frac{-ix(g-1)}{g^k}} + e^{\frac{ix(g-3)}{g^k}} + e^{\frac{-ix(g-3)}{g^k}} + \\ e^{\frac{ix(g-5)}{g^k}} + e^{\frac{-ix(g-5)}{g^k}} + \cdots + e^{\frac{2ix}{g^k}} + e^{\frac{-2ix}{g^k}} + 1 \end{array}\right)\right] \\
&= \frac{1}{g}\left(2\sum_{m=1}^{\frac{g-1}{2}} \cos\left[\frac{2mx}{g^k}\right] + 1\right).
\end{aligned}$$

Equation (12) follows from a trivial substitution in equation (9), and finally, equation (13) is the where we use the independence property of the Rademacher functions.

Since the exponential function is continuous, we have

$$\int_0^1 \exp\left[(g-1)ix \lim_{n\to\infty}\sum_{k=1}^{n}\frac{r_k(t)}{g^k}\right]dt = \int_0^1 \lim_{n\to\infty} \exp\left[(g-1)ix\sum_{k=1}^{n}\frac{r_k(t)}{g^k}\right]dt.$$

Also since $\exp\left[(g-1)ix\sum_{k=1}^{n}\frac{r_k(t)}{g^k}\right]$ is a sequence of integrable functions on $[0,1]$ and converges uniformly on $[0,1]$ to the function $\exp[ix(1-2t)]$, we can interchange the limit with the integral. Therefore we have

$$\begin{aligned}
\int_0^1 \lim_{n\to\infty} \exp\left[(g-1)ix\sum_{k=1}^{n}\frac{r_k(t)}{g^k}\right]dt &= \lim_{n\to\infty}\int_0^1 \exp\left[(g-1)ix\sum_{k=1}^{n}\frac{r_k(t)}{g^k}\right]dt. \\
&= \lim_{n\to\infty}\int_0^1 \prod_{k=1}^{n} \exp\left[(g-1)ix\frac{r_k(t)}{g^k}\right]dt.
\end{aligned}$$

Now we show that

$$\lim_{n\to\infty}\int_0^1 \exp\left[(g-1)ix\sum_{k=1}^{n}\frac{r_k(t)}{g^k}\right]dt = \lim_{n\to\infty}\prod_{k=1}^{n}\int_0^1 \exp\left[(g-1)ix\frac{r_k(t)}{g^k}\right]dt$$

by evaluating the integral over all possible values of the function $r_k(t)$ as we did to prove equation (11) and using the independence property of the Rademacher functions. Taking $\delta_1, \cdots, \delta_n$ to be the possible values of the function $r_k(t)$ we have

$$\begin{aligned}
&\int_0^1 \exp\left[(g-1)ix\sum_{k=1}^{n}\frac{r_k(t)}{g^k}\right]dt \\
&= \frac{1}{g^n}\sum \exp\left[ix\sum_{k=1}^{n}(g-1)\frac{r_k(t)}{g^k}\right] \\
&= \sum_{\delta_1,\cdots,\delta_n} \exp\left[ix\sum_{k=1}^{n}(g-1)\frac{\delta_k}{g^k}\right]\mu\{r_1(t)=\delta_1, r_2(t)=\delta_2, \cdots, r_n(t)=\delta_n\} \\
&= \sum_{\delta_1,\cdots,\delta_n} \exp\left[i\sum_{k=1}^{n}(g-1)\frac{\delta_k}{g^k}\right]\prod_{k=1}^{n}\mu\{r_k(t)=\delta_k\}.
\end{aligned}$$

Simple calculations give

$$\begin{aligned}
&\sum_{\delta_1,\cdots,\delta_n} \exp\left[ix\sum_{k=1}^{n}(g-1)\frac{\delta_k}{g^k}\right]\prod_{k=1}^{n}\mu\left\{r_k\left(t\right)=\delta_k\right\} \\
=&\sum_{\delta_1,\cdots,\delta_n}\prod_{k=1}^{n}\exp\left[ix(g-1)\frac{\delta_k}{g^k}\right]\mu\left\{r_k\left(t\right)=\delta_k\right\} \\
=&\prod_{k=1}^{n}\sum_{\delta_k}\exp\left[ix(g-1)\frac{\delta_k}{g^k}\right]\mu\left\{r_k\left(t\right)=\delta_k\right\} \\
=&\prod_{k=1}^{n}\int_0^1\exp\left[(g-1)ix\frac{r_k\left(t\right)}{g^k}\right]dt.
\end{aligned}$$

Now using equation (11) we obtain formula (8). The formula for the even case can be proved using very similar steps as in the odd case. The only difference is that in step (3) we need to consider the values of the function $r_k\left(t\right)$ in $\{1, \frac{g-3}{g-1}, \frac{g-5}{g-1}, \cdots, \frac{1}{g-1}, \frac{-1}{g-1}, \cdots, \frac{-(g-5)}{g-1}, \frac{-(g-3)}{g-1}, -1\}$. ∎

If we take the value of g from 2 to 10 alternatively in formula (7) and formula (8) we get

$$\begin{aligned}
\frac{\sin x}{x} &= \prod_{k=1}^{\infty}\cos\frac{x}{2^k} \\
&= \prod_{k=1}^{\infty}\frac{1}{3}\left(2\cos\frac{2x}{3^k}+1\right) \\
&= \prod_{k=1}^{\infty}\frac{1}{2}\left[\cos\frac{x}{4^k}+\cos\frac{3x}{4^k}\right] \\
&= \prod_{k=1}^{\infty}\frac{1}{5}\left(2\left[\cos\frac{2x}{5^k}+\cos\frac{4x}{5^k}\right]+1\right) \\
&= \prod_{k=1}^{\infty}\frac{1}{3}\left[\cos\frac{x}{6^k}+\cos\frac{3x}{6^k}+\cos\frac{5x}{6^k}\right] \\
&= \prod_{k=1}^{\infty}\frac{1}{7}\left(2\left[\cos\frac{2x}{7^k}+\cos\frac{4x}{7^k}+\cos\frac{6x}{7^k}\right]+1\right) \\
&= \prod_{k=1}^{\infty}\frac{1}{4}\left[\cos\frac{x}{8^k}+\cos\frac{3x}{8^k}+\cos\frac{5x}{8^k}+\cos\frac{7x}{8^k}\right] \\
&= \prod_{k=1}^{\infty}\frac{1}{9}\left(2\left[\cos\frac{2x}{9^k}+\cos\frac{4x}{9^k}+\cos\frac{6x}{9^k}+\cos\frac{8x}{9^k}\right]+1\right) \\
&= \prod_{k=1}^{\infty}\frac{1}{5}\left[\cos\frac{x}{10^k}+\cos\frac{3x}{10^k}+\cos\frac{5x}{10^k}+\cos\frac{7x}{10^k}+\cos\frac{9x}{10^k}\right]
\end{aligned}$$

We notice that in general the formulas are infinite products of sum of cosines. The main difference is that in the odd case we have the term "1" which appears due to the value 0 of $r_k(t)$.

In the next section we are going to use these formulas to approximate the value of π by setting $x = \frac{\pi}{2}$.

5 Approximating the value of π using the general formulas

The following table is obtained using a computer program written in C++. The program uses formulas (7) and (8) to get approximations of the value of π up to 15 decimal places of different bases. In this table we report only the base value from 2 to 13, but the program can give an approximation of the value of π up to 15 decimal places to any base $g \geq 2$. The first row gives the base value to be substituted in the formula. The second row gives the number of iterations needed to compute π up to 15 decimal places. The third row gives the total number of cosines needed to compute π up to 15 decimal places. The values of the third row can be obtained by multiplying the number of cosines appearing in the formula corresponding to the base value, by the number of iterations needed to compute π up to 15 decimal places. For example, when we use $g = 4$ in the formula (7), the number of cosines are 2, and from the second row, the number of iterations needed to compute π up to 15 decimal places is 12. Therefore the number of cosines needed to compute π up to 15 decimal places are $2 \times 12 = 24$.

base value (g)	2	3	4	5	6	7	8	9	10	11	12	13
I(g)	24	15	12	11	10	9	8	8	8	7	7	7
C(g)	24	15	24	22	30	27	32	32	40	35	42	42

From the table we can observe the following:

1. As the base value g get larger, the number of iterations needed to compute π up to 15 decimal places decreases. This seems to be clear from the formulas, since as the base value gets larger the number of operations increases. Therefore we will need fewer iterations to reach the approximated value of π up to 15 decimal places.

2. We see that among all base values appearing in the table, base 3 gives an approximation to the value of π up to 15 decimal places with least number of cosines. So, we can say that the formula for base 3 is the best since it gives the value of π up to 15 decimal places with minimal cost in terms of operations. Moreover, the odd base gives the approximation with fewer number of cosines compared to the even base. In general, as we go higher in the base (odd or even), the number of cosines increases.

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