[Rose-Hulman Undergraduate Mathematics Journal](https://scholar.rose-hulman.edu/rhumj)

Volume 7
Issue 1

Article 10

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April 20, 2006

Abstract

In this paper we consider the set of positive points at which a polynomial with positive coefficients, arbitrary dimension n, and real exponents is equal to a fixed positive constant c. We find that when this set is non-empty and is bestowed with the relative Euclidean topology coming from $\mathbf{R^n}$, it is homeomorphic to a codimension one piecewise linear set that depends only on the polynomial's exponents. This piecewise linear set can in a certain sense be interpreted as a bijectively mapped version of the original set as the constant c approaches infinity. In addition to this result, we provide a condition on the polynomial exponents for testing if the solution space is homeomorphic to the n-1 dimensional sphere $\mathbf{S^{n-1}}$, and derive piecewise linear inner and outer bounds for our solution set. Each point in our solution set that lies on a fixed ray originating at the origin is trapped between a unique inner and outer bound point also on that ray. While this paper provides insight into the level sets of only a specific type of polynomial, an appropriate generalization of these observations might one day lead to improved techniques for analyzing level sets of high dimension polynomials in general, objects which appear frequently throughout mathematics.

Acknowledgments

I would like to thank Professor Charles Doran for acting as my advisor on this paper, and Professor Stanislav Jabuka for his time and advice.

Let T_c be the set of positive real points where a multivariate polynomial in n variables, with positive coefficients c_k and real exponent vectors p_k, equals a positive constant c. That is,

$$T_c \equiv \{t > 0 \mid \sum_{k=1}^{m} c_k \ t^{p_k} = c\} \subset \mathbf{R}^n \tag{1}$$

where we require that $c > 0$, $c_k > 0$, $p_k \neq (0, 0, \ldots, 0)$, $m > 0$ and use the following conventions:

$$t \equiv (t_1, t_2, \ldots, t_n) \in \mathbf{R}^n$$

$$p_k \equiv (p_{(k,1)}, p_{(k,2)}, \ldots, p_{(k,n)}) \in \mathbf{R}^n$$

$$t^{p_k} \equiv t_1^{p_{(k,1)}} \cdot t_2^{p_{(k,2)}} \cdots t_n^{p_{(k,n)}} \in \mathbf{R}$$

$$t > 0 \iff t_1 > 0, t_2 > 0, \ldots, t_n > 0.$$

We will also assume, without loss of generality, that $c > 1$. This can always be achieved without changing our set T_c if we multiply both our constant c and each of our coefficients c_k by the same sufficiently large value. It is worth noting that the "polynomials" referred to in this paper are not true polynomials, but rather generalizations of them which allow for real valued exponents.

The focus of this paper will be the set T_c when it is thought of as a topological space under the relative Euclidean topology induced by $\mathbf{R}^n$. To facilitate our later proofs, we will start out by applying a homeomorphic transformation to our space, converting T_c into a slightly simpler space Z_c . When this is complete, we will define a new topological space Z_∞ which is piecewise linear and can be thought of as a certain limit of Z_c as c approaches infinity. Theorem 1 on page 5 will then demonstrate that whenever Z_c is not empty, it is homeomorphic to Z_∞. Next, theorem 2 on page 13 will clarify when Z_c (and therefore Z_∞ and T_c) are topologically equivalent to a sphere. Finally, theorem 3 on page 14 will establish two piecewise linear spaces between which every point in a special translation of Z_c is trapped. This provides a subset of Euclidean space in which Z_c is completely contained.

You may find it useful at this point to examine the figures on pages 18, 19 and 20 which should give you some idea of what the spaces Z_c can look like in dimension $n = 2$. The sets In_c and Out_c shown in these plots are examples of the piecewise linear spaces which we will use to bound Z_c.

To avoid studying the properties of T_c directly, let us now formally introduce the simpler to analyze but topologically equivalent space

$$Z_c \equiv \log_c T_c = \{z \mid \sum_{k=1}^{m} c_k \ c^{p_k \cdot z} = c\} \subset \mathbf{R}^n \tag{2}$$

where

$$\log_c : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$$

is defined as

$$\log_c t = (\log_c t_1, \log_c t_2, \ldots, \log_c t_n) = (z_1, z_2, \ldots, z_n) = z$$

and we use the dot $(\cdot)$ to denote the dot product so,

$$p_k \cdot z \ = \ p_{(k,1)}\ z_1 + p_{(k,2)}\ z_2 + \ldots + p_{(k,n)}\ z_n.$$

Since we are only considering positive t, $\log_c$ is clearly continuous and bijective with a continuous inverse, and thus is a homeomorphism, even when restricted to

$$\log_c : T_c \longrightarrow \log_c(T_c) = Z_c.$$

Let us now define

$$k_z \equiv \text{argmax}_k \ \ p_k \cdot z \tag{3}$$

by which we mean that, for $z \in \mathbf{R}^n$ fixed, k_z is any predetermined choice of k that maximizes $p_k \cdot z$. Notice that

$$Z_c \equiv \{z \mid \sum_{k=1}^{m} c_k\ c^{p_k \cdot z} = c\} = \{z \mid \ \log_c(\sum_{k=1}^{m} c_k\ c^{p_k \cdot z}) = \log_c(c)\}$$

$$= \{z \mid \ \log_c(\sum_{k=1}^{m} c_k\ c^{p_k \cdot z} c^{-p_{k_z} \cdot z} c^{p_{k_z} \cdot z}) = 1\}$$

$$= \{z \mid p_{k_z} \cdot z = 1 - \log_c(\sum_{k=1}^{m} c_k\ c^{(p_k - p_{k_z}) \cdot z})\}$$

$$= \{z \mid p_{k_z} \cdot z = 1 - \log_c(c_{k_z} + \sum_{k \neq k_z} c_k\ c^{(p_k - p_{k_z}) \cdot z})\} \tag{4}$$

where $(p_k - p_{k_z}) \cdot z \leq 0$ by the construction of k_z. This non-positivity implies that

$$0 \leq \lim_{c \to \infty} c_k\ c^{(p_k - p_{k_z}) \cdot z} \leq c_k \quad \forall k$$

and thus that

$$\log(c_{k_z}) \leq \lim_{c \to \infty} \log(c_{k_z} + \sum_{k \neq k_z} c_k\ c^{(p_k - p_{k_z}) \cdot z}) \leq \log(\sum_{k=1}^{m} c_k) \tag{5}$$

where you should notice that in the above inequalities the base of the logarithms used does not depend on c. These inequalities will come into play again later on when we attempt to bound our solution set, but for now they just demonstrate to us that

$$Z_\infty \equiv \{z \mid p_{k_z} \cdot z = 1 - \lim_{c\to\infty} \log_c(\sum_{k=1}^{m} c_k \ c^{(p_k - p_{k_z})\cdot z})\}$$

$$= \{z \mid p_{k_z} \cdot z = 1 - \lim_{c\to\infty} \frac{\log(\sum_{k=1}^{m} c_k \ c^{(p_k - p_{k_z})\cdot z})}{\log(c)}\}$$

$$= \{z \mid p_{k_z} \cdot z = 1\} = \{z \mid \max_k \ p_k \cdot z = 1\}. \tag{6}$$

It is worth noting that in some cases

$$\lim_{c\to\infty} Z_c \ = \ \{z \mid p_{k_z} \cdot z = 1 - \lim_{c\to\infty} \log_c(\sum_{k=1}^{m} c_k \ c^{(p_k - p_{k_z})\cdot z})\}$$

and when that is so, we can use the elegant definition

$$Z_\infty \equiv \lim_{c\to\infty} Z_c. \tag{7}$$

Not only is Z_∞ a piecewise linear set, but it also satisfies the rather strong condition that it is the boundary of the convex region

$$R_\infty \equiv \{z \mid \max_k \ p_k \cdot z \le 1\}. \tag{8}$$

To see that R_∞ is convex, we need only observe that for $z \equiv \mu \ z_1 + (1-\mu) \ z_2$ where z_1 and z_2 are points in R_∞ and $0 \le \mu \le 1$, we have:

$$\max_k \ p_k \cdot z \equiv \max_k \ p_k \cdot (\mu \ z_1 + (1-\mu) \ z_2)$$

$$\le \mu \ \max_k \ p_k \cdot z_1 + (1-\mu) \ \max_k \ p_k \cdot z_2$$

$$\le \mu \ (1) + (1-\mu) \ (1) = 1$$

implying that $z \in R_\infty$. In other words, given any two points in R_∞, we know that all points on the line segment between these two points are also in R_∞.

As it turns out, even for finite c, Z_c is always the boundary of the convex region

$$R_c \equiv \{z \mid \sum_{k=1}^{m} c_k \ c^{p_k\cdot z} \le c\}. \tag{9}$$

To demonstrate that R_c is convex, we again define $z \equiv \mu \ z_1 + (1-\mu) \ z_2$ with $0 \le \mu \le 1$, where this time z_1 and z_2 are points in R_c. We now need only show that

$$\sum_{k=1}^{m} c_k \ c^{p_k\cdot z} \equiv \sum_{k=1}^{m} c_k \ c^{\mu \ p_k\cdot z_1} c^{(1-\mu) \ p_k\cdot z_2} \le c$$

to demonstrate that $z \in R_c$. But oddly enough, this is just a consequence of Hölder's famous inequality for sums, which states that for real numbers $a_k, b_k \geq 0$, with $0 \leq \mu \leq 1$,

$$\sum_{k=1}^{m} (a_k)^{\mu} (b_k)^{1-\mu} \leq \left(\sum_{k=1}^{m} a_k \right)^{\mu} \left(\sum_{k=1}^{m} b_k \right)^{1-\mu} . \tag{10}$$

Using $a_k \equiv c_k\ c^{p_k \cdot z_1}$ and $b_k \equiv c_k\ c^{p_k \cdot z_2}$ gives us:

$$\sum_{k=1}^{m} c_k\ c^{p_k \cdot z} \equiv \sum_{k=1}^{m} c_k\ c^{\mu\ p_k \cdot z_1}\ c^{(1-\mu)\ p_k \cdot z_2}$$

$$\leq \left(\sum_{k=1}^{m} c_k\ c^{p_k \cdot z_1} \right)^{\mu} \left(\sum_{k=1}^{m} c_k\ c^{p_k \cdot z_2} \right)^{1-\mu} .$$

But now, since z_1 and z_2 are elements of R_c, we have (by the definition of R_c) that:

$$\left(\sum_{k=1}^{m} c_k\ c^{p_k \cdot z_1} \right)^{\mu} \left(\sum_{k=1}^{m} c_k\ c^{p_k \cdot z_2} \right)^{1-\mu} \leq (c)^{\mu} (c)^{1-\mu} = c$$

completing our proof that Z_c is the boundary of the convex region R_c.

Theorem 1 *If Z_c does not equal the empty set $\emptyset$ then Z_c is homeomorphic to the piecewise linear set*

$$Z_\infty \equiv \{z \mid \max_k\ p_k \cdot z = 1\}$$

where both Z_c and Z_∞ are considered topological spaces under the induced Euclidean topologies coming from $\boldsymbol{R}^n$.

The five lemmas that follow will play an integral part in proving this theorem. Instead of working with Z_c as it stands, however, we will work with the topological space Z_c^s, which is just Z_c translated in $\mathbf{R}^n$ by $s = (s_1, s_2, \ldots, s_n) \in \mathbf{R}^n$ units. Thus,

$$Z_c^s\ \equiv\ \{z \mid \sum_{k=1}^{m} c_k\ c^{p_k \cdot (z+s)} = c\}\ =\ \{z \mid \sum_{k=1}^{m} c_k\ c^{p_k \cdot s}\ c^{p_k \cdot z} = c\}.$$

Z_c^s and Z_c are homeomorphic, so anything that we prove about the homeomorphism type of Z_c^s for any $s \in \mathbf{R}^n$ will also be true of Z_c and T_c. We will identify a special translation vector $s = s_0$ such that $Z_c^{s_0}$ contains at most one point lying on each ray emanating from the origin, which, as we will see, is a very desirable property.

To address this problem more formally, consider the intersection of Z_c^s with a ray which starts at the origin and is directed at an angle $\theta = (\theta_1, \theta_2, \ldots, \theta_{n-1}) \in [0, 2\pi)^{n-1}$. Any $z \in Z_c^s$ on this ray can be written $z = r\,\alpha(\theta)$, where r is z's Euclidean distance from the origin ($r = |z|$), and α is a smooth, n-valued function of θ satisfying $|\alpha(\theta)| = 1$. Thus, for example,

$$\alpha(\theta_1) = (\cos(\theta_1),\, \sin(\theta_1))$$

in dimension n = 2, and

$$\alpha(\theta_1, \theta_2) = (\cos(\theta_1)\sin(\theta_2),\, \sin(\theta_1)\sin(\theta_2),\, \cos(\theta_2))$$

in dimension n = 3. Dimension one is something of a special case since there we require that θ is of dimension zero, taking on the two values θ_- and θ_+ where $\alpha(\theta_-) = -1$ and $\alpha(\theta_+) = 1$.

To continue with our definitions, for fixed s and θ let $f_\theta^s : (0, \infty) \longrightarrow (0, \infty)$ be the smooth function of r

$$f_\theta^s(r) \equiv \sum_{k=1}^{m} c_k\ c^{p_k \cdot s}\ c^{p_k \cdot z} = \sum_{k=1}^{m} c_k\ c^{p_k \cdot s}\ c^{r(p_k \cdot \alpha(\theta))} \tag{11}$$

which is found in the definition of Z_c^s. Notice that if we define $Z_c^{s,\theta}$ to be the points in Z_c^s which lie on the ray starting at the origin and directed at angle θ, then we have

$$Z_c^{s,\theta} \equiv \{z = r\ \alpha(\theta) \mid f_\theta^s(r) = c\ \ ,\ \ r > 0\} \tag{12}$$

and

$$Z_c^s = \bigcup_{\theta \in [0,2\pi)^{n-1}} Z_c^{s,\theta} \tag{13}$$

so, the points $r > 0$ where $f_\theta^s(r) = c$ for each θ completely characterize the set Z_c^s.

The proof of theorem 1 will proceed as follows: In lemma 1 we will identify and prove the existence of our special translation vector s_0, by which we will translate Z_c, producing $Z_c^{s_0}$.

In lemma 2 we will show that on each ray emanating from the origin of Euclidean space there can exist no more than one point of $Z_c^{s_0}$. The translation vector s_0 is essential in this argument because, if necessary, it shifts Z_c so that Z_c "surrounds" the origin. For example, when Z_c is roughly spherical, $Z_c^{s_0}$ will have the origin of Euclidean space trapped within the volume it encloses, even if Z_c does not.

In lemma 3 we will demonstrate that on each ray emanating from the origin $Z_c^{s_0}$ contains the same number of points as Z_∞. On all rays where $Z_c^{s_0}$ and Z_∞ contain exactly one point, we can then treat the parameterized sets $Z_c^{s_0,\theta}$ and $Z_\infty^{0,\theta}$ as functions, each taking a ray angle θ and producing the point where the ray at that angle intersects $Z_c^{s_0}$ or Z_∞.

Lemma 4 and lemma 5 will go on to show that $Z_c^{s_0,\theta}$ and $Z_\infty^{0,\theta}$ are continuous functions of θ. At that point, we will combine our lemmas by constructing a bijection from $Z_c^{s_0}$ to Z_∞ which explicitly takes $Z_c^{s_0,\theta}$ to $Z_\infty^{0,\theta}$ for each θ. The continuity of the functions $Z_c^{s_0,\theta}$ and $Z_\infty^{0,\theta}$ in θ will lend continuity to our bijection in both directions, implying that it is a homeomorphism, and completing the proof of our theorem.

Lemma 1 *If Z_c is not the empty set, then there exists a translation vector $s = s_0$ such that*

$$f_\theta^{s_0}(0) \equiv \lim_{r\to 0} f_\theta^{s_0}(r) < c$$

for all θ, where

$$f_\theta^s(r) \equiv \sum_{k=1}^m c_k \ c^{p_k \cdot s} \ c^{r(p_k\cdot\alpha(\theta))}$$

is the function arising in the construction

$$Z_c^s \equiv \bigcup_{\theta\in[0,2\pi)^{n-1}} \{r\ \alpha(\theta) \mid f_\theta^s(r) = c \ \ , \ \ r > 0\}.$$

Proof: Notice that

$$\lim_{r\to 0} f_\theta^s(r) = \sum_{k=1}^m c_k \ c^{p_k\cdot s} \tag{14}$$

so $f_\theta^{s_0}(0)$ is less than or equal to c for all θ if and only if

$$\sum_{k=1}^m c_k \ c^{p_k\cdot s} \le c.$$

To begin with, lets find an $s = s_0$ that satisfies the above non-strict version of the inequality. Well, clearly

$$\exists s_0 \in \mathbf{R}^n \ \text{ such that } \ \sum_{k=1}^m c_k \ c^{p_k\cdot s_0} \le c \iff \exists z \in \mathbf{R}^n \ \text{ such that } \ \sum_{k=1}^m c_k \ c^{p_k\cdot z} \le c$$

and so, since $\sum_{k=1}^m c_k \ c^{p_k\cdot z} = c$ for all z $\in Z_c$ by the definition of Z_c, we will always be able to find our required s_0 by choosing any $s_0 \in Z_c$. We then need only use the assumption that $Z_c \neq \emptyset$ to be sure that some such s_0 exists.

But now, the question arises as to whether we can complete the more difficult task of finding an s_0 such that $f_\theta^{s_0}(0)$ is *strictly* less than c for all θ. Well, suppose not, and choose an $s_0 \in Z_c$ so that

$$\sum_{k=1}^m c_k \ c^{p_k\cdot s_0} = c$$

and so

$$f_\theta^{s_0}(r) = \sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0} c^{r \; p_k \cdot \alpha(\theta)}.$$

Now, if $f_\theta^{s_0}(r) < c$ for some $r = r_0$ and $\theta = \theta_0$ then the translation vector $s_1 = s_0 + r_0 \; \alpha(\theta_0)$ satisfies

$$f_\theta^{s_1}(0) < c$$

contradicting our assumption that no translation vector with this property exists. Thus, we must have that $f_\theta^{s_0}(r) \geq c$ for all θ and r. Observe though that since each $c_k > 0$ by definition, our function f_θ^s for any s satisfies

$$\frac{d^2}{dr^2} f_\theta^s(r) = \sum_{k=1}^{m} c_k \; c^{p_k \cdot s} (\log(c) \; p_k \cdot \alpha(\theta))^2 \; c^{r(p_k \cdot \alpha(\theta))} \geq 0 \tag{15}$$

and so $f_\theta^s(r)$ is convex in the variable r. In addition, since $f_\theta^s(r)$ is analytic and non-constant by construction (it is just a sum of exponentials in r), it cannot be constant on any open interval in the variable r. This, together with convexity, tells us that if $f_\theta^s(r)$ ever begins to increase as $r > 0$ increases, then it can never again decrease. Therefore, $f_\theta^s(r)$ must either increase forever, decrease forever, or initially decrease and then increase forever after. In our case, however, since $f_\theta^{s_0}(0) = c$ and $f_\theta^{s_0}(r) \geq c$ (as demonstrated above), our function must initially increase and therefore must simply increase forever as r increases on the region where $r > 0$. But this implies that $f_\theta^{s_0}(r)$ can never attain the value c for any θ and any $r > 0$, so $Z_c^{s_0}$ (by its definition) must be empty, producing a contradiction when combined with the fact that the set Z_c which is homeomorphic to $Z_c^{s_0}$ was assumed to be non-empty. This guarantees that a translation vector s_0 with $f_\theta^{s_0}(0) < c$ exists, so our lemma is proved.

Note in particular that if $\sum_{k=1}^{m} c_k < c$ we need only choose $s_0 = (0, 0, \ldots, 0)$ to satisfy our condition $f_\theta^{s_0}(0) < c$. This will come into play later on when we consider bounding our solution set Z_c.

Lemma 2 *If $Z_c \neq \emptyset$ and we choose an s_0 such that $f_\theta^{s_0}(0) < c$, then for all fixed θ, the set*

$$Z_c^{s_0,\theta} \equiv \{r \cdot \alpha(\theta) \mid f_\theta^{s_0}(r) = c \; , \; r > 0\}$$

$$\equiv \{r \cdot \alpha(\theta) \mid \sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0} \; c^{r \; p_k \cdot \alpha(\theta)} = c \; , \; r > 0\}$$

contains one element if

$$p_k \cdot \alpha(\theta) > 0$$

for some k, and zero elements otherwise.

Proof: As we have mentioned before, the convexity and non-constantness of $f_\theta^{s_0}(r)$ in the variable r place tight restrictions on its behavior. Since $f_\theta^{s_0}(0) < c$, we must have one of the following scenarios as $r > 0$ increases:

1. $f_\theta^{s_0}(r)$ increases forever and thus attains c exactly once.

2. $f_\theta^{s_0}(r)$ decreases initially but then levels out and increases forever after, again attaining c exactly once.

3. $f_\theta^{s_0}(r)$ is non-increasing forever, and thus never attains the value c.

The number of times that c is reached therefore depends only on the long run behavior of $f_\theta^{s_0}(r)$. In particular, if $p_k \cdot \alpha(\theta) > 0$ for any k then our function will have at least one term with a positive exponent and thus will eventually increase forever and attain c exactly once. On the other hand, if this inequality is not satisfied for any k, then our function will have all non-positive exponents and therefore will never increase and never reach c. This completes the proof of lemma 2.

Lemma 3 *If we choose an s_0 such that $f_\theta^{s_0}(0) < c$ then, for all fixed θ, the set $Z_c^{s_0,\theta}$, contains the same number of elements as*

$$Z_\infty^{0,\theta} = \{z = r\ \alpha(\theta) \mid \max_k\ p_k \cdot z = 1\ ,\ r > 0\}.$$

Proof: Suppose that $Z_c^{s_0,\theta}$ contains one element for some fixed θ. As we have seen in lemma 2, this implies that there exists some $k = \tilde{k}$ such that $p_{\tilde{k}} \cdot \alpha(\theta) > 0$. But that means that $z = r\ \alpha(\theta)$ satisfies $p_{\tilde{k}} \cdot z > 0$ for all $r > 0$. Note however that

$$\max_k\ p_k \cdot z\ \geq\ p_{\tilde{k}} \cdot z > 0.$$

Thus, since $\max_k\ p_k \cdot z = r \max_k\ p_k \cdot \alpha(\theta)$ is positive for all positive r, and since $\max_k\ p_k \cdot \alpha(\theta)$ is constant due to our choice of a constant θ, that means that there is a unique $r > 0$ such that

$$\max_k\ p_k \cdot z\ = r\ \max_k\ p_k \cdot \alpha(\theta)\ = 1.$$

In particular, it is just

$$r = \frac{1}{\max_k\ p_k \cdot \alpha(\theta)}.$$

But this implies that the set $Z_\infty^{0,\theta}$ contains exactly one element.

On the other hand, suppose that $Z_c^{s_0,\theta}$ has no elements (which we know, by lemma 2 is the only other possibility for $Z_c^{s_0,\theta}$). In this case, lemma 2 implies that there is no k such that $p_k \cdot \alpha(\theta) > 0$. But, in particular, this means that $\max_k\ p_k \cdot \alpha(\theta) \leq 0$. Specifically, this tells us that there is no $z = r\ \alpha(\theta)$ for $r > 0$ satisfying $r\ \max_k\ p_k \cdot \alpha(\theta) = 1$, so $Z_\infty^{0,\theta}$ is an empty set. Thus the proof of lemma 3 is complete.

Lemma 4 *For fixed c and any choice of s_0 such that $f_\theta^{s_0}(0) < c$, the parameterized set $Z_c^{s_0,\theta} = \{r\ \alpha(\theta) \mid f_\theta^{s_0}(r) = c\ , r > 0\}$ is a continuous function of θ at all points θ_0 where $Z_c^{s_0,\theta_0} \neq \emptyset$.*

Proof: From lemma 2 we know that $Z_c^{s_0,\theta}$ always has zero or one element, and thus can be thought of as a function of θ so long as we restrict ourselves to those points θ_0 where it is not empty. Then, all we need to show is that $Z_c^{s_0,\theta}$ is continuous in θ at these special points. But this is equivalent to demonstrating that

$$\lim_{\theta\to\theta_0} Z_c^{s_0,\theta} = Z_c^{s_0,\theta_0} \tag{16}$$

for all directions at which θ can approach θ_0.

Consider the graphs of $f_\theta^{s_0}(r)$ and $f_{\theta_0}^{s_0}(r)$ as functions of r. By the continuity, convexity and non-constantness of $f_{\theta_0}^{s_0}(r)$, and our choice of s_0 so that $f_{\theta_0}^{s_0}(0) < c$, it must be the case that $f_{\theta_0}^{s_0}(r)$ is increasing on some open interval around the unique point r_0 satisfying $f_{\theta_0}^{s_0}(r_0) = c$. Now, choose an $\epsilon > 0$ small enough so that $r_0 + \epsilon$ and $r_0 - \epsilon$ lie in this interval. Since $f_\theta^{s_0}(r)$ is continuous in θ we must have that

$$\lim_{\theta\to\theta_0} f_\theta^{s_0}(r) = f_{\theta_0}^{s_0}(r) \quad \forall r > 0. \tag{17}$$

Thus, in particular, we can choose our vector θ to be close enough to θ_0 (component wise) so that

$$|\ f_\theta^{s_0}(r+\epsilon) - f_{\theta_0}^{s_0}(r+\epsilon)\ | \quad \text{and} \quad |\ f_\theta^{s_0}(r-\epsilon) - f_{\theta_0}^{s_0}(r-\epsilon)\ |$$

are as close to zero as we like. By the increasingness of $f_{\theta_0}^{s_0}(r)$ on our interval, we have that $f_{\theta_0}^{s_0}(r_0-\epsilon) < c$ and $f_{\theta_0}^{s_0}(r_0+\epsilon) > c$ and thus, by making our vector θ sufficiently close to θ_0, we can ensure that $f_\theta^{s_0}(r_0-\epsilon) < c$ and $f_\theta^{s_0}(r_0+\epsilon) > c$. By the mean value theorem $f_\theta^{s_0}(r)$ must then attain the value c at some point r_1 in the interval $[r_0-\epsilon, r_0+\epsilon]$. But now, since we can choose any $\epsilon > 0$ that we like merely by making θ close enough to θ_0, it is clear that as $\theta \to \theta_0$ we have $r_1 \to r_0$ and so

$$\begin{aligned}\lim_{\theta\to\theta_0} Z_c^{s_0,\theta} &= \lim_{\theta\to\theta_0} \{r\ \alpha(\theta)\ | f_\theta^{s_0}(r) = c\,,\ r > 0\,\} \\ &= \{r\ \alpha(\theta_0)\ | f_{\theta_0}^{s_0}(r) = c\ ,\ r > 0\ \} = Z_c^{s_0,\theta_0}\end{aligned} \tag{18}$$

by the continuity of α. This holds regardless of the direction from which θ approaches θ_0, thus completing the proof of lemma 4.

Lemma 5 *The parameterized set $Z_\infty^{0,\theta}$ is a continuous function of θ at all points θ_0 where $Z_\infty^{0,\theta_0} \neq \emptyset$.*

Proof: Since $Z_\infty^{0,\theta}$ has at most one element (as demonstrated by lemma 3 in conjunction with lemma 2), it can be thought of as a function at those points θ_0 where it is non-empty. To prove continuity, we must show that

$$\lim_{\theta\to\theta_0} Z_\infty^{0,\theta} = \lim_{\theta\to\theta_0} \{z = r\ \alpha(\theta) \mid r\ \max_k\ p_k \cdot \alpha(\theta) = 1\} = Z_\infty^{0,\theta_0}$$

for all directions at which θ can approach θ_0. However, for any fixed θ, we can express $Z_\infty^{0,\theta}$ succinctly as

$$Z_\infty^{0,\theta} = \begin{cases} \{\frac{\alpha(\theta)}{\max_k p_k \cdot \alpha(\theta)}\} & , \max_k p_k \cdot \alpha(\theta) > 0 \\ \emptyset & , \text{otherwise} \end{cases} .$$

Thus, we need only show that

$$\lim_{\theta\to\theta_0} \frac{\alpha(\theta)}{\max_k\ p_k \cdot \alpha(\theta)} = \frac{\alpha(\theta_0)}{\max_k\ p_k \cdot \alpha(\theta_0)}.$$

But note that $\alpha(\theta)$ is bounded since, by construction, $|\alpha(\theta)| = 1$. This, together with the continuity of α, implies that it suffices to show that

$$\lim_{\theta\to\theta_0} \max_k\ p_k \cdot \alpha(\theta) = \max_k\ p_k \cdot \alpha(\theta_0).$$

Well, for $k_{r\cdot\alpha(\theta_0)} \equiv \text{argmax}_k\ p_k \cdot r\ \alpha(\theta_0)$ consider any k_0 satisfying

$$(p_{k_{r\cdot\alpha(\theta_0)}} -\ p_{k_0}) \cdot \alpha(\theta_0) > 0.$$

Then, since α varies continuously in θ, any θ_1 sufficiently close to θ_0 (component wise), will satisfy

$$(p_{k_{r\cdot\alpha(\theta_0)}} -\ p_{k_0}) \cdot \alpha(\theta_1) > 0.$$

But that implies that in our limit as θ approaches θ_0, the vector $p_{k_{r\cdot\alpha(\theta)}}$ eventually satisfies

$$p_{k_{r\cdot\alpha(\theta)}} \cdot \alpha(\theta_0) = p_{k_{r\cdot\alpha(\theta_0)}} \cdot \alpha(\theta_0)$$

giving us that

$$\lim_{\theta\to\theta_0} \max_k\ p_k \cdot \alpha(\theta) \equiv \lim_{\theta\to\theta_0}\ p_{k_{r\cdot\alpha(\theta)}} \cdot \alpha(\theta)$$

$$= (\lim_{\theta\to\theta_0}\ p_{k_{r\cdot\alpha(\theta)}}) \cdot (\lim_{\theta\to\theta_0}\ \alpha(\theta)) = (\lim_{\theta\to\theta_0}\ p_{k_{r\cdot\alpha(\theta)}}) \cdot \alpha(\theta_0)$$

$$= \lim_{\theta\to\theta_0}\ p_{k_{r\cdot\alpha(\theta)}} \cdot\ \alpha(\theta_0)\ =\ p_{k_{r\cdot\alpha(\theta_0)}} \cdot\ \alpha(\theta_0)$$

$$= \max_k \; p_k \cdot \alpha(\theta_0)$$

completing the proof of lemma 5.

At long last we are in a position to prove theorem 1, which states that Z_∞ and Z_c are homeomorphic whenever $Z_c \neq \emptyset$. By lemma 1 we know that there will always be some translation vector s_0 satisfying $f_\theta^{s_0}(0) < c$. Lemma 2 and 3 then tell us that $Z_\infty^{0,\theta}$ and $Z_c^{s_0,\theta}$ always have the same number of elements (zero or one). Finally, lemma 4 and 5 tell us that $Z_\infty^{0,\theta}$ and $Z_c^{s_0,\theta}$ vary continuously (when thought of as functions) in the variable θ at all points θ_0 where they are non-empty. Now, to prove our theorem, we consider the map

$$\psi : Z_c \longrightarrow Z_\infty$$
$$\psi(Z_c^{s_0,\theta_0}) = Z_\infty^{0,\theta_0} \tag{19}$$

The inverse of ψ clearly also exists, and is just given by

$$\psi^{-1} : Z_\infty \longrightarrow Z_c$$
$$\psi^{-1}(Z_\infty^{0,\theta_0}) = Z_c^{s_0,\theta_0} \tag{20}$$

so ψ is bijective.

To prove that ψ is a homeomorphism, all that remains to be shown is that it and its inverse are continuous at all points θ_0 where they are defined. Thus we need to demonstrate that

$$\lim_{Z_c^{s_0,\theta} \to Z_c^{s_0,\theta_0}} \psi(Z_c^{s_0,\theta}) = \psi(Z_c^{s_0,\theta_0})$$

and

$$\lim_{Z_\infty^{0,\theta} \to Z_\infty^{0,\theta_0}} \psi^{-1}(Z_\infty^{0,\theta}) = \psi^{-1}(Z_\infty^{0,\theta_0})$$

Note, however, that points in Z_c or Z_∞ which have differing θ lie on different rays shooting out from the origin. In particular, since each angle θ is associated with at most one unique point in each of our two sets, the only possible way that $Z_c^{s_0,\theta}$ can approach $Z_c^{s_0,\theta_0}$ or $Z_\infty^{0,\theta}$ can approach Z_∞^{0,θ_0} is if θ approaches θ_0. Thus we have

$$\lim_{Z_c^{s_0,\theta} \to Z_c^{s_0,\theta_0}} \psi(Z_c^{s_0,\theta}) = \lim_{\theta \to \theta_0} \psi(Z_c^{s_0,\theta})$$
$$= \lim_{\theta \to \theta_0} Z_\infty^{0,\theta} = Z_\infty^{0,\theta_0} = \psi(Z_c^{s_0,\theta_0})$$

and

$$\lim_{Z_\infty^{0,\theta} \to Z_\infty^{0,\theta_0}} \psi^{-1}(Z_\infty^{0,\theta}) = \lim_{\theta \to \theta_0} \psi^{-1}(Z_\infty^{0,\theta})$$
$$= \lim_{\theta \to \theta_0} Z_c^{s_0,\theta} = Z_c^{s_0,\theta_0} = \psi^{-1}(Z_\infty^{0,\theta_0})$$

due to the continuity of $Z_\infty^{0,\theta}$ and $Z_c^{s_0,\theta}$ in θ. Therefore, we have that ψ is a homeomorphism from Z_c to Z_∞, completing the proof of theorem 1.

There is an important special case of the homeomorphism type of Z_c that is worth addressing, namely when Z_c is homeomorphic to $\mathbf{S}^{n-1}$, the n-1 dimensional sphere. To investigate the circumstances under which this occurs, we define

$$\Delta = \text{ConvexHull}(\ \bigcup_{k=1}^{m} \{p_k\}\ ) \tag{21}$$

by which we mean that Δ is the minimal convex polytope (with its interior filled) that contains all the points in $\bigcup_{k=1}^{m}\{p_k\}$. When the exponent vectors p_k are integer valued, Δ is commonly known as the Newton polytope of the polynomial $\sum_{k=1}^{m} c_k\ t^{p_k}$. As it turns out, the set $Z_\infty = \{z \mid \max_k\ p_k \cdot z = 1\}$ has a very similar construction to the boundary of what is known as the "dual" polytope associated with the Newton polytope Δ.

Theorem 2 *Z_c is homeomorphic to $\mathbf{S}^{n-1}$ if and only if Δ contains the origin $(0, 0, \ldots, 0) \in \mathbf{R}^n$ in its interior.*

Proof: If $Z_c^{s_0,\theta}$ is non-empty for all θ, then due to lemmas 1 and 2 there is exactly one element of $Z_c^{s_0,\theta}$ lying on each ray protruding outward from the origin. But then, since $Z_c^{s_0,\theta}$ varies continuously in θ, and since two values $Z_c^{s_0,\theta_a}$ and $Z_c^{s_0,\theta_b}$ are close to each other if and only if θ_a is close to θ_b, $Z_c^{s_0,\theta}$ interpreted as a function of θ is itself a homeomorphism between $Z_c^{s_0}$ and $\bigcup_{\theta\in[0,2\pi)^{n-1}} = \mathbf{S}^{n-1}$. If, on the other hand, $Z_c^{s_0,\theta}$ is empty for any θ, then Z_c is homeomorphic to the n-1 sphere with some points removed, and thus can never be homeomorphic to the sphere itself.

But when will this condition that $Z_c^{s_0,\theta}$ is always non-empty be satisfied? Lemma 2 tells us that $Z_c^{s_0,\theta}$ will be empty if and only if $\alpha(\theta)$ satisfies $p_k \cdot \alpha(\theta) \le 0$ for all k. In other words, $Z_c^{s_0,\theta}$ will be empty for some θ if and only if there exists some vector $z = r\ \alpha(\theta) \neq (0, 0, \ldots, 0)$ with $r > 0$ and $p_k \cdot z \le 0$ for all k.

Assume that we can find such a vector z. Well, the fact that $p_k \cdot z \le 0$ for all k means in particular that the minimal angle measured between the vector z and each of the vectors p_k is not in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. But this means that we can construct a codimension one plane in $\mathbf{R}^n$ passing through the origin such that all of the points p_k lie on the opposite side of the plane as z, where we consider any points lying on the plane itself to be "on the opposite side of the plane as" all other points. However, this means that the convex hull of $\bigcup_{k=1}^{m}\{p_k\}$ cannot cross through our plane passing through the origin, which means that no points lying on this plane can be in the interior of our convex hull. In particular, this implies that the origin is not in $\Delta = \text{ConvexHull}(\ \bigcup_{k=1}^{m}\{p_k\}\ )$.

On the other hand, assume that Δ contains the origin. By the definition of the convex hull, the points $d \in \Delta$ are just those that can be written $d =$

$\sum_{k=1}^{m} w_k\, p_k$ for some weights $w_k \in \mathbf{R}$ with $\sum_{k=1}^{m} w_k = 1$ and $w_k \geq 0$. Then, since Δ contains the origin by assumption, there must be some set of weights w_k satisfying

$$\sum_{k=1}^{m} w_k\, p_k = (0, 0, \ldots, 0)$$

But that means that

$$p_1 = -\frac{1}{w_1} \sum_{k=2}^{m} w_k\, p_k$$

and so, for any vector z

$$p_1 \cdot z = -\frac{1}{w_1} \sum_{k=2}^{m} w_k\, p_k \cdot z \tag{22}$$

Assume that there exists a vector z satisfying $p_k \cdot z \leq 0$ for all k. This implies that $p_1 \cdot z$ is non-positive and $-\frac{1}{w_1} \sum_{k=2}^{m} w_k\, p_k \cdot z$ is non-negative. But due to equation 22 above, both of these conditions can only be satisfied if $p_k \cdot z = 0$ for all k. But this implies that the vectors p_k all lie on the same side of an n-1 dimensional plane that passes through the origin (and is perpendicular to z), prohibiting the convex hull of the p_k from containing the origin in its interior, which contradicts our assumption. Thus we are guaranteed that for every vector $z = r\,\alpha(\theta)$, there exists some k so that $p_k \cdot z > 0$, which, due to lemma 2, implies that there is exactly one point in the set $Z_c^{s_0,\theta}$ for each θ, giving us that $Z_c^{s_0}$ is homeomorphic to $\mathbf{S}^{n-1}$. This completes the proof of theorem 2.

As it turns out, there is more that we can say about $Z_c^{s_0}$ than just its homeomorphism type. Consider our final theorem which places restrictions on how far the points of $Z_c^{s_0,\theta}$ can and must lie from the origin.

Theorem 3 *For fixed c and any choice of s_0 such that $f_\theta^{s_0}(0) < c$ for all fixed θ, we have that each non-empty set*

$$Z_c^{s_0,\theta} \equiv \{r \cdot \alpha(\theta) \mid f_\theta^{s_0}(r) = c\ ,\ r > 0\}$$
$$\equiv \{r \cdot \alpha(\theta) \mid \sum_{k=1}^{m} c_k\ c^{p_k \cdot s_0}\ c^{r\ p_k \cdot \alpha(\theta)} = c\ ,\ r > 0\}$$

consists of a single point that lies between the unique points in the sets

$$\mathrm{In}_c^\theta \equiv \{\frac{1 - \log_c(\sum_{k=1}^{m} c_k\ c^{p_k \cdot s_0})}{\max_k\ p_k \cdot \alpha(\theta)}\ \alpha(\theta)\}$$

and

$$\mathrm{Out}_c^\theta \equiv \{\frac{1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0})}{\max_k\ p_k \cdot \alpha(\theta)}\ \alpha(\theta)\}$$

on a line passing through the origin.

Proof: Consider equation 4 on page 3, which tells us that

$$Z_c = \{z \mid p_{k_z} \cdot z = 1 - \log_c(c_{k_z} + \sum_{k \neq k_z} c_k \; c^{(p_k - p_{k_z}) \cdot z})\}.$$

Extending this result to $Z_c^{s_0}$ we have

$$Z_c^{s_0} = \{z \mid p_{k_z} \cdot z = 1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0} + \sum_{k \neq k_z} c_k \; c^{p_k \cdot s_0} c^{(p_k - p_{k_z}) \cdot z})\}. \tag{23}$$

Now, since $(p_k - p_{k_z}) \cdot z \leq 0$ by the construction of k_z, and we can always guarantee that $c > 1$, we have that $0 < c^{(p_k - p_{k_z}) \cdot z} \leq 1$. Since $\log_c$ is a strictly increasing function for positive inputs and each term in our sum is positive, this tells us that:

$$\log_c(c_{k_z} c^{p_{k_z} \cdot s_0}) \leq \log_c(\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0} c^{(p_k - p_{k_z}) \cdot z}) \leq \log_c(\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0}). \tag{24}$$

However, our choice of s_0 was such that $\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0} < c$. This implies that $\log_c(\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0}) < 1$, which gives us the inequalities

$$0 < 1 - \log_c(\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0})$$

$$\leq 1 - \log_c(\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0} c^{(p_k - p_{k_z}) \cdot z}) \leq 1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0}). \tag{25}$$

Equation 23 then tells us that for all $z \in Z_c^{s_0}$

$$0 < 1 - \log_c(\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0}) \leq \; p_{k_z} \cdot z \; \leq 1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0}). \tag{26}$$

And thus, if we fix θ to some value θ_0 such that $z = r \, \alpha(\theta_0) \in Z_c$ for some $r > 0$, then

$$0 < \frac{1 - \log_c(\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0})}{p_{k_z} \cdot \alpha(\theta_0)} \leq \; r \; \leq \frac{1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0})}{p_{k_z} \cdot \alpha(\theta_0)}. \tag{27}$$

This motivates us to construct the sets,

$$\mathrm{Out}_c = \{z \mid \max_k \; p_k \cdot z = 1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0})\} \tag{28}$$

and

$$\mathrm{In}_c = \{z \mid \max_k \; p_k \cdot z = 1 - \log_c(\sum_{k=1}^{m} c_k \; c^{p_k \cdot s_0})\} \tag{29}$$

which naturally provide an outer and inner bound for Z_c. Consider

$$\mathrm{Out}_c^{\theta} = \{r \; \alpha(\theta) \mid r \; \max_k \; p_k \cdot \alpha(\theta) = 1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0}) \; , r > 0 \; \}$$

$$= \{ \frac{1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0})}{\max_k \ p_k \cdot \alpha(\theta)} \ \alpha(\theta) \mid \frac{1 - \log_c(c_{k_z} c^{p_{k_z} \cdot s_0})}{\max_k \ p_k \cdot \alpha(\theta)} \ > 0 \}$$

and

$$\mathrm{In}_c^\theta = \{ r \ \alpha(\theta) \mid r \ \max_k \ p_k \cdot \alpha(\theta) = 1 - \log_c(\sum_{k=1}^m c_k \ c^{p_k \cdot s_0}) \ , r > 0 \ \}$$

$$= \{ \frac{1 - \log_c(\sum_{k=1}^m c_k \ c^{p_k \cdot s_0})}{\max_k \ p_k \cdot \alpha(\theta)} \ \alpha(\theta) \mid \frac{1 - \log_c(\sum_{k=1}^m c_k \ c^{p_k \cdot s_0})}{\max_k \ p_k \cdot \alpha(\theta)} > 0 \}.$$

By the same argument given in lemma 3 on page 9 which explains why Z_∞^θ and Z_c^θ must have the same number of elements, we can see that Out_c^θ and In_c^θ have exactly one element if and only if Z_c^θ has exactly one element, and zero elements otherwise. Thus, if for all sets $A \subset \mathbf{R^n}$ containing just one element we define $|A|$ to be the euclidean distance of the (unique) point in the set A from the origin, then at angles θ_0 where $Z_c^{\theta_0} \neq \emptyset$ we have (by equation 27)

$$|\mathrm{In}_c^{\theta_0}| = | \frac{1 - \log_c(\sum_{k=1}^m c_k \ c^{p_k \cdot s_0})}{\max_k \ p_k \cdot \alpha(\theta_0)} \ \alpha(\theta_0) | = | \frac{1 - \log_c(\sum_{k=1}^m c_k \ c^{p_k \cdot s_0})}{\max_k \ p_k \cdot \alpha(\theta_0)} |$$

$$\leq |Z_c^{\theta_0}| \leq$$

$$| \frac{1 - \log_c(c_{k_z} \ c^{p_{k_z} \cdot s_0})}{\max_k \ p_k \cdot \alpha(\theta)} | = | \frac{1 - \log_c(c_{k_z} \ c^{p_{k_z} \cdot s_0})}{\max_k \ p_k \cdot \alpha(\theta)} \ \alpha(\theta) | = |\mathrm{Out}_c^{\theta_0}|.$$

Therefore, a ray projecting outwards from the origin along any such angle θ_0 will first hit the unique point in the set $\mathrm{In}_c^{\theta_0}$, then the unique point in the set $Z_c^{s_0, \theta_0}$, and finally the unique point in the set $\mathrm{Out}_c^{\theta_0}$. In this sense, Z_c, when shifted appropriately by a value s_0, "lies between" In_c and Out_c. This completes the proof of theorem 3.

It is worth reiterating that when the sum of our coefficients c_k is less than c, we can choose $s_0 = (0, 0, \ldots, 0)$, so in this case no translation of our initial set is necessary. However, even when the sum of our coefficients is larger than c, computing a suitable s_0 should not be terribly difficult. In fact, it seems that an uncountable infinity of them lie in every open neighborhood of each point in Z_c. Thus, as long as we can compute at least a single point $z_0 \in Z_c$ we know where to look. We might consider a tiny codimension 1 ball around z_0, and choose values s_1 at random from this ball until one satisfying $\sum_{k=1}^m c_k \ c^{p_k \cdot s_1} < c$ is found. This algorithm will probably halt rapidly in practice, so long as our ball is chosen to have a sufficiently small radius, and our value z_0 is computed with sufficient accuracy. We might even try shrinking the diameter of our ball after each failed iteration of the algorithm to cause it to halt faster.

Conclusions

We have demonstrated that the polynomial level set T_c is homeomorphic to c^{Z_∞} and that it is trapped between the curves c^{In_c} and c^{Out_c}. In addition, we have seen that T_c is homeomorphic to $\mathbf{S}^{n-1}$ if and only if Δ contains the origin.

There are a number of directions in which this research could be extended. For example, we might consider the case where our c_k are allowed to be negative or even complex. In addition, we might relax the restriction that we are only considering our polynomial slice for positive real inputs. Other possibilities would be to search for tighter piecewise linear bounds on our solution set T_c than c^{In_c} and c^{Out_c} provide, or to study the points of T_c that stretch out towards infinity. It is the author's hope that if the properties discovered in this paper generalize elegantly to include negative coefficients and negative points of polynomials then they might lead to improved techniques for studying level sets of general high dimensional polynomials, and perhaps even assist in solving systems of such polynomials.

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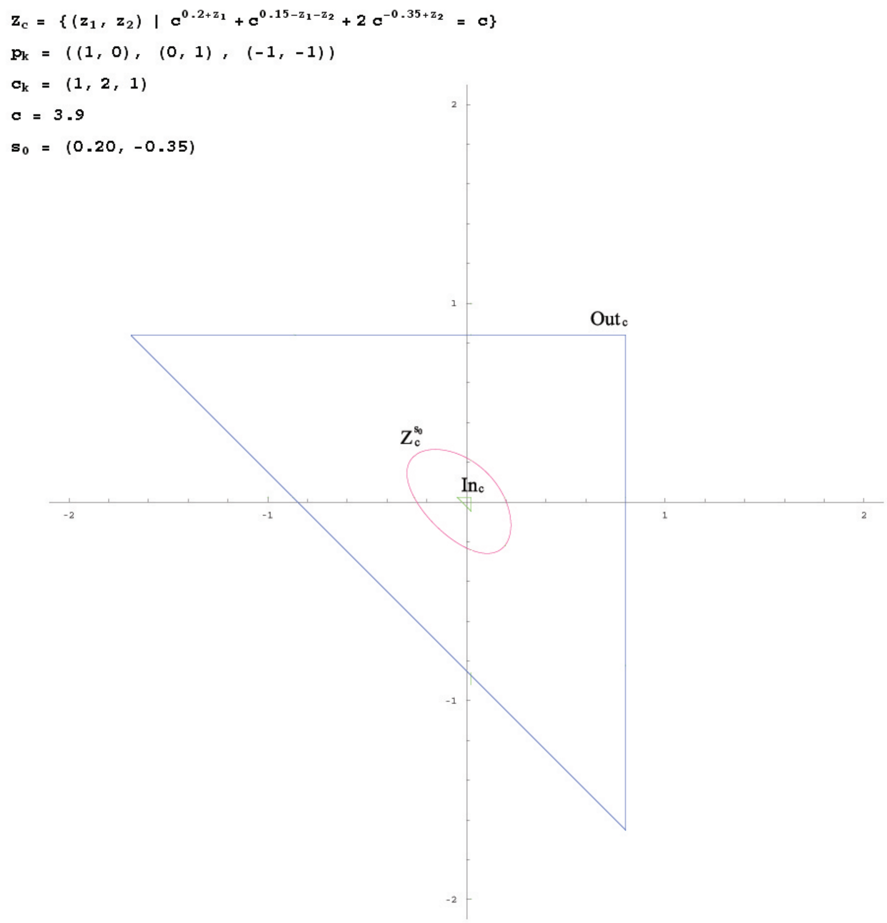

Figure 1: Sphere-like example curve with shifting.

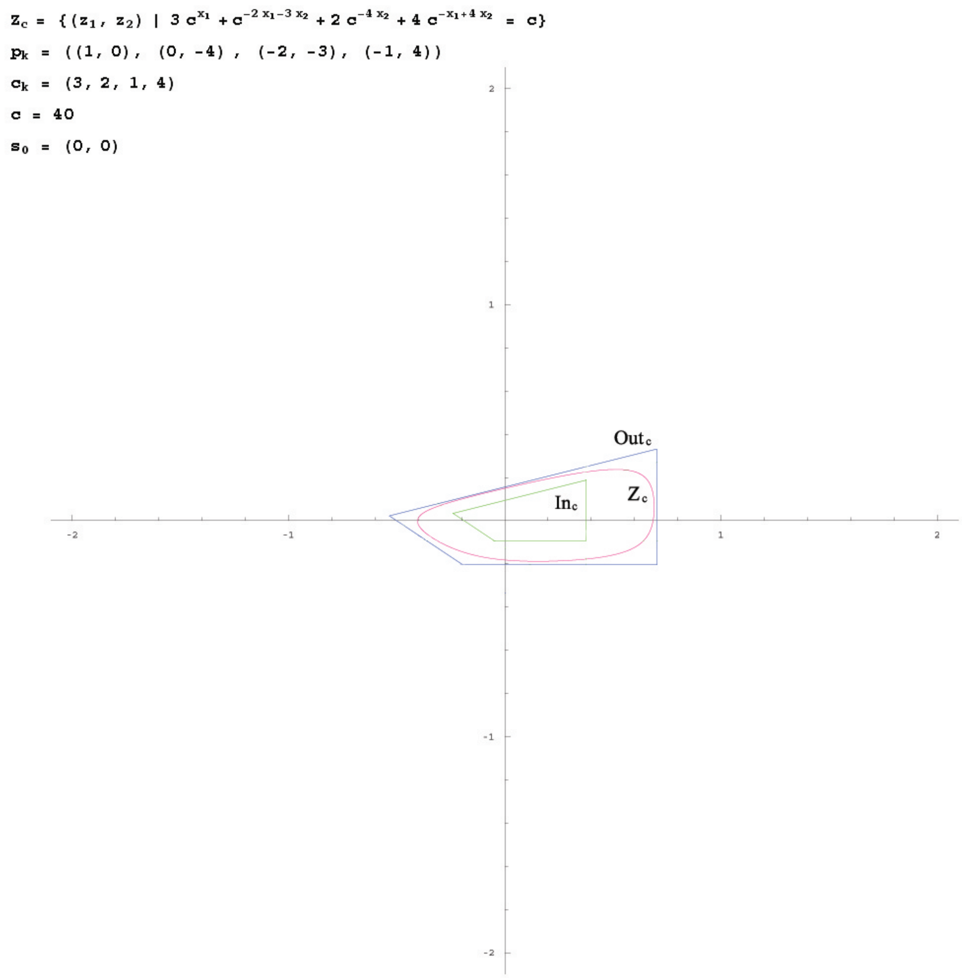

Figure 2: Sphere-like example curve without shifting.

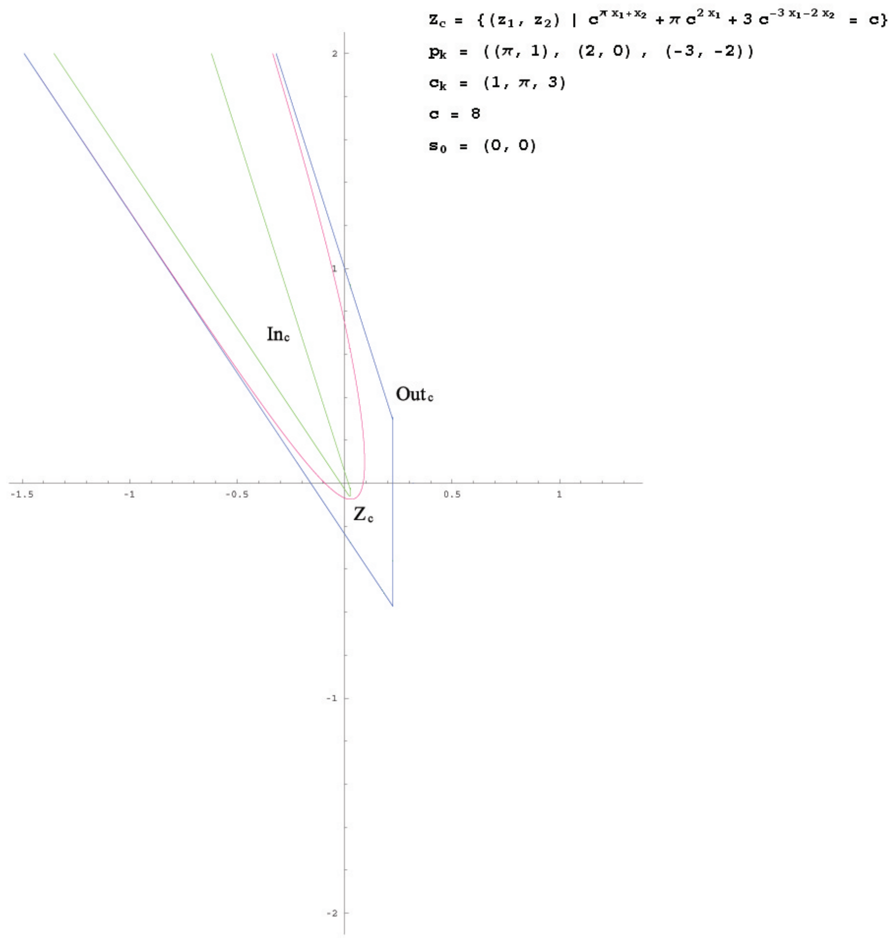

Figure 3: Line-like example curve without shifting.