Level Sets of Arbitrary Dimension Polynomials with Positive Coefficients and Real Exponents

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Level Sets of Arbitrary Dimension Polynomials with Positive Coefficients and Real Exponents

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Abstract

In this paper we consider the set of positive points at which a polynomial with positive coefficients, arbitrary dimension $n$, and real exponents is equal to a fixed positive constant $c$. We find that when this set is non-empty and is bestowed with the relative Euclidean topology coming from $\mathbb{R}^n$, it is homeomorphic to a codimension one piecewise linear set that depends only on the polynomial’s exponents. This piecewise linear set can in a certain sense be interpreted as a bijectively mapped version of the original set as the constant $c$ approaches infinity. In addition to this result, we provide a condition on the polynomial exponents for testing if the solution space is homeomorphic to the $n$-1 dimensional sphere $S^{n-1}$, and derive piecewise linear inner and outer bounds for our solution set. Each point in our solution set that lies on a fixed ray originating at the origin is trapped between a unique inner and outer bound point also on that ray. While this paper provides insight into the level sets of only a specific type of polynomial, an appropriate generalization of these observations might one day lead to improved techniques for analyzing level sets of high dimension polynomials in general, objects which appear frequently throughout mathematics.

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Let \( T_c \) be the set of positive real points where a multivariate polynomial in \( n \) variables, with positive coefficients \( c_k \) and real exponent vectors \( p_k \), equals a positive constant \( c \). That is,

\[
T_c \equiv \{ t > 0 \mid \sum_{k=1}^{m} c_k t^{p_k} = c \} \subset \mathbb{R}^n \tag{1}
\]

where we require that \( c > 0, \ c_k > 0, \ p_k \neq (0,0,\ldots,0), \ m > 0 \) and use the following conventions:

\[
t \equiv (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n
\]

\[
p_k \equiv (p_{(k,1)}, p_{(k,2)}, \ldots, p_{(k,n)}) \in \mathbb{R}^n
\]

\[
t^{p_k} \equiv t_1^{p_{(k,1)}} \cdot t_2^{p_{(k,2)}} \cdots t_n^{p_{(k,n)}} \in \mathbb{R}
\]

\[
t > 0 \iff t_1 > 0, t_2 > 0, \ldots, t_n > 0.
\]

We will also assume, without loss of generality, that \( c > 1 \). This can always be achieved without changing our set \( T_c \) if we multiply both our constant \( c \) and each of our coefficients \( c_k \) by the same sufficiently large value. It is worth noting that the “polynomials” referred to in this paper are not true polynomials, but rather generalizations of them which allow for real valued exponents.

The focus of this paper will be the set \( T_c \) when it is thought of as a topological space under the relative Euclidean topology induced by \( \mathbb{R}^n \). To facilitate our later proofs, we will start out by applying a homeomorphic transformation to our space, converting \( T_c \) into a slightly simpler space \( Z_c \). When this is complete, we will define a new topological space \( Z_\infty \) which is piecewise linear and can be thought of as a certain limit of \( Z_c \) as \( c \) approaches infinity. Theorem 1 on page 5 will then demonstrate that whenever \( Z_c \) is not empty, it is homeomorphic to \( Z_\infty \). Next, theorem 2 on page 13 will clarify when \( Z_c \) (and therefore \( Z_\infty \) and \( T_c \)) are topologically equivalent to a sphere. Finally, theorem 3 on page 14 will establish two piecewise linear spaces between which every point in a special translation of \( Z_c \) is trapped. This provides a subset of Euclidean space in which \( Z_c \) is completely contained.

You may find it useful at this point to examine the figures on pages 18, 19 and 20 which should give you some idea of what the spaces \( Z_c \) can look like in dimension \( n = 2 \). The sets \( \text{In}_c \) and \( \text{Out}_c \) shown in these plots are examples of the piecewise linear spaces which we will use to bound \( Z_c \).

To avoid studying the properties of \( T_c \) directly, let us now formally introduce the simpler to analyze but topologically equivalent space

\[
Z_c \equiv \log_c T_c = \{ z \mid \sum_{k=1}^{m} c_k \ c^{p_k \cdot z} = c \} \subset \mathbb{R}^n \tag{2}
\]
where $$\log_c : \mathbb{R}_+^n \rightarrow \mathbb{R}^n$$ is defined as

$$\log_c t = (\log_c t_1, \log_c t_2, \ldots, \log_c t_n) = (z_1, z_2, \ldots, z_n) = z$$

and we use the dot ($\cdot$) to denote the dot product so,

$$p_k \cdot z = p_{(k,1)} z_1 + p_{(k,2)} z_2 + \ldots + p_{(k,n)} z_n.$$ 

Since we are only considering positive $$t$$, $$\log_c$$ is clearly continuous and bijective with a continuous inverse, and thus is a homeomorphism, even when restricted to $$\log_c : T_c \rightarrow \log_c(T_c) = Z_c$$.

Let us now define

$$k_z \equiv \text{argmax}_k p_k \cdot z$$

by which we mean that, for $$z \in \mathbb{R}^n$$ fixed, $$k_z$$ is any predetermined choice of $$k$$ that maximizes $$p_k \cdot z$$. Notice that

$$Z_c \equiv \{ z \mid m \sum_{k=1}^m c_k e^{p_k \cdot z} = c \} = \{ z \mid \log_c(\sum_{k=1}^m c_k e^{p_k \cdot z}) = \log_c(c) \}$$

$$= \{ z \mid \log_c(\sum_{k=1}^m c_k e^{p_k \cdot z} e^{-p_k \cdot z} e^{p_k \cdot z}) = 1 \}$$

$$= \{ z \mid p_{k_z} \cdot z = 1 - \log_c(\sum_{k=1}^m c_k e^{(p_k - p_{k_z}) \cdot z}) \}$$

$$= \{ z \mid p_{k_z} \cdot z = 1 - \log_c(c_{k_z} + \sum_{k \neq k_z} c_k e^{(p_k - p_{k_z}) \cdot z}) \} \quad (4)$$

where $$(p_k - p_{k_z}) \cdot z \leq 0$$ by the construction of $$k_z$$. This non-positivity implies that

$$0 \leq \lim_{c \rightarrow \infty} c_k e^{(p_k - p_{k_z}) \cdot z} \leq c_k \quad \forall k$$

and thus that

$$\log(c_{k_z}) \leq \lim_{c \rightarrow \infty} \log(c_{k_z} + \sum_{k \neq k_z} c_k e^{(p_k - p_{k_z}) \cdot z}) \leq \log(\sum_{k=1}^m c_k) \quad (5)$$

where you should notice that in the above inequalities the base of the logarithms used does not depend on $$c$$. These inequalities will come into play again later on.
when we attempt to bound our solution set, but for now they just demonstrate to us that
\[
Z_\infty \equiv \{ z \mid p_k \cdot z = 1 - \lim_{c \to \infty} \log_c(\sum_{k=1}^m c_k e^{(p_k - p_{kz})z}) \}
\]
\[
= \{ z \mid p_k \cdot z = 1 - \lim_{c \to \infty} \log(\sum_{k=1}^m c_k e^{(p_k - p_{kz})z}) \}
\]
\[
= \{ z \mid p_k \cdot z = 1 \} = \{ z \mid \max_k p_k \cdot z = 1 \}. \quad (6)
\]
It is worth noting that in some cases
\[
\lim_{c \to \infty} Z_c = \{ z \mid p_k \cdot z = 1 - \lim_{c \to \infty} \log(\sum_{k=1}^m c_k e^{(p_k - p_{kz})z}) \}
\]
and when that is so, we can use the elegant definition
\[
Z_\infty \equiv \lim_{c \to \infty} Z_c. \quad (7)
\]
Not only is \(Z_\infty\) a piecewise linear set, but it also satisfies the rather strong condition that it is the boundary of the convex region
\[
R_\infty \equiv \{ z \mid \max_k p_k \cdot z \leq 1 \}. \quad (8)
\]
To see that \(R_\infty\) is convex, we need only observe that for \(z \equiv \mu z_1 + (1 - \mu) z_2\) where \(z_1\) and \(z_2\) are points in \(R_\infty\) and \(0 \leq \mu \leq 1\), we have:
\[
\max_k p_k \cdot z \equiv \max_k p_k \cdot (\mu z_1 + (1 - \mu) z_2)
\]
\[
\leq \mu \max_k p_k \cdot z_1 + (1 - \mu) \max_k p_k \cdot z_2
\]
\[
\leq \mu (1) + (1 - \mu) (1) = 1
\]
implying that \(z \in R_\infty\). In other words, given any two points in \(R_\infty\), we know that all points on the line segment between these two points are also in \(R_\infty\).

As it turns out, even for finite \(c\), \(Z_c\) is always the boundary of the convex region
\[
R_c \equiv \{ z \mid \sum_{k=1}^m c_k e^{p_k \cdot z} \leq c \}. \quad (9)
\]
To demonstrate that \(R_c\) is convex, we again define \(z \equiv \mu z_1 + (1 - \mu) z_2\) with \(0 \leq \mu \leq 1\), where this time \(z_1\) and \(z_2\) are points in \(R_c\). We now need only show that
\[
\sum_{k=1}^m c_k e^{p_k \cdot z} \equiv \sum_{k=1}^m c_k e^{\mu p_k \cdot z_1 e^{(1-\mu) p_k \cdot z_2} \leq c}
\]
to demonstrate that \( z \in R_c \). But oddly enough, this is just a consequence of Hölder’s famous inequality for sums, which states that for real numbers \( a_k, b_k \geq 0 \), with \( 0 \leq \mu \leq 1 \),

\[
\sum_{k=1}^{m} (a_k)^\mu (b_k)^{1-\mu} \leq \left( \sum_{k=1}^{m} a_k \right)^\mu \left( \sum_{k=1}^{m} b_k \right)^{1-\mu}.
\] (10)

Using \( a_k \equiv c_k e^{p_k \cdot z_1} \) and \( b_k \equiv c_k e^{p_k \cdot z_2} \) gives us:

\[
\sum_{k=1}^{m} c_k e^{p_k \cdot z} \equiv \sum_{k=1}^{m} c_k e^{\mu P_k \cdot z_1} e^{(1-\mu) P_k \cdot z_2} \leq \left( \sum_{k=1}^{m} c_k e^{P_k \cdot z_1} \right)^\mu \left( \sum_{k=1}^{m} c_k e^{P_k \cdot z_2} \right)^{1-\mu}.
\]

But now, since \( z_1 \) and \( z_2 \) are elements of \( R_c \), we have (by the definition of \( R_c \)) that:

\[
\left( \sum_{k=1}^{m} c_k e^{P_k \cdot z_1} \right)^\mu \left( \sum_{k=1}^{m} c_k e^{P_k \cdot z_2} \right)^{1-\mu} \leq (c)^\mu (c)^{1-\mu} = c
\]

completing our proof that \( Z_c \) is the boundary of the convex region \( R_c \).

---

**Theorem 1** If \( Z_c \) does not equal the empty set \( \emptyset \) then \( Z_c \) is homeomorphic to the piecewise linear set

\[
Z_\infty \equiv \{ z \mid \max_k p_k \cdot z = 1 \}
\]

where both \( Z_c \) and \( Z_\infty \) are considered topological spaces under the induced Euclidean topologies coming from \( \mathbb{R}^n \).

The five lemmas that follow will play an integral part in proving this theorem. Instead of working with \( Z_c \) as it stands, however, we will work with the topological space \( Z_c^s \), which is just \( Z_c \) translated in \( \mathbb{R}^n \) by \( s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n \) units. Thus,

\[
Z_c^s \equiv \{ z \mid \sum_{k=1}^{m} c_k e^{p_k \cdot (z+s)} = c \} = \{ z \mid \sum_{k=1}^{m} c_k e^{p_k \cdot s} e^{p_k \cdot z} = c \}.
\]

\( Z_c^s \) and \( Z_c \) are homeomorphic, so anything that we prove about the homeomorphism type of \( Z_c^s \) for any \( s \in \mathbb{R}^n \) will also be true of \( Z_c \) and \( T_c \). We will identify a special translation vector \( s = s_0 \) such that \( Z_c^{s_0} \) contains at most one point lying on each ray emanating from the origin, which, as we will see, is a very desirable property.
To address this problem more formally, consider the intersection of $Z^s_c$ with a ray which starts at the origin and is directed at an angle $\theta = (\theta_1, \theta_2, \ldots, \theta_{n-1}) \in [0, 2\pi)^{n-1}$. Any $z \in Z^s_c$ on this ray can be written $z = r \alpha(\theta)$, where $r$ is $z$’s Euclidean distance from the origin ($r = |z|$), and $\alpha$ is a smooth, $n$-valued function of $\theta$ satisfying $|\alpha(\theta)| = 1$. Thus, for example,

$$\alpha(\theta_1) = (\cos(\theta_1), \sin(\theta_1))$$

in dimension $n = 2$, and

$$\alpha(\theta_1, \theta_2) = (\cos(\theta_1)\sin(\theta_2), \sin(\theta_1)\sin(\theta_2), \cos(\theta_2))$$

in dimension $n = 3$. Dimension one is something of a special case since there we require that $\theta$ is of dimension zero, taking on the two values $\theta_-$ and $\theta_+$ where $\alpha(\theta_-) = -1$ and $\alpha(\theta_+) = 1$.

To continue with our definitions, for fixed $s$ and $\theta$ let $f^s_\theta : (0, \infty) \rightarrow (0, \infty)$ be the smooth function of $r$

$$f^s_\theta(r) \equiv \sum_{k=1}^m c_k e^{p_k \cdot s} e^{p_k \cdot z} = \sum_{k=1}^m c_k e^{p_k \cdot s} e^{r(p_k \cdot \alpha(\theta))}$$

(11)

which is found in the definition of $Z^s_c$. Notice that if we define $Z^s,c^{,\theta}$ to be the points in $Z^s_c$ which lie on the ray starting at the origin and directed at angle $\theta$, then we have

$$Z^s,c^{,\theta} \equiv \{ z = r \alpha(\theta) \mid f^s_\theta(r) = c \ , \ r > 0 \}$$

(12)

and

$$Z^s_c = \bigcup_{\theta \in [0, 2\pi)^{n-1}} Z^s,c^{,\theta}$$

(13)

so, the points $r > 0$ where $f^s_\theta(r) = c$ for each $\theta$ completely characterize the set $Z^s_c$.

The proof of theorem 1 will proceed as follows: In lemma 1 we will identify and prove the existence of our special translation vector $s_0$, by which we will translate $Z_c$, producing $Z^{s_0}_c$.

In lemma 2 we will show that on each ray emanating from the origin of Euclidean space there can exist no more than one point of $Z^{s_0}_c$. The translation vector $s_0$ is essential in this argument because, if necessary, it shifts $Z_c$ so that $Z_c$ “surrounds” the origin. For example, when $Z_c$ is roughly spherical, $Z^{s_0}_c$ will have the origin of Euclidean space trapped within the volume it encloses, even if $Z_c$ does not.

In lemma 3 we will demonstrate that on each ray emanating from the origin $Z^{s_0}_c$ contains the same number of points as $Z^{s_0}_\infty$. On all rays where $Z^{s_0}_c$ and $Z^{s_0}_\infty$ contain exactly one point, we can then treat the parameterized sets $Z^{s_0,c^{,\theta}}$ and $Z^{s_0,\theta}$ as functions, each taking a ray angle $\theta$ and producing the point where the ray at that angle intersects $Z^{s_0}_c$ or $Z^{s_0}_\infty$. 6
Lemma 4 and lemma 5 will go on to show that $Z_{c,0}^{\theta}$ and $Z_{\infty,0}^{\theta}$ are continuous functions of $\theta$. At that point, we will combine our lemmas by constructing a bijection from $Z_{c,0}^{\theta}$ to $Z_{\infty}^{\theta}$ which explicitly takes $Z_{c,0}^{\theta}$ to $Z_{0,\infty}^{0,\theta}$ for each $\theta$. The continuity of the functions $Z_{c,0}^{\theta}$ and $Z_{0,\infty}^{0,\theta}$ in $\theta$ will lend continuity to our bijection in both directions, implying that it is a homeomorphism, and completing the proof of our theorem.

Lemma 1: If $Z_c$ is not the empty set, then there exists a translation vector $s = s_0$ such that

$$f_\theta^{s_0}(0) \equiv \lim_{r \to 0} f_\theta^{s_0}(r) < c$$

for all $\theta$, where

$$f_\theta^{s}(r) \equiv \sum_{k=1}^{m} c_k e^{p_k \cdot s} e^{r(p_k \cdot \alpha(\theta))}$$

is the function arising in the construction

$$Z_c^* \equiv \bigcup_{\theta \in [0,2\pi)^{n-1}} \{ r \alpha(\theta) \mid f_\theta^{s}(r) = c , \ r > 0 \}.$$ 

Proof: Notice that

$$\lim_{r \to 0} f_\theta^{s}(r) = \sum_{k=1}^{m} c_k e^{p_k \cdot s}$$

so $f_\theta^{s_0}(0)$ is less than or equal to $c$ for all $\theta$ if and only if

$$\sum_{k=1}^{m} c_k e^{p_k \cdot s} \leq c.$$

To begin with, lets find an $s = s_0$ that satisfies the above non-strict version of the inequality. Well, clearly

$$\exists s_0 \in \mathbb{R}^n \text{ such that } \sum_{k=1}^{m} c_k e^{p_k \cdot s_0} \leq c \iff \exists z \in \mathbb{R}^n \text{ such that } \sum_{k=1}^{m} c_k e^{p_k \cdot z} \leq c$$

and so, since $\sum_{k=1}^{m} c_k e^{p_k \cdot z} = c$ for all $z \in Z_c$ by the definition of $Z_c$, we will always be able to find our required $s_0$ by choosing any $s_0 \in Z_c$. We then need only use the assumption that $Z_c \neq \emptyset$ to be sure that some such $s_0$ exists.

But now, the question arises as to whether we can complete the more difficult task of finding an $s_0$ such that $f_\theta^{s_0}(0)$ is strictly less than $c$ for all $\theta$. Well, suppose not, and choose an $s_0 \in Z_c$ so that

$$\sum_{k=1}^{m} c_k e^{p_k \cdot s_0} = c$$
and so
\[ f_{\theta}^{s_0}(r) = \sum_{k=1}^{m} c_k \, e^{p_k \cdot s_0} e^{r \cdot p_k \cdot \alpha(\theta)}. \]

Now, if \( f_{\theta}^{s_0}(r) < c \) for some \( r = r_0 \) and \( \theta = \theta_0 \) then the translation vector \( s_1 = s_0 + r_0 \alpha(\theta_0) \) satisfies
\[ f_{\theta}^{s_1}(0) < c \]
contradicting our assumption that no translation vector with this property exists. Thus, we must have that \( f_{\theta}^{s_0}(r) \geq c \) for all \( \theta \) and \( r \). Observe though that since each \( c_k > 0 \) by definition, our function \( f_{\theta}^{s_0} \) for any \( s \) satisfies
\[
\frac{d^2}{dr^2} f_{\theta}^{s}(r) = \sum_{k=1}^{m} c_k \, e^{p_k \cdot s}(\log(c) \cdot p_k \cdot \alpha(\theta))^2 \cdot c \cdot r \cdot \alpha(\theta)) \geq 0 \quad (15)
\]
and so \( f_{\theta}^{s}(r) \) is convex in the variable \( r \). In addition, since \( f_{\theta}^{s}(r) \) is analytic and non-constant by construction (it is just a sum of exponentials in \( r \)), it cannot be constant on any open interval in the variable \( r \). This, together with convexity, tells us that if \( f_{\theta}^{s}(r) \) ever begins to increase as \( r > 0 \) increases, then it can never again decrease. Therefore, \( f_{\theta}^{s}(r) \) must either increase forever, decrease forever, or initially decrease and then increase forever after. In our case, however, since \( f_{\theta}^{s_0}(0) = c \) and \( f_{\theta}^{s_0}(r) \geq c \) (as demonstrated above), our function must initially increase and therefore must simply increase forever as \( r \) increases on the region where \( r > 0 \). But this implies that \( f_{\theta}^{s_0}(r) \) can never attain the value \( c \) for any \( \theta \) and any \( r > 0 \), so \( Z_{s_0}^{c} \) (by its definition) must be empty, producing a contradiction when combined with the fact that the set \( Z_{c} \) which is homeomorphic to \( Z_{s_0}^{c} \) was assumed to be non-empty. This guarantees that a translation vector \( s_0 \) with \( f_{\theta}^{s_0}(0) < c \) exists, so our lemma is proved.

Note in particular that if \( \sum_{k=1}^{m} c_k < c \) we need only choose \( s_0 = (0,0,\ldots,0) \) to satisfy our condition \( f_{\theta}^{s_0}(0) < c \). This will come into play later on when we consider bounding our solution set \( Z_{c} \).

**Lemma 2** If \( Z_{c} \neq \emptyset \) and we choose an \( s_0 \) such that \( f_{\theta}^{s_0}(0) < c \), then for all fixed \( \theta \), the set
\[
Z_{c}^{s_0,\theta} \equiv \{ r \cdot \alpha(\theta) \mid f_{\theta}^{s_0}(r) = c, \ r > 0 \}
\]
\[\equiv \{ r \cdot \alpha(\theta) \mid \sum_{k=1}^{m} c_k \, e^{p_k \cdot s_0} e^{r \cdot p_k \cdot \alpha(\theta)} = c, \ r > 0 \}\]
contains one element if
\[ p_k \cdot \alpha(\theta) > 0 \]
for some \( k \), and zero elements otherwise.
Proof: As we have mentioned before, the convexity and non-constantness of $f_{s_0}^s(r)$ in the variable $r$ place tight restrictions on its behavior. Since $f_{s_0}^s(0) < c$, we must have one of the following scenarios as $r > 0$ increases:

1. $f_{s_0}^s(r)$ increases forever and thus attains $c$ exactly once.

2. $f_{s_0}^s(r)$ decreases initially but then levels out and increases forever after, again attaining $c$ exactly once.

3. $f_{s_0}^s(r)$ is non-increasing forever, and thus never attains the value $c$.

The number of times that $c$ is reached therefore depends only on the long run behavior of $f_{s_0}^s(r)$. In particular, if $p_k \cdot \alpha(\theta) > 0$ for any $k$ then our function will have at least one term with a positive exponent and thus will eventually increase forever and attain $c$ exactly once. On the other hand, if this inequality is not satisfied for any $k$, then our function will have all non-positive exponents and therefore will never increase and never reach $c$. This completes the proof of lemma 2.

**Lemma 3** If we choose an $s_0$ such that $f_{s_0}^s(0) < c$ then, for all fixed $\theta$, the set $Z_{s_0}^{c,\theta}$, contains the same number of elements as

$$Z_{\infty}^{c,\theta} = \{ z = r \cdot \alpha(\theta) \mid \max_k p_k \cdot z = 1 \text{, } r > 0 \}.$$ 

Proof: Suppose that $Z_{s_0}^{c,\theta}$ contains one element for some fixed $\theta$. As we have seen in lemma 2, this implies that there exists some $k = \tilde{k}$ such that $p_{\tilde{k}} \cdot \alpha(\theta) > 0$. But that means that $z = r \cdot \alpha(\theta)$ satisfies $p_{\tilde{k}} \cdot z > 0$ for all $r > 0$. Note however that

$$\max_k p_k \cdot z \geq p_{\tilde{k}} \cdot z > 0.$$ 

Thus, since $\max_k p_k \cdot z = r \max_k p_k \cdot \alpha(\theta)$ is positive for all positive $r$, and since $\max_k p_k \cdot \alpha(\theta)$ is constant due to our choice of a constant $\theta$, that means that there is a unique $r > 0$ such that

$$\max_k p_k \cdot z = r \max_k p_k \cdot \alpha(\theta) = 1.$$ 

In particular, it is just

$$r = \frac{1}{\max_k p_k \cdot \alpha(\theta)}.$$ 

But this implies that the set $Z_{\infty}^{c,\theta}$ contains exactly one element.
On the other hand, suppose that $Z_{c, \theta}^{s_0}$ has no elements (which we know, by lemma 2 is the only other possibility for $Z_{c, \theta}^{s_0}$). In this case, lemma 2 implies that there is no $k$ such that $p_k \cdot \alpha(\theta) > 0$. But, in particular, this means that $\max_k p_k \cdot \alpha(\theta) \leq 0$. Specifically, this tells us that there is no $z = r \alpha(\theta)$ for $r > 0$ satisfying $r \max_k p_k \cdot \alpha(\theta) = 1$, so $Z_{c, \theta}^{s_0}$ is an empty set. Thus the proof of lemma 3 is complete.

**Lemma 4** For fixed $c$ and any choice of $s_0$ such that $f_{\theta}^{s_0}(0) < c$, the parameterized set $Z_{c, \theta}^{s_0} = \{ r \alpha(\theta) \mid f_{\theta}^{s_0}(r) = c, r > 0 \}$ is a continuous function of $\theta$ at all points $\theta_0$ where $Z_{c, \theta}^{s_0, \theta_0} \neq \emptyset$.

Proof: From lemma 2 we know that $Z_{c, \theta}^{s_0}$ always has zero or one element, and thus can be thought of as a function of $\theta$ so long as we restrict ourselves to those points $\theta_0$ where it is not empty. Then, all we need to show is that $Z_{c, \theta}^{s_0}$ is continuous in $\theta$ at these special points. But this is equivalent to demonstrating that

$$
\lim_{\theta \to \theta_0} Z_{c, \theta}^{s_0} = Z_{c, \theta}^{s_0, \theta_0}
$$

(16)

for all directions at which $\theta$ can approach $\theta_0$.

Consider the graphs of $f_{\theta}^{s_0}(r)$ and $f_{\theta_0}^{s_0}(r)$ as functions of $r$. By the continuity, convexity and non-constantness of $f_{\theta_0}^{s_0}(r)$, and our choice of $s_0$ so that $f_{\theta_0}^{s_0}(0) < c$, it must be the case that $f_{\theta_0}^{s_0}(r)$ is increasing on some open interval around the unique point $r_0$ satisfying $f_{\theta_0}^{s_0}(r_0) = c$. Now, choose an $\epsilon > 0$ small enough so that $r_0 + \epsilon$ and $r_0 - \epsilon$ lie in this interval. Since $f_{\theta_0}^{s_0}(r)$ is continuous in $\theta$ we must have that

$$
\lim_{\theta \to \theta_0} f_{\theta}^{s_0}(r) = f_{\theta_0}^{s_0}(r) \ \forall r > 0.
$$

(17)

Thus, in particular, we can choose our vector $\theta$ to be close enough to $\theta_0$ (component wise) so that

$$
| f_{\theta}^{s_0}(r + \epsilon) - f_{\theta_0}^{s_0}(r + \epsilon) | \quad \text{and} \quad | f_{\theta}^{s_0}(r - \epsilon) - f_{\theta_0}^{s_0}(r - \epsilon) |
$$

are as close to zero as we like. By the increasingness of $f_{\theta_0}^{s_0}(r)$ on our interval, we have that $f_{\theta_0}^{s_0}(r_0 - \epsilon) < c$ and $f_{\theta_0}^{s_0}(r_0 + \epsilon) > c$ and thus, by making our vector $\theta$ sufficiently close to $\theta_0$, we can ensure that $f_{\theta}^{s_0}(r_0 - \epsilon) < c$ and $f_{\theta}^{s_0}(r_0 + \epsilon) > c$. By the mean value theorem $f_{\theta}^{s_0}(r)$ must then attain the value $c$ at some point $r_1$ in the interval $[r_0 - \epsilon, r_0 + \epsilon]$. But now, since we can choose any $\epsilon > 0$ that we like merely by making $\theta$ close enough to $\theta_0$, it is clear that as $\theta \to \theta_0$ we have $r_1 \to r_0$ and so

$$
\lim_{\theta \to \theta_0} Z_{c, \theta}^{s_0} = \lim_{\theta \to \theta_0} \{ r \alpha(\theta) \mid f_{\theta}^{s_0}(r) = c, r > 0 \}
$$

$$
= \{ r \alpha(\theta_0) \mid f_{\theta_0}^{s_0}(r) = c, r > 0 \} = Z_{c, \theta_0}^{s_0}
$$

(18)
by the continuity of $\alpha$. This holds regardless of the direction from which $\theta$ approaches $\theta_0$, thus completing the proof of lemma 4.

Lemma 5 The parameterized set $Z_{\infty}^{0, \theta}$ is a continuous function of $\theta$ at all points $\theta_0$ where $Z_{\infty}^{0, \theta_0} \neq \emptyset$.

Proof: Since $Z_{\infty}^{0, \theta}$ has at most one element (as demonstrated by lemma 3 in conjunction with lemma 2), it can be thought of as a function at those points $\theta_0$ where it is non-empty. To prove continuity, we must show that

$$\lim_{\theta \to \theta_0} Z_{\infty}^{0, \theta} = \lim_{\theta \to \theta_0} \{ z = r \alpha(\theta) \mid r \max_k p_k \cdot \alpha(\theta) = 1 \} = Z_{\infty}^{0, \theta_0}$$

for all directions at which $\theta$ can approach $\theta_0$. However, for any fixed $\theta$, we can express $Z_{\infty}^{0, \theta}$ succinctly as

$$Z_{\infty}^{0, \theta} = \left\{ \{ \max_k p_k \cdot \alpha(\theta) \}, \max_k p_k \cdot \alpha(\theta) > 0 \right\}.$$

Thus, we need only show that

$$\lim_{\theta \to \theta_0} \max_k p_k \cdot \alpha(\theta) = \max_k p_k \cdot \alpha(\theta_0).$$

But note that $\alpha(\theta)$ is bounded since, by construction, $|\alpha(\theta)| = 1$. This, together with the continuity of $\alpha$, implies that it suffices to show that

$$\lim_{\theta \to \theta_0} \max_k p_k \cdot \alpha(\theta) = \max_k p_k \cdot \alpha(\theta_0).$$

Well, for $k_{r, \alpha(\theta_0)} \equiv \arg\max_k p_k \cdot r \alpha(\theta_0)$ consider any $k_0$ satisfying

$$(p_{k_{r, \alpha(\theta_0)}} - p_{k_0}) \cdot \alpha(\theta_0) > 0.$$
completing the proof of lemma 5.

At long last we are in a position to prove theorem 1, which states that $Z_0$ and $Z_c$ are homeomorphic whenever $Z_c \neq \emptyset$. By lemma 1 we know that there will always be some translation vector $s_0$ satisfying $f_{s_0}^\theta(0) < c$. Lemma 2 and 3 then tell us that $Z_0,\theta$ and $Z_c,\theta$ always have the same number of elements (zero or one). Finally, lemma 4 and 5 tell us that $Z_0,\theta$ and $Z_c,\theta$ vary continuously (when thought of as functions) in the variable $\theta$ at all points $\theta_0$ where they are non-empty. Now, to prove our theorem, we consider the map

$$\psi: Z_c \longrightarrow Z_\infty$$

$$\psi(Z_c,\theta_0) = Z_0,\theta_0$$

(19)

The inverse of $\psi$ clearly also exists, and is just given by

$$\psi^{-1}: Z_\infty \longrightarrow Z_c$$

$$\psi^{-1}(Z_0,\theta_0) = Z_c,\theta_0$$

(20)

so $\psi$ is bijective.

To prove that $\psi$ is a homeomorphism, all that remains to be shown is that it and its inverse are continuous at all points $\theta_0$ where they are defined. Thus we need to demonstrate that

$$\lim_{Z_c,\theta_0 \rightarrow Z_0,\theta_0} \psi(Z_c,\theta) = \psi(Z_0,\theta_0)$$

and

$$\lim_{Z_\infty,\theta_0 \rightarrow Z_0,\theta_0} \psi^{-1}(Z_\infty,\theta) = \psi^{-1}(Z_0,\theta_0)$$

Note, however, that points in $Z_c$ or $Z_\infty$ which have differing $\theta$ lie on different rays shooting out from the origin. In particular, since each angle $\theta$ is associated with at most one unique point in each of our two sets, the only possible way that $Z_c,\theta$ can approach $Z_c,\theta_0$ or $Z_\infty,\theta$ can approach $Z_\infty,\theta_0$ is if $\theta$ approaches $\theta_0$. Thus we have

$$\lim_{Z_c,\theta_0 \rightarrow Z_0,\theta_0} \psi(Z_c,\theta) = \lim_{\theta \rightarrow \theta_0} \psi(Z_c,\theta)$$

$$= \lim_{\theta \rightarrow \theta_0} Z_0,\theta = Z_0,\theta_0 = \psi(Z_c,\theta_0)$$

and

$$\lim_{Z_\infty,\theta_0 \rightarrow Z_0,\theta_0} \psi^{-1}(Z_\infty,\theta) = \lim_{\theta \rightarrow \theta_0} \psi^{-1}(Z_\infty,\theta)$$

$$= \lim_{\theta \rightarrow \theta_0} Z_c,\theta = Z_c,\theta_0 = \psi^{-1}(Z_\infty,\theta_0)$$

12
due to the continuity of $Z_\infty^0, \theta$ and $Z_\infty^{s_0}, \theta$ in $\theta$. Therefore, we have that $\psi$ is a homeomorphism from $Z_c$ to $Z_\infty$, completing the proof of theorem 1.

There is an important special case of the homeomorphism type of $Z_c$ that is worth addressing, namely when $Z_c$ is homeomorphic to $S^{n-1}$, the $n$-dimensional sphere. To investigate the circumstances under which this occurs, we define

$$\Delta = \text{ConvexHull}(\bigcup_{k=1}^m \{p_k\})$$

by which we mean that $\Delta$ is the minimal convex polytope (with its interior filled) that contains all the points in $\bigcup_{k=1}^m \{p_k\}$. When the exponent vectors $p_k$ are integer valued, $\Delta$ is commonly known as the Newton polytope of the polynomial $\sum_{k=1}^m c_k t^{p_k}$. As it turns out, the set $Z_\infty = \{z | \max_k p_k \cdot z = 1\}$ has a very similar construction to the boundary of what is known as the “dual” polytope associated with the Newton polytope $\Delta$.

**Theorem 2** $Z_c$ is homeomorphic to $S^{n-1}$ if and only if $\Delta$ contains the origin $(0, 0, \ldots, 0) \in \mathbb{R}^n$ in its interior.

Proof: If $Z_c^{s_0, \theta}$ is non-empty for all $\theta$, then due to lemmas 1 and 2 there is exactly one element of $Z_c^{s_0, \theta}$ lying on each ray protruding outward from the origin. But then, since $Z_c^{s_0, \theta}$ varies continuously in $\theta$, and since two values $Z_c^{s_0, \theta_a}$ and $Z_c^{s_0, \theta_b}$ are close to each other if and only if $\theta_a$ is close to $\theta_b$, $Z_c^{s_0, \theta}$ interpreted as a function of $\theta$ is itself a homeomorphism between $Z_c^{s_0}$ and $\bigcup_{\theta \in [0, 2\pi]} S^{n-1} = S^{n-1}$.

If, on the other hand, $Z_c^{s_0, \theta}$ is empty for any $\theta$, then $Z_c$ is homeomorphic to the $n$-sphere with some points removed, and thus can never be homeomorphic to the sphere itself.

But when will this condition that $Z_c^{s_0, \theta}$ is always non-empty be satisfied? Lemma 2 tells us that $Z_c^{s_0, \theta}$ will be empty if and only if $\alpha(\theta)$ satisfies $p_k \cdot \alpha(\theta) \leq 0$ for all $k$. In other words, $Z_c^{s_0, \theta}$ will be empty for some $\theta$ if and only if there exists some vector $z = r \cdot \alpha(\theta) \neq (0, 0, \ldots, 0)$ with $r > 0$ and $p_k \cdot z \leq 0$ for all $k$.

Assume that we can find such a vector $z$. Well, the fact that $p_k \cdot z \leq 0$ for all $k$ means in particular that the minimal angle measured between the vector $z$ and each of the vectors $p_k$ is not in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. But this means that we can construct a codimension one plane in $\mathbb{R}^n$ passing through the origin such that all of the points $p_k$ lie on the opposite side of the plane as $z$, where we consider any points lying on the plane itself to be “on the opposite side of the plane as” all other points. However, this means that the convex hull of $\bigcup_{k=1}^m \{p_k\}$ cannot cross through our plane passing through the origin, which means that no points lying on this plane can be in the interior of our convex hull. In particular, this implies that the origin is not in $\Delta = \text{ConvexHull}(\bigcup_{k=1}^m \{p_k\})$.

On the other hand, assume that $\Delta$ contains the origin. By the definition of the convex hull, the points $d \in \Delta$ are just those that can be written $d =$
$$\sum_{k=1}^{m} w_k p_k$$ for some weights $w_k \in \mathbb{R}$ with $\sum_{k=1}^{m} w_k = 1$ and $w_k \geq 0$. Then, since $\Delta$ contains the origin by assumption, there must be some set of weights $w_k$ satisfying

$$\sum_{k=1}^{m} w_k p_k = (0, 0, \ldots, 0)$$

But that means that

$$p_1 = -\frac{1}{w_1} \sum_{k=2}^{m} w_k p_k$$

and so, for any vector $z$

$$p_1 \cdot z = -\frac{1}{w_1} \sum_{k=2}^{m} w_k p_k \cdot z$$ (22)

Assume that there exists a vector $z$ satisfying $p_k \cdot z \leq 0$ for all $k$. This implies that $p_1 \cdot z$ is non-positive and $-\frac{1}{w_1} \sum_{k=2}^{m} w_k p_k \cdot z$ is non-negative. But due to equation 22 above, both of these conditions can only be satisfied if $p_k \cdot z = 0$ for all $k$. But this implies that the vectors $p_k$ all lie on the same side of an $n-1$ dimensional plane that passes through the origin (and is perpendicular to $z$), prohibiting the convex hull of the $p_k$ from containing the origin in its interior, which contradicts our assumption. Thus we are guaranteed that for every vector $z = r \alpha(\theta)$, there exists some $k$ so that $p_k \cdot z > 0$, which, due to lemma 2, implies that there is exactly one point in the set $Z_{c_0}^{s_0, \theta}$ for each $\theta$, giving us that $Z_{c_0}$ is homeomorphic to $S^{n-1}$. This completes the proof of theorem 2.

As it turns out, there is more that we can say about $Z_{c_0}$ than just its homeomorphism type. Consider our final theorem which places restrictions on how far the points of $Z_{c_0}^{s_0, \theta}$ can and must lie from the origin.

**Theorem 3** For fixed $c$ and any choice of $s_0$ such that $f^{s_0}_\theta(0) < c$ for all fixed $\theta$, we have that each non-empty set

$$Z_{c_0}^{s_0, \theta} \equiv \{ r \cdot \alpha(\theta) \mid f^{s_0}_\theta(r) = c, \ r > 0 \}$$

$$\equiv \{ r \cdot \alpha(\theta) \mid \sum_{k=1}^{m} c_k e^{p_k \cdot s_0} e^{r \cdot p_k \cdot \alpha(\theta)} = c, \ r > 0 \}$$

consists of a single point that lies between the unique points in the sets

$$\text{In}_{c_0}^\theta \equiv \left\{ \frac{1 - \log(c \sum_{k=1}^{m} c_k e^{p_k \cdot s_0})}{\max_k p_k \cdot \alpha(\theta)} \alpha(\theta) \right\}$$

and

$$\text{Out}_{c_0}^\theta \equiv \left\{ \frac{1 - \log(c_k e^{p_k \cdot s_0})}{\max_k p_k \cdot \alpha(\theta)} \alpha(\theta) \right\}$$

on a line passing through the origin.
Proof: Consider equation 4 on page 3, which tells us that

\[ Z_c = \{ z \mid p_{k_z} \cdot z = 1 - \log_c(c_{k_z} + \sum_{k \neq k_z} c_k e^{(p_k - p_{k_z}) \cdot z}) \}. \]

Extending this result to \( Z_c^{s_0} \) we have

\[ Z_c^{s_0} = \{ z \mid p_{k_z} \cdot z = 1 - \log_c(c_{k_z} e^{p_{k_z} \cdot s_0} + \sum_{k \neq k_z} c_k e^{p_k \cdot s_0 e^{(p_k - p_{k_z}) \cdot z}}) \}. \] (23)

Now, since \((p_k - p_{k_z}) \cdot z \leq 0\) by the construction of \(k_z\), and we can always guarantee that \(c > 1\), we have that \(0 < e^{(p_k - p_{k_z}) \cdot z} \leq 1\). Since \(\log_c\) is a strictly increasing function for positive inputs and each term in our sum is positive, this tells us that:

\[ \log_c(c_{k_z} e^{p_{k_z} \cdot s_0}) \leq \log_c(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0 e^{(p_k - p_{k_z}) \cdot z}}) \leq \log_c(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0}). \] (24)

However, our choice of \(s_0\) was such that \(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0} < c\). This implies that \(\log_c(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0}) < 1\), which gives us the inequalities

\[ 0 < 1 - \log_c(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0}) \]

\[ \leq 1 - \log_c(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0 e^{(p_k - p_{k_z}) \cdot z}}) \leq 1 - \log_c(c_{k_z} e^{p_{k_z} \cdot s_0}). \] (25)

Equation 23 then tells us that for all \(z \in Z_c^{s_0}\)

\[ 0 < 1 - \log_c(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0}) \leq p_{k_z} \cdot z \leq 1 - \log_c(c_{k_z} e^{p_{k_z} \cdot s_0}). \] (26)

And thus, if we fix \(\theta\) to some value \(\theta_0\) such that \(z = r \alpha(\theta_0) \in Z_c\) for some \(r > 0\), then

\[ 0 < \frac{1 - \log_c(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0})}{p_{k_z} \cdot \alpha(\theta_0)} \leq r \leq \frac{1 - \log_c(c_{k_z} e^{p_{k_z} \cdot s_0})}{p_{k_z} \cdot \alpha(\theta_0)}. \] (27)

This motivates us to construct the sets,

\[ \text{Out}_c = \{ z \mid \max_k p_k \cdot z = 1 - \log_c(c_{k_z} e^{p_{k_z} \cdot s_0}) \} \] (28)

and

\[ \text{In}_c = \{ z \mid \max_k p_k \cdot z = 1 - \log_c(\sum_{k=1}^{m} c_k e^{p_k \cdot s_0}) \} \] (29)

which naturally provide an outer and inner bound for \(Z_c\). Consider

\[ \text{Out}^\theta_c = \{ r \alpha(\theta) \mid r \max_k p_k \cdot \alpha(\theta) = 1 - \log_c(c_{k_z} e^{p_{k_z} \cdot s_0}) , r > 0 \} \]
\[
\left\{ \frac{1 - \log_c(c_{k} \cdot c^{p_{k} \cdot s_{0}})}{\max_k p_k \cdot \alpha(\theta)} \alpha(\theta) \left| \frac{1 - \log_c(c_{k} \cdot c^{p_{k} \cdot s_{0}})}{\max_k p_k \cdot \alpha(\theta)} > 0 \right\}
\]

and

\[
\text{In}^\theta = \{ r \alpha(\theta) \mid r \max_k p_k \cdot \alpha(\theta) = 1 - \log_c\left(\sum_{k=1}^{m} c_k \cdot c^{p_{k} \cdot s_{0}}\right), r > 0 \}
\]

\[
= \left\{ \frac{1 - \log_c\left(\sum_{k=1}^{m} c_k \cdot c^{p_{k} \cdot s_{0}}\right)}{\max_k p_k \cdot \alpha(\theta)} \alpha(\theta) \left| \frac{1 - \log_c\left(\sum_{k=1}^{m} c_k \cdot c^{p_{k} \cdot s_{0}}\right)}{\max_k p_k \cdot \alpha(\theta)} > 0 \right\}.
\]

By the same argument given in lemma 3 on page 9 which explains why \(Z_{\infty}^0\)
and \(Z_{c}^0\) must have the same number of elements, we can see that \(\text{Out}_{e}^0\) and \(\text{In}_{e}^0\)
have exactly one element if and only if \(Z_{c}^0\) has exactly one element, and zero elements otherwise. Thus, if for all sets \(A \subset \mathbb{R}^n\) containing just one element we define \(|A|\) to be the euclidean distance of the (unique) point in the set \(A\) from the origin, then at angles \(\theta_0\) where \(Z_{c}^0 \neq \emptyset\) we have (by equation 27)

\[
|\text{In}_{e}^\theta| = |\frac{1 - \log_c\left(\sum_{k=1}^{m} c_k \cdot c^{p_{k} \cdot s_{0}}\right)}{\max_k p_k \cdot \alpha(\theta_0)} \alpha(\theta_0)| = \left| \frac{1 - \log_c\left(\sum_{k=1}^{m} c_k \cdot c^{p_{k} \cdot s_{0}}\right)}{\max_k p_k \cdot \alpha(\theta_0)} \alpha(\theta) \right|
\]

\[
\leq |Z_{c}^0| \leq \left| \frac{1 - \log_c(c_{k} \cdot c^{p_{k} \cdot s_{0}})}{\max_k p_k \cdot \alpha(\theta)} \right| = \left| \frac{1 - \log_c(c_{k} \cdot c^{p_{k} \cdot s_{0}})}{\max_k p_k \cdot \alpha(\theta)} \right| = |\text{Out}_{e}^\theta|.
\]

Therefore, a ray projecting outwards from the origin along any such angle \(\theta_0\) will first hit the unique point in the set \(\text{In}_{e}^\theta\), then the unique point in the set \(Z_{c,0}^0\), and finally the unique point in the set \(\text{Out}_{e}^\theta\). In this sense, \(Z_{c}\), when shifted appropriately by a value \(s_0\), “lies between” \(\text{In}_{c}\) and \(\text{Out}_{c}\). This completes the proof of theorem 3.

It is worth reiterating that when the sum of our coefficients \(c_k\) is less than \(c\), we can choose \(s_0 = (0, 0, \ldots, 0)\), so in this case no translation of our initial set is necessary. However, even when the sum of our coefficients is larger than \(c\), computing a suitable \(s_0\) should not be terribly difficult. In fact, it seems that an uncountable infinity of them lie in every open neighborhood of each point in \(Z_c\). Thus, as long as we can compute at least a single point \(z_0 \in Z_c\) we know where to look. We might consider a tiny codimension 1 ball around \(z_0\), and choose values \(s_1\) at random from this ball until one satisfying \(\sum_{k=1}^{m} c_k \cdot c^{p_{k} \cdot s_1} < c\) is found. This algorithm will probably halt rapidly in practice, so long as our ball is chosen to have a sufficiently small radius, and our value \(z_0\) is computed with sufficient accuracy. We might even try shrinking the diameter of our ball after each failed iteration of the algorithm to cause it to halt faster.
Conclusions

We have demonstrated that the polynomial level set $T_c$ is homeomorphic to $c^{2\infty}$ and that it is trapped between the curves $c^{\text{In}}$ and $c^{\text{Out}}$. In addition, we have seen that $T_c$ is homeomorphic to $S^{n-1}$ if and only if $\Delta$ contains the origin.

There are a number of directions in which this research could be extended. For example, we might consider the case where our $c_k$ are allowed to be negative or even complex. In addition, we might relax the restriction that we are only considering our polynomial slice for positive real inputs. Other possibilities would be to search for tighter piecewise linear bounds on our solution set $T_c$ than $c^{\text{In}}$ and $c^{\text{Out}}$ provide, or to study the points of $T_c$ that stretch out towards infinity. It is the author’s hope that if the properties discovered in this paper generalize elegantly to include negative coefficients and negative points of polynomials then they might lead to improved techniques for studying level sets of general high dimensional polynomials, and perhaps even assist in solving systems of such polynomials.

References


$\mathbb{Z}_n = \{ (z_1, z_2) \mid c^{2 \cdot 1 + 1} + c^{0 \cdot 0 - 2 - 1} + 2 c^{4 \cdot 3 - 2} = 0 \}$

$p_n = (1, 0), (0, 1), (-1, -1))

q_n = (1, 2, 1)

$c = 3.9$

$s_n = (0.20, -0.35)$

Figure 1: Sphere-like example curve with shifting.
\( Z_n = \{ (x_1, x_2) \mid 3 \sigma(x_1^4 + \sigma x_1^3 x_2 + 2 \sigma^2 x_1^2 x_2 + 4 \sigma^3 x_1 x_2^2) = 0 \} \)

\( \mathcal{B}_n = \{ (1, 0), (0, -4) , (-2, -3), (-1, 4) \} \)

\( \mathcal{C}_k = \{ (3, 2, 1, 6) \} \)

\( \sigma = 40 \)

\( \pi_0 = (0, 0) \)

Figure 2: Sphere-like example curve without shifting.
$\mathbb{Z}_c = \{ (x_1, x_2) \mid x_1^2 + x_2^2 + 2x_1 + 3x_2 - 2 = 0 \}$

$\mathbb{P} = \{ (\pi, 1), (2, 0), (-3, -2) \}$

$\mathbb{C}_b = (1, \pi, 3)$

$\varphi = 0$

$\mathbb{E}_0 = (0, 0)$

Figure 3: Line-like example curve without shifting.