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# Defining a Zeta Function for Cell Products of Graphs

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# DEFINING A ZETA FUNCTION FOR CELL PRODUCTS OF GRAPHS

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Abstract. The Riemann Zeta Function has been successfully and promisingly generalized in various ways so that the concept of zeta functions has become important in many different areas of research. In particular, work done by Y. Ihara in the 1960s led to the definition of an Ihara Zeta Function for finite graphs. The Ihara Zeta Function has the nice property of having three equivalent expressions: an Euler product form over "primes" of the graph, an expression in terms of vertex operators on the graph, and an expression in terms of arc operators on the graph. In this paper we present two possibilities for generalizing the Ihara Zeta Function to cell products of graphs. We start with a background discussion of the Ihara Zeta Function and cell products. Then we present our generalized zeta functions and prove some properties about them. Our hope is that the ideas presented in this paper will stimulate further ideas about using the nice properties of the Ihara Zeta Function as a model for defining zeta functions more generally on higher dimensional geometric objects.

## 1. "Primes" of a Graph

In this paper [1], we define a graph $X$ to be a finite set of vertices such that pairs of vertices may be connected by edges. We allow multiple (but finitely many) edges between two vertices and one or more (but finitely many) loops at a single vertex.

Each edge of $X$, including loops, can be given two orientations, because we consider travelling from vertex $v_i$ to vertex $v_j$ along an edge to be different from travelling along the same edge from vertex $v_j$ to vertex $v_i$. For loops, we need the analogous argument that a loop can be traversed in two different directions (*e.g.* graph $X_2$ in Section 3). So, given that $X$ has $m$ edges, it has $2m$ oriented edges. We shall call these oriented edges *arcs*. If $\alpha$ is an arc, then $\bar{\alpha}$ will denote its oppositely oriented partner, *i.e.* $\alpha$ and $\bar{\alpha}$ are the two possible orientations of a single edge in $X$. We shall refer to $\alpha$ and $\bar{\alpha}$ together as a pair of *conjugates*.

A *walk* in $X$ is a sequence of arcs such that every arc begins where the previous arc ends. A walk is *closed* (C) if the last arc of the walk ends where the first arc begins. [2]

A closed walk is

(i) *backtrack-less* (B) if, during the closed walk, an arc $\alpha$ is never immediately followed by $\bar{\alpha}$;

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[2]We do not consider empty sequences to be closed walks.

(ii) *tail-less* (T) if the first and last arcs of the closed walk are not conjugates, *i.e.* the closed walk does *not* look like $\alpha\beta ... \bar{\alpha}$;
(iii) *primitive* (P) if the closed walk is not the power of another closed walk, *i.e.* a primitive closed walk does *not* simply result from traversing another closed walk several times.

Let $\mathcal{S} = \{$the set of all backtrack-less, tail-less, primitive closed (BTPC) walks in $X\}$, and define an equivalence relation on the elements of $\mathcal{S}$ in the following manner: $s_1 \sim s_2$ if $s_2$ can be achieved by cyclically permuting the sequence of arcs of $s_1$. [3]

For example,

$$s_1 = \alpha\beta\gamma\delta \quad \sim \quad s_2 = \beta\gamma\delta\alpha \quad \sim \quad s_3 = \gamma\delta\alpha\beta \quad \sim \quad s_4 = \delta\alpha\beta\gamma.$$

*Primes* of a graph $X$ are defined to be the equivalence classes of this relation. The *degree* of a prime is defined to be the number of arcs traversed during any BTPC walk representing the prime.

## 2. Ihara Zeta Function of a Graph

In 1966, in a paper entitled "On discrete subgroups of the two by two projective linear group over $p$-adic fields," [3] Y. Ihara defined a zeta function for the discrete subgroups mentioned in the title of his paper. This zeta function definition was in the form of an Euler product. In the paper, Ihara proved a formula that equated the Euler product form with an expression in terms of related group operators. It is now known that Ihara's Euler product definition of the zeta function can be interpreted in terms of primes of a finite graph, and this fact is used to define a zeta function directly for finite graphs, hence the terminology "Ihara Zeta Function of a graph." Moreover, the formula proven by Ihara carries over to the graph interpretation. Below, we define the Ihara Zeta Function of a finite graph $X$ and state Ihara's Formula as it applies in this case.

*Definition.* Let $X$ be a finite graph. Then the Ihara Zeta Function of $X$ is

$$Z_X(u) = \prod_{\gamma \in \text{primes}(X)} \frac{1}{1 - u^{\deg \gamma}}, \tag{2.1}$$

where $\deg \gamma$ denotes the degree of the prime $\gamma$ as defined in Section 1.

*Ihara's Formula* [3]. For a finite graph $X$ the following equality holds,

$$Z_X(u) = \prod_{\gamma \in \text{primes}(X)} \frac{1}{1 - u^{\deg \gamma}} = \frac{(1-u^2)^{n-m}}{\det(I_n - uA_X + u^2 Q_X)}, \tag{2.2}$$

where $n$ is the number of vertices of $X$, and $m$ is the number of edges; $A_X$ is the adjacency matrix of $X$; $I_n$ is the $n \times n$ identity matrix; and $Q_X$ is the "degree

[3]Note, then, that $\alpha\beta\gamma\delta$ and $\bar{\alpha}\bar{\delta}\bar{\gamma}\bar{\beta}$ are not equivalent.

minus one" matrix $Q_X = (q_{ij})$ with $q_{ii} = \deg(v_i) - 1$, and $q_{ij} = 0$ for $i \neq$j. Note that $\deg(v_i)$ denotes the degree of the vertex $v_i$, which is defined as the number of edges incident with $v_i$ (loops counted twice). For example, for a graph with $n = 4$ vertices $Q_X$ would look like:

$$Q_X = \begin{bmatrix} \deg(v_1) - 1 & 0 & 0 & 0 \\ 0 & \deg(v_2) - 1 & 0 & 0 \\ 0 & 0 & \deg(v_3) - 1 & 0 \\ 0 & 0 & 0 & \deg(v_4) - 1 \end{bmatrix}.$$

For the remainder of this paper, the term "Ihara Zeta Function" will refer strictly to the definition given by (2.1) for a finite graph and its equivalent forms. In 1989, in a paper entitled "Zeta Functions of Finite Graphs and Representations of $p$-Adic Groups," [2] K. Hashimoto proved the existence of another expression for the Ihara Zeta Function in terms of an operator, $T_X$, which acts on the arcs of a finite graph $X$. In the next section, we describe this operator and some of its properties.

## 3. Definition of the $T$ Operator

For an arc $\alpha$ of a graph $X$, let $O(\alpha) = \{$arcs in $X$, excluding $\bar{\alpha}$, that flow out of $\alpha\}$. Here, an arc $\beta$ flows out of $\alpha$ if $\beta$ begins where $\alpha$ ends.

*Definition.* For a finite graph $X$ with $2m$ arcs, the operator $T_X$ acts on arcs of $X$ as follows,

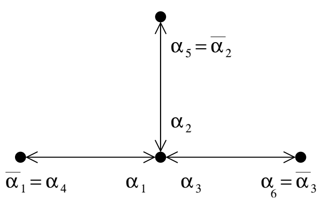
$$T_X(\alpha) = \sum_{\alpha' \in O(\alpha)} \alpha'. \tag{3.1}$$

If $W$ denotes the vector space (over $\mathbb{C}$) whose basis elements are the $2m$ arcs of $X$, then $T_X : W \to W$. Specifically, $T_X$ takes an arc of $X$ to a sum over its outflowing arcs, but does not allow backtracking. This definition is somewhat abstract, and it is easier to work with $T_X$ as a matrix. We can view $T_X$ as a $2m$ x $2m$ matrix, $T_X = (t_{ij})$, with

$$(t_{ij}) = \{1 \text{ if } \alpha_i \in O(\alpha_j);\ 0 \text{ otherwise}\}.$$

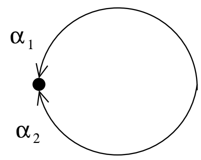
Below are four examples of graphs and their corresponding $T$ matrices. The labelled arrows on each graph represent the arcs of the graph.

$X_1$



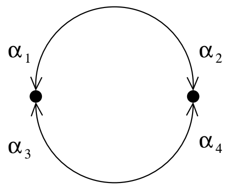

$$T_{X_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$X_2$



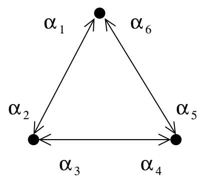

$$T_{X_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$X_3$




$$T_{X_3} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$X_4$




$$T_{X_4} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Viewing $T_X$ as a matrix makes its abstract definition easier to understand. Consider the graph $X_1$ above. From the definition of $T_X$, (3.1), we have $T_{X_1}(\alpha_1) = \alpha_6 + \alpha_5$. On the other hand, in the vector space spanned by the arcs of $X_1$, we can write $\alpha_1$ as the vector $(1, 0, 0, 0, 0, 0)$, since it is the first of six arcs. When we act (*i.e.* multiply) this vector by the $T_{X_1}$ matrix, we get the vector $(0, 0, 0, 0, 1, 1)$, which matches the abstract definition.

The trace of the $T$-matrix provides a way of counting the total number of BTC walks (primitive and non-primitive) in a graph. In particular, as the examples above suggest, $\mathrm{Tr}(T_{X_i})$ is the number of BTC walks of degree 1 (1 arc traversed) in $X_i$ (two for $X_2$ and zero for the others). Moreover, $\mathrm{Tr}[(T_{X_i})^n]$ is the number of BTC walks of degree $n$ in $X_i$. This idea from Hashimoto's paper is stated as a lemma below.

*Lemma* (*Hashimoto*)[2].

$$\mathrm{Tr}[(T_{X_i})^n] = \text{BTC walks of degree } n \text{ in } X_i. \tag{3.2}$$

We omit the proof and instead continue working with the examples above to illustrate the lemma. Consider, for example, squaring the $T$-matrices of the four graphs above:

$$(T_{X_1})^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \text{trace} = 0.$$

$$(T_{X_2})^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ trace} = 2.$$

$$(T_{X_3})^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ trace} = 4.$$

$$(T_{X_4})^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ trace} = 0.$$

$\mathrm{Tr}[(T_{X_1})^2] = \mathrm{Tr}[(T_{X_4})^2] = 0$, which corresponds to $X_1$ and $X_4$ having no BTC walks of degree 2. $\mathrm{Tr}[(T_{X_2})^2] = 2$, corresponding to $X_2$ having two BTC walks of degree 2. And $\mathrm{Tr}[(T_{X_3})^2] = 4$, corresponding to $X_3$ having four BTC walks of degree 2. Note that $\mathrm{Tr}[T^n]$ does not consider equivalence of BTC walks under cyclic permutation of constituent arcs. Thus, the degree-2 BTC walk $\alpha_1\alpha_4$ in $X_3$ is counted once, as is the degree-2 BTC walk $\alpha_4\alpha_1$.

We now state (and quickly summarize the proof of) a result from Hashimoto's paper. The proof utilizes the observations made above about the $T$-operator. We will use the result in our own proofs in Sections 11 and 14.

*Proposition (Hashimoto)*[2]. Let $\gamma \in \text{primes}(X)$. Then,

$$Tr[(T_X)^n] = \sum_{(\deg\gamma)|n} \deg\gamma, \tag{3.3}$$

where the sum is over all $\gamma \in \text{primes}(X)$ whose degree divides $n$.

*Proof.* From (3.2), $Tr[(T_X)^n]$ counts the number of degree-$n$ BTC walks in $X$. Every degree-$n$ BTC walk can be written uniquely as a power (perhaps 1) of a BTPC walk representing a prime, so to count degree-$n$ BTC walks it is sufficient to consider the primes of $X$ whose degrees divide $n$. For each such prime, $\gamma$, there exist $\deg\gamma$ BTC walks of degree $n$, one starting at each constituent arc of $\gamma$.

## 4. Three Expressions for the Ihara Zeta Function

In this section, we state Hashimoto's Formula relating the Ihara Zeta Function to the $T$ operator, and then restate for emphasis the three equivalent expressions for the Ihara Zeta Function.

*Hashimoto's Formula* [2]. For a finite graph $X$ the following equality holds,

$$Z_X(u) = \prod_{\gamma \in \text{primes}(X)} \frac{1}{1 - u^{\deg \gamma}} = \frac{1}{\det(I_{2m} - uT_X)}, \tag{4.1}$$

where $m$ is again the number of edges of $X$, and $I_{2m}$ is the $2m \times 2m$ identity matrix.

Therefore, the Zeta Function of a graph has the following equivalent expressions:

(4.2)
$$Z_X(u) = \prod_{\gamma \in \text{primes}(X)} \frac{1}{1 - u^{\deg \gamma}} = \frac{(1-u^2)^{n-m}}{\det(I_n - uA_X + u^2 Q_X)} = \frac{1}{\det(I_{2m} - uT_X)}.$$

So the Ihara Zeta Function has a form related to its primes, a form related to its vertices, and a form related to its arcs.

## 5. Example: Determining the Zeta Function of a Graph

Let $X$ be the graph $X_3$ from Section 3. First, we find $Z_X(u)$ using the Euler product expression. $X$ has two primes, represented by the BTPC walks $\alpha_1\alpha_4$ and $\alpha_2\alpha_3$. Each of these primes has degree 2. So,

$$Z_X(u) = \prod_{\gamma \in \text{primes}(X)} \frac{1}{1 - u^{\deg \gamma}} = (\frac{1}{1-u^2})(\frac{1}{1-u^2}) = \frac{1}{u^4 - 2u^2 + 1}. \tag{5.1}$$

Next, we find $Z_X(u)$ using the vertex-operator expression and show that we get the same result. $X$ has $n = 2$ vertices, and $m = 2$ edges. Also, we have

$$A_X = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

$$Q_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$I_2 - uA_X + u^2 Q_X = \begin{bmatrix} 1+u^2 & -2u \\ -2u & 1+u^2 \end{bmatrix}.$$

So from the vertex-operator form, we get

$$Z_X(u) = \frac{(1-u^2)^{2-2}}{\det(I_2 - uA_X + u^2 Q_X)} = \frac{1}{(1+u^2)^2 - 4u^2} = \frac{1}{u^4 - 2u^2 + 1}. \tag{5.2}$$

Finally, we find $Z_X(u)$ using Hashimoto's $T$-operator expression. From Section 3,

$$T_X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$I_4 - uT_X = \begin{bmatrix} 1 & 0 & 0 & -u \\ 0 & 1 & -u & 0 \\ 0 & -u & 1 & 0 \\ -u & 0 & 0 & 1 \end{bmatrix}.$$

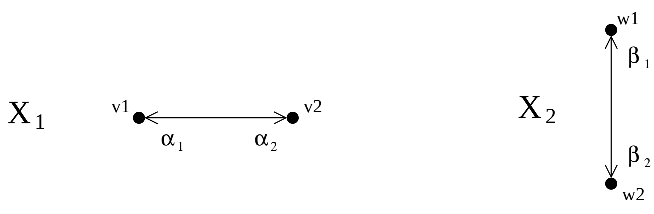
So from Hashimoto's Formula we get

$$Z_X(u) = \frac{1}{\det(I_4 - uT_X)} = \frac{1}{u^4 - 2u^2 + 1}. \tag{5.3}$$

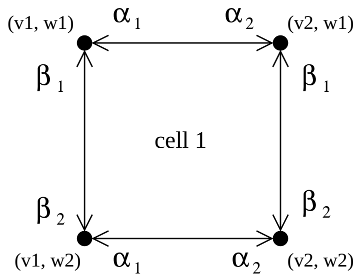
## 6. Cell Product of Graphs

In this paper, we deal with two different ways of taking products of graphs. In Section 9, we briefly discuss the Kronecker product, also called the tensor or graph-theoretic product. The Kronecker product of two graphs is another graph. However, in this section and in Sections 5-8, we consider a topological product called the cell product: Start with two graphs $X_1$ and $X_2$ considered as 1-dimensional simplicial complexes. One way to think of this is to regard $X_i$, $i = (1, 2)$, as a subset of Euclidean 3-space, $\mathbb{R}^3$. Then, edges between vertices are taken as actual curves in $\mathbb{R}^3$ (for the moment, we are not worrying about arcs). So $X_i$ inherits a topology from $\mathbb{R}^3$, which makes it into a 1-dimensional simplicial complex. The cartesian product $X_1 \times X_2$ of these 1-dimensional simplicial complexes now yields a cell complex of dimension 2. In other words, pairs of edges, one from $X_1$ and one from $X_2$, "multiply" together to yield a two-dimensional cell in $\mathbb{R}^3$. Thus if $X_1$ has $m_1$ edges and $X_2$ has $m_2$ edges, the cell product will consist of $m_1 m_2$ cells (how these cells "glue" together is discussed below). We call this the cell product. Since pairs of edges, one from each original graph, yield 2-dimensional cells, the cell product of two graphs is not a graph, but rather a 2-dimensional surface. Below are two examples showing the cell product of two graphs, $X_1$ and $X_2$. The first example is simple. Here, both $X_1$ and $X_2$ have exactly one edge, so the cell product consists of only one cell. The second example is more complicated. In that example, $X_1$ has two edges while $X_2$ has one, so the cell product consists of two cells, and we must ascertain how the cells should glue together.

Example 1.

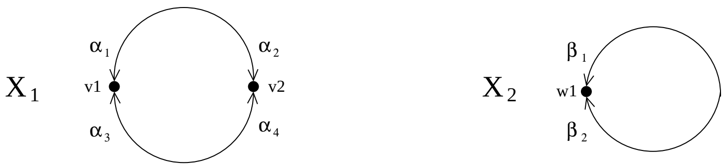


Cell Product:

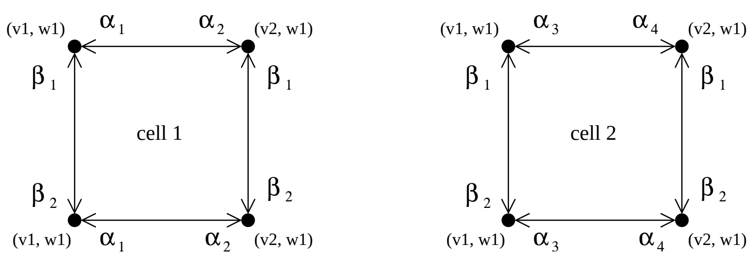



Notice that we have taken the effort to label the vertices and arcs of $X_1$ and $X_2$ and then to relabel them on the boundary of the 2-dimensional cell. To be more specific, then, we should say that for the cell product, pairs of *labelled* edges, one from $X_1$ and one from $X_2$, yield *labelled cells*. The reason behind keeping track of vertex and arc labels is perhaps not apparent at the moment, but as the following example will illustrate, they show us whether and how cells must be glued together when there are more than one of them.

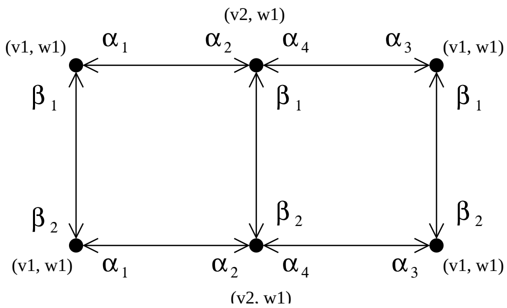
Example 2.




Cells:




Having kept track of the labels, we can now determine whether and how cells 1 and 2 should be glued together. By matching vertex and arc labels, we see, for instance, that the right vertical side of cell 1 should be glued to the right vertical side of cell 2. This gives us a rectangle,




Continuing this matching procedure, we see that the leftmost and rightmost vertical sides of the rectangle match up, as do top and bottom horizontal ones. We conclude that the cell product of $X_1$ and $X_2$ is the (labelled) surface of a torus. Looking back at Example 1, we see that no gluing took place there because the four vertices of cell 1 had distinct labels.

## 7. Outline of Results

It is the cell product of two graphs $X_1$ and $X_2$ for which we wish to define a zeta function. We would like our zeta function to be a generalization of the Ihara Zeta Function in the sense that we would like it to possess an Euler product form, a generalized vertex-operator form, and a generalized arc-operator form. In the following sections, we propose two zeta functions for cell products, $Z_{X_1,X_2}(u)$ and $\tilde{Z}_{X_1,X_2}(u)$. The motivation for the first zeta function, $Z_{X_1,X_2}(u)$, comes from a paper written in 1992 by H. Bass entitled "The Ihara-Selberg Zeta Function of a Tree Lattice" [1]. In a segment of this paper, Bass provides a proof for the equality,

$$\frac{(1-u^2)^{n-m}}{\det(I_n - uA_X + u^2 Q_X)} = \frac{1}{\det(I_{2m} - uT_X)}, \tag{7.1}$$

which holds in the case of a single graph. We stated this equality in (4.2). The equality there followed transitively from (2.2) and (4.1). Bass supplies a more direct proof of equality via a consideration of operators on graphs. In Section 8 we define some of Bass' operators on single graphs, and in Section 9 we generalize those operators for cell products. This enables us to define $Z_{X_1,X_2}(u)$ in Section 10 so that a generalized version of (7.1) holds. This gives us generalized vertex-operator and arc-operator forms for $Z_{X_1,X_2}(u)$. In Section 11 we then show that an Euler product form exists for $Z_{X_1,X_2}(u)$ if we define primes on a cell product in a certain way. Ultimately, as we will discuss in Section 13, it turns out that $Z_{X_1,X_2}(u)$ is precisely the Ihara Zeta Function for the Kronecker product of $X_1$ and $X_2$, which gives us a second interpretation for our Euler product.

In Section 14 we present a second possibility for a zeta function for cell products, $\tilde{Z}_{X_1,X_2}(u)$. This possibility was proposed by J. Hoffman. Given a generalized arc-operator definition for $\tilde{Z}_{X_1,X_2}(u)$, we show that it has an Euler product form in terms of primes of $X_1$ and $X_2$. However, we were unable to find a generalized vertex-operator form for $\tilde{Z}_{X_1,X_2}(u)$. Also, the relationship between $\tilde{Z}_{X_1,X_2}(u)$ and

$Z_{X_1,X_2}(u)$ remains to be explored.

We turn now to operators on graphs and cell products of graphs, which will help us to define $Z_{X_1,X_2}(u)$.

## 8. The $\overset{*}{T}$, J, and D Operators

For an arc $\alpha$ of a graph $X$, let $\overset{*}{O}(\alpha) = \{$arcs in $X$, including $\bar{\alpha}$, that flow out of $\alpha\}$.

*Definition.* For a finite graph $X$ with $2m$ arcs, $\overset{*}{T}_X$ acts on arcs of $X$ as follows,

$$\overset{*}{T}_X(\alpha) = \sum_{\alpha' \in \overset{*}{O}(\alpha)} \alpha'. \tag{8.1}$$

$\overset{*}{T}_X$ closely resembles $T_X$. $\overset{*}{T}_X$ takes an arc of $X$ to a sum over its outflowing arcs, but unlike $T_X$, $\overset{*}{T}_X$ *allows* backtracking. We can view $\overset{*}{T}$ as a $2m$ x $2m$ matrix too, $\overset{*}{T}_X = (\overset{*}{t}_{ij})$, now with

$$(\overset{*}{t}_{ij}) = \{1 \text{ if } \alpha_i \in \overset{*}{O}(\alpha_j);\ 0 \text{ otherwise}\}.$$

For example, for the graph $X_3$ from Section 3,

$$\overset{*}{T}_X = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The J and D operators are more straightforward to define.

*Definition.* For a finite graph $X$ with $2m$ arcs, $J_X$ acts on arcs of $X$ as follows,

$$J_X(\alpha) = \bar{\alpha}. \tag{8.2}$$

We can view $J_X$ as a $2m$ x $2m$ matrix, $J = (j_{ik})$, with

$$(j_{ik}) = \{1 \text{ if } \alpha_i = \bar{\alpha}_k;\ 0 \text{ otherwise}\}.$$

For example, for the graph $X_3$ from Section 3,

$$J_X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

*Definition.* For a finite graph $X$ with $n$ vertices, the operator $D_X$ acts on vertices of $X$ as follows,

$$D_X(v) = (\deg v) \cdot v. \tag{8.3}$$

We can view $D_X$ as an $n$ x $n$ matrix; it is simply $Q_X + I$. Recall that $Q_X$ was defined in Section 2. For example, for the graph $X_3$ from Section 3,

$$D_X = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

## 9. Operators on Cell Products of Graphs

Let $X_1$ and $X_2$ be two finite graphs with respective vertex sets $V_1$ and $V_2$, arc sets $R_1$ and $R_2$, adjacency matrices $A_1$ and $A_2$, and so on. Define $C_0 \equiv \mathbb{C} \cdot (V_1 \times V_2)$. $C_0$ is the vector space over $\mathbb{C}$ with basis $V_1 \times V_2$, the standard cartesian product of $V_1$ and $V_2$. So, the basis elements of $C_0$ are all pairs of vertices $(v, w)$ such that $v \in V_1$ and $w \in V_2$. Similarly, define $C_1 \equiv \mathbb{C} \cdot (R_1 \times R_2)$. $C_1$ is the vector space over $\mathbb{C}$ with basis $R_1 \times R_2$. So, the basis elements of $C_1$ are all pairs of arcs $(\alpha, \beta)$ such that $\alpha \in R_1$ and $\beta \in R_2$.

Define the following operators on cell products of graphs:

(i) $\partial_0, \partial_1 : C_1 \longrightarrow C_0$,
$\partial_0(\alpha, \beta) = (\text{tail vertex of } \alpha, \text{tail vertex of } \beta)$,
$\partial_1(\alpha, \beta) = (\text{head vertex of } \alpha, \text{head vertex of } \beta)$.

Example. For the cell product in Example 1 of Section 6,
$\partial_0(\alpha_1, \beta_1) = (v_2, w_2)$,
$\partial_1(\alpha_1, \beta_1) = (v_1, w_1)$.

(ii) $\sigma_0, \sigma_1 : C_0 \longrightarrow C_1$,
$\sigma_0(v, w) = \sum_\alpha \sum_\beta (\alpha, \beta)$, where the sum is over all $\alpha$ and $\beta$ such that $\alpha$ flows out of $v$ and $\beta$ flows out of $w$,
$\sigma_1(v, w) = \sum_\alpha \sum_\beta (\alpha, \beta)$, where the sum is over all $\alpha$ and $\beta$ such that $\alpha$ flows into $v$ and $\beta$ flows into $w$.

Example. For the cell product in Example 2 of Section 6,
$\sigma_0(v1, w1) = (\alpha_2, \beta_1) + (\alpha_2, \beta_2) + (\alpha_4, \beta_1) + (\alpha_4, \beta_2)$,
$\sigma_1(v1, w1) = (\alpha_1, \beta_1) + (\alpha_1, \beta_2) + (\alpha_3, \beta_1) + (\alpha_3, \beta_2)$.

The following identities follow immediately from the definitions above:

(iii) $\partial_1\sigma_0,\ \partial_0\sigma_1 : C_0 \longrightarrow C_0$.
$\partial_1\sigma_0(v, w) = \partial_0\sigma_1(v, w) = \sum_{v' \in V_1} \sum_{w' \in V_2} [M(v', w')] \cdot (v', w')$,

where $[M(v', w')]$ is equal to (the number of arcs from $v$ to $v'$ in $X_1$) times (the number of arcs from $w$ to $w'$ in $X_2$). Therefore, in the vector space

$C_0$, $\partial_1\sigma_0$ is simply the matrix $A_1 \otimes A_2$, the standard Kronecker product of the adjacency matrices of $X_1$ and $X_2$. Therefore, we write the identity as:

$$\partial_1\sigma_0 = \partial_0\sigma_1 = A_1 \otimes A_2.$$

Example. For the cell product in Example 2, Section 6, we found, in (ii):
$\sigma_0(v_1, w_1) = (\alpha_2, \beta_1) + (\alpha_2, \beta_2) + (\alpha_4, \beta_1) + (\alpha_4, \beta_2)$. So,
$\partial_1\sigma_0(v_1, w_1) = \partial_1(\alpha_2, \beta_1) + \partial_1(\alpha_2, \beta_2) + \partial_1(\alpha_4, \beta_1) + \partial_1(\alpha_4, \beta_2) = 4 \cdot (v_1, w_1)$.
Meanwhile,

$$A_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} 2 \end{bmatrix}.$$

$$A_1 \otimes A_2 = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}.$$

Now, $C_0$ is spanned by $(v_1, w_1)$ and $(v_2, w_1)$, which are denoted by the vectors $(1, 0)$ and $(0, 1)$, and $(A_1 \otimes A_2)$ acting on $(1, 0)$ gives $4 \cdot (0, 1)$, which matches the action of $\partial_1\sigma_0$.

(iv) $\partial_0\sigma_0$, $\partial_1\sigma_1 : C_0 \longrightarrow C_0$.
$\partial_0\sigma_0(v, w) = \partial_1\sigma_1(v, w) = [\deg(v) \cdot \deg(w)] \cdot (v, w)$. In the vector space $C_0$, then, $\partial_0\sigma_0$ is simply the matrix $D_1 \otimes D_2$. We write the identity as:

$$\partial_0\sigma_0 = \partial_1\sigma_1 = D_1 \otimes D_2.$$

(v) $\sigma_0\partial_1 : C_1 \longrightarrow C_1$.
$\sigma_0\partial_1(\alpha, \beta) = \sum_{\alpha' \in \overset{*}{O}(\alpha)} \sum_{\beta' \in \overset{*}{O}(\beta)} (\alpha', \beta')$.

In the vector space $C_1$, then, $\sigma_0\partial_1$ is the matrix $\overset{*}{T}_1 \otimes \overset{*}{T}_2$, so we write the identity as:

$$\sigma_0\partial_1 = \overset{*}{T}_1 \otimes \overset{*}{T}_2\ .$$

(vi) $\sigma_0\partial_0$, $\sigma_0\partial_1 : C_1 \longrightarrow C_1$.
$\sigma_0\partial_0(\alpha, \beta) = \sigma_0\partial_1(\bar{\alpha}, \bar{\beta}) = \sum_{\alpha' \in \overset{*}{O}(\bar{\alpha})} \sum_{\beta' \in \overset{*}{O}(\bar{\beta})} (\alpha', \beta')$.

In the vector space $C_1$, $\sigma_0\partial_0$ is the matrix $(\overset{*}{T}_1 \otimes \overset{*}{T}_2)(J_1 \otimes J_2)$. This is because $(J_1 \otimes J_2)$ takes $(\alpha, \beta)$ to $(\bar{\alpha}, \bar{\beta})$, and then $(\overset{*}{T}_1 \otimes \overset{*}{T}_2)$ acts on $(\bar{\alpha}, \bar{\beta})$ as in (v). We write the identity as:

$$\sigma_0\partial_0 = (\overset{*}{T}_1 \otimes \overset{*}{T}_2)(J_1 \otimes J_2).$$

For notational simplification in the upcoming section, we list a few more identities, which are direct consequences of (iii)-(vi). In the following identities, $u$ is a (scalar) variable:

(vii) Define $\partial(u) \equiv \partial_0 u - \partial_1$. Then $\partial(u)\sigma_0 = (D_1 \otimes D_2)u - A_1 \otimes A_2$.

(viii) Define $\sigma(u) \equiv \sigma_0 u$. Define $\triangle(u) \equiv I_0 - (A_1 \otimes A_2)u + (D_1 \otimes D_2 - I_0)u^2$. Here, $I_0$ is the identity operator on $C_0$. Then $[\partial(u)][\sigma(u)] = \triangle(u) - (1-u^2)I_0$.

(ix) $\sigma_0\partial(u) = [\overset{*}{T}_1 \otimes \overset{*}{T}_2][(J_1 \otimes J_2)u - I_1]$. Here, $I_1$ is the identity operator on $C_1$.

## 10. A Zeta Function for Cell Products

*Definition.* Let $X_1$ and $X_2$ be two finite graphs with $\overset{*}{T}$-matrices $\overset{*}{T}_1$ and $\overset{*}{T}_2$ and $J$-matrices $J_1$ and $J_2$, respectively. Let $I_1$ be the identity operator on $C_1$ as described in Section 9 (ix). Then define a zeta function for the cell product of the graphs as follows,

$$Z_{X_1,X_2}(u) = \frac{1}{\det[I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)u]}. \tag{10.1}$$

Note that $(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)$ is not the same as $(T_1 \otimes T_2)$. Both take $(\alpha, \beta) \in C_1$ to a sum over outflowing pairs $(\alpha', \beta')$. However, $(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)$ only disallows simultaneous backtracking in $\alpha$ and $\beta$, whereas $(T_1 \otimes T_2)$ disallows any backtracking in either $\alpha$ or $\beta$. In fact, our alternative definition for a zeta function for cell products, which we define in Section 14, will use $(T_1 \otimes T_2)$.

Our definition, (10.1), is a generalized arc-operator expression analogous to the RHS of (4.2) for the Ihara Zeta Function. Unlike the Ihara Zeta Function, though, which was defined initially as an Euler product, we define our zeta function as above and proceed to look for a generalized vertex-operator and an Euler product form. We start with a generalization of (7.1) to achieve a generalized vertex-operator form.

*Proposition.* Let $X_1$ and $X_2$ be two finite graphs with adjacency matrices $A_1$ and $A_2$ and degree matrices $D_1$ and $D_2$, respectively. Define $C_0$ and $C_1$ as in Section 9, and let $r_0 = \dim(C_0)$ and $2r_1 = \dim(C_1)$, where dim denotes the dimension of the vector space (*i.e*, the number of basis elements). As before, let $I_0$ and $I_1$ be the identity operators on $C_0$ and $C_1$, respectively. Then,

(10.2)
$$Z_{X_1,X_2}(u) = \frac{1}{\det[I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)u]} = \frac{(1-u^2)^{r_0 - r_1}}{\det[I_0 - (A_1 \otimes A_2)u + (D_1 \otimes D_2 - I_0)u^2]}.$$

*Proof.* The proof consists of five steps and follows directly from the single-graph version carried out by Bass [1].

(i) For finite graphs $X_1$ and $X_2$, let
$C = C_0 \oplus C_1$. $C$ is the vector space with basis consisting of all $(v, w) \in C_0$ and all $(\alpha, \beta) \in C_1$.
$E_i = \mathrm{End}(C_i), (i = 0, 1)$. $E_i$ is the collection of maps that take $C_i \to C_i$. For instance, $\partial_1\sigma_0 : C_0 \to C_0$ (see Section 9 (iii)) belongs to $E_0$.
$H_i = \mathrm{Hom}(C_i, C_{1-i}), (i = 0, 1)$. $H_i$ is the collection of maps that take $C_i \to C_{1-i}$. For instance, $\partial_0 : C_1 \to C_0$ (see Section 9 (i)) belongs to $H_1$.
$E = \mathrm{End}(C) = \begin{bmatrix} E_0 & H_1 \\ H_0 & E_1 \end{bmatrix}$. $E$ is the collection of maps that take $C \to C$.
For instance, $\begin{bmatrix} I_0 & \partial_0 \\ 0 & I_1 \end{bmatrix}$. belongs to $E$.

(ii) Let $E[u]$ denote polynomials in the variable $u$ whose coefficients are matrices that belong to $E$. Define $L$, $M \in E[u]$ to be

$$L = \begin{bmatrix} (1-u^2)I_0 & \partial(u) \\ 0 & I_1 \end{bmatrix} = \begin{bmatrix} I_0 & -\partial_1 \\ 0 & I_1 \end{bmatrix} + \begin{bmatrix} 0 & \partial_0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} -I_0 & 0 \\ 0 & 0 \end{bmatrix} u^2,$$

$$M = \begin{bmatrix} I_0 & -\partial(u) \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix} = \begin{bmatrix} I_0 & \partial_1 \\ 0 & I_1 \end{bmatrix} + \begin{bmatrix} 0 & -\partial_0 \\ \sigma_0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & -I_1 \end{bmatrix} u^2.$$

Using the operator identities of Section 9, we find

$$LM = \begin{bmatrix} \triangle(u) & 0 \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix},$$

$$ML = \begin{bmatrix} (1-u^2)I_0 & 0 \\ \sigma(u)(1-u^2) & [(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)u - I_1][(J_1 \otimes J_2)u - I_1] \end{bmatrix}.$$

(iii) We now claim that $LM$ and $ML$ have the same determinant. Formally, $ML = M(LM)M^{-1}$, and so to show that $LM$ and $ML$ have the same determinant, we just need to check that $M$ is in fact invertible. For this, we need a lemma from Bass' paper, which we state without proof:

*Lemma* [1]. Let $P$ be a polynomial in the variable $u$ with coefficients that are square matrices. If the constant matrix of $P$ (no $u$) is invertible, then $P$ is invertible.

The constant matrix of $M$ is

$$\begin{bmatrix} I_0 & \partial_1 \\ 0 & I_1 \end{bmatrix}, \text{whose inverse is simply} \begin{bmatrix} I_0 & -\partial_1 \\ 0 & I_1 \end{bmatrix}.$$

Therefore, $M$ is indeed invertible, and we conclude that $LM$ and $ML$ have the same determinant.

(iv) Calculate the determinants:

$$\begin{aligned}&\det(LM)\\ &= \det(\triangle(u)) \cdot \det((1-u^2)I_1)\\ &= (1-u^2)^{2r_1} \cdot \det(\triangle(u)).\end{aligned}$$

And this is equal to

$$\begin{aligned}&\det(ML)\\ &= \det[(1-u^2)I_0] \cdot \det[(I_1 - [\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2]u)(I_1 - [J_1 \otimes J_2]u)]\\ &= (1-u^2)^{r_0} \cdot \det[I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)u] \cdot \det[I_1 - (J_1 \otimes J_2)u]\end{aligned}$$

(v) The proof will be complete once we show that $\det[I_1 - (J_1 \otimes J_2)u] = (1-u^2)^{r_1}$. This can be seen as follows. Labelling the basis elements of $C_1$ appropriately, we can write

$$I_1 - (J_1 \otimes J_2)u = \begin{bmatrix} I_2 & -I_2u \\ -I_2u & I_2 \end{bmatrix}, \tag{10.3}$$

where $I_2$ is the $r_1 \times r_1$ identity matrix (recall that $2r_1 = \dim(C_1)$).

To see why this is true, it is easiest to consider an example. Suppose $C_1$ has only four basis elements: $e_1 = (\alpha, \beta), e_2 = (\alpha, \bar{\beta}), e_3 = (\bar{\alpha}, \bar{\beta}), e_4 = (\bar{\alpha}, \beta)$. Then, $I_1$ maps $e_1 \to e_1, e_2 \to e_2, e_3 \to e_3, e_4 \to e_4$. Meanwhile, recalling the definition of the $J$-matrix, (8.2), we see that $(J_1 \otimes J_2)$ takes $e_1 \to e_3, e_2 \to e_4, e_3 \to e_1, e_4 \to e_2$. Therefore, with this basis, we have as matrices

$$I_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ while } J_1 \otimes J_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

So in accordance with (10.3) we have

$$I_1 - (J_1 \otimes J_2)u = \begin{bmatrix} 1 & 0 & -u & 0 \\ 0 & 1 & 0 & -u \\ -u & 0 & 1 & 0 \\ 0 & -u & 0 & 1 \end{bmatrix}.$$

Of course, we could have labelled the basis elements differently. For example, we could have let $e_1 = (\alpha, \bar{\beta}), e_2 = (\bar{\alpha}, \bar{\beta}), e_3 = (\alpha, \beta), e_4 = (\bar{\alpha}, \beta)$, and then $(I_1 - J_1 \otimes J_2)$ would not match the form in (10.3). By "labelling the basis elements of $C_1$ appropriately," then, we mean assigning the $e_i$'s so that 1's appear in the $I_1$ and $(J_1 \otimes J_2)$ matrices where we want them to appear. From basic linear algebra, we know that reassigning the $e_i$'s might change the matrix $(I_1 - J_1 \otimes J_2)$, but not its determinant. And since we only care about the determinant, we are justified in working exclusively with the "appropriately labelled" basis. When $C_1$ has more basis elements, it is easy to see how, keeping the actions of $I_1$ and $J_1 \otimes J_2$ in mind, we can label the basis elements so that $I_1 - J_1 \otimes J_2$ matches (10.3).

So $\det[I_1 - J_1 \otimes J_2] = \det(z)$, where

$$z = \begin{bmatrix} I_2 & -I_2 u \\ -I_2 u & I_2 \end{bmatrix}.$$

In calculating $\det(z)$, it helps to first multiply $z$ by $y$, where

$$y = \begin{bmatrix} I_2 & I_2 u \\ 0 & I_2 \end{bmatrix}.$$

Note that $\det(y) = 1$, so $\det(yz) = \det(z)$. In particular,

$$yz = \begin{bmatrix} (1-u^2)I_2 & 0 \\ -I_2 u & I_2 \end{bmatrix}.$$

So, $\det[I_1 - (J_1 \otimes J_2)u] = \det(z)\det(yz) = \det[(1-u^2)I_2] = (1-u^2)^{r_1}$, which completes the proof.

## 11. An Euler Product Form

Referring to the definitions of the $\overset{*}{T}$ and $J$ operators in Section 8, we see that the operator $(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)$, which appears in (10.1), acts on $(\alpha, \beta) \in C_1$ in the following manner:

$$(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)(\alpha, \beta) = [\sum (\alpha', \beta')] - (\bar{\alpha}, \bar{\beta}), \tag{11.1}$$

where the sum is over all $(\alpha', \beta') \in C_1$ such that $\alpha'$ flows out of $\alpha$, $\beta'$ flows out of $\beta$, and backtracking is allowed in both $\alpha$ and $\beta$. Now recall that $\alpha$ and its conjugate, $\bar{\alpha}$, represent a unique edge, $e$, in $X_1$, while $\beta$ and $\bar{\beta}$ represent a unique edge, $f$, in $X_2$. Recall also that there exists a unique cell in the cell product of $X_1$ and $X_2$ that results from "multiplying" $e$ from $X_1$ with $f$ from $X_2$. Having retained arc labels on the boundary of this cell, we can view $(\alpha, \beta)$ as a directed diagonal of the cell. For instance, consider Figure A below (we omit vertex labels in these examples).

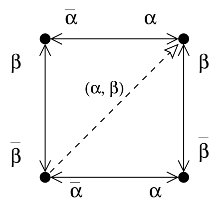



FIGURE A

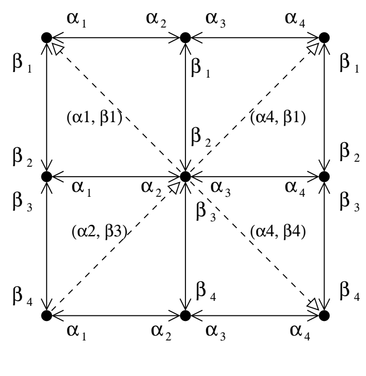



FIGURE B

Interpreting (11.1) geometrically, we see that $(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)$ takes such a directed diagonal to a sum over all outflowing directed diagonals on the cell product. So, for the example in Figure B,

$$(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)(\alpha_2, \beta_3) = (\alpha_1, \beta_1) + (\alpha_4, \beta_1) + (\alpha_4, \beta_4).$$

This suggests that we define a walk along the cell product of two graphs to be a sequence of directed diagonals such that every directed diagonal begins where the previous one ends. Then, definitions analogous to the ones in Section 1 follow. A walk is *closed* if the last directed diagonal ends where the first one begins. *Backtracking* occurs when the directed diagonal $(\alpha, \beta)$ is followed immediately by $(\bar{\alpha}, \bar{\beta})$, its *conjugate*. A *tail* occurs when the first and last directed diagonals of a closed walk are conjugates. A closed walk fails to be *primitive* if it simply results from traversing another closed walk several times.

Define $\mathcal{S} = \{$BTPC walks on the cell product$\}$. [4] Then, exactly as was done for the case of single graphs in Section 1, we can introduce an equivalence relation among the walks in $\mathcal{S}$ based on cyclic permutation of constituent directed diagonals and let primes on the cell product be the equivalence classes of this relation, with the degree of a prime defined to be the number of directed diagonals traversed by a BTPC walk representing the prime. With this definition of primes (we will refer to them below as primes on cells), we can formulate an Euler product form for $Z_{X_1,X_2}(u)$.

*Proposition.* $Z_{X_1,X_2}(u)$ has the following Euler product expansion,

$$Z_{X_1,X_2}(u) = \frac{1}{\det[I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)u]} = \prod_{\xi \in \text{primes}(cells)} \frac{1}{1 - u^{\deg \xi}}. \tag{11.2}$$

*Proof.* From the RHS of (10.2), we see that $Z_{X_1,X_2}(u)$ is equal to 1 divided by a polynomial in $u$. This polynomial has constant term 1, and so the power series expansion in the variable $u$ of the LHS of (11.2) has constant term 1. Similarly, each term in the RHS product of (11.2) is 1 over a polynomial with constant term 1, so the power series expansion in the variable u of the RHS of (11.2) also has constant term 1. Since the power series expansions of both the RHS and LHS of (11.2) have 1 as the constant term, to show equality between the two expressions, it suffices to show that taking $u\dfrac{d}{du}\log$ of both sides yields equivalent expressions. Starting with the RHS, we get

$$u\frac{d}{du}\log \prod_{\xi \in \text{primes}(cells)} \frac{1}{1 - u^{\deg \xi}}$$

$= -u\frac{d}{du}\sum_{\xi \in \text{primes}(cells)} \log(1 - u^{\deg \xi})$

[4] Recall that BTPC refers to walks that are backtrack-less, tail-less, primitive, and closed.

$= \sum_{\xi \in \text{primes}(cells)} \dfrac{(\deg \xi) u^{\deg \xi}}{1 - u^{\deg \xi}}$

$= \sum_{\xi \in \text{primes}(cells)} \sum_{s=1}^{\infty} (\deg \xi) u^{s \cdot \deg \xi}$

Making the substitution $n = s \cdot \deg \xi$, the sum becomes

$= \sum_{n=1}^{\infty} (\sum_{\xi \in \text{primes}(cells),\, \deg \xi | n} \deg \xi)\, u^n.$

Meanwhile, taking $u \dfrac{d}{du} \log$ of the LHS yields (via a standard result of linear algebra for square matrices like $(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)$; for instance, see Hashimoto [2]):

$$u \frac{d}{du} \log \frac{1}{\det[I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2) u]}$$

$= \sum_{n=1}^{\infty} \operatorname{Tr}[(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)^n]\, u^n.$

To show that equality holds in (11.2), then, we need only show equality between $(\sum_{\xi \in \text{primes}(cells),\, \deg \xi | n} \deg \xi)$ and $\operatorname{Tr}[(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)^n]$ so that the coefficients of the derived RHS and LHS expressions are equal. So we ask,

$$\operatorname{Tr}[(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)^n] \overset{?}{=} \sum_{\deg \xi | n} \deg \xi, \tag{11.3}$$

where the sum is over all $\xi \in$ primes($cells$) whose degree divides $n$. To complete the proof, we need to show that (11.3) is true. To this end, recall that a walk along the cell product of two graphs was defined to be a sequence of directed diagonals on the cell product such that each directed diagonal begins where the previous one ends. While the cells themselves are 2-dimensional, the diagonals across them are 1-dimensional, so we can consider the 1-dimensional graph, call it $X_d$, formed by the directed diagonals. Then, we see that $(\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)$, which takes a directed diagonal to a sum over outflowing directed diagonals without allowing backtracking, is precisely the $T$-operator on $X_d$. Consequently, (11.3) follows from Hashimoto's results, (3.2) and (3.3), in 1-dimension. This completes the proof.

Therefore, our zeta function has the following equivalent expressions,

(11.4)

$$Z_{X_1,X_2}(u) = \frac{1}{\det[I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2) u]} = \frac{(1-u^2)^{r_0 - r_1}}{\det[I_0 - (A_1 \otimes A_2) u + (D_1 \otimes D_2 - I_0) u^2]}$$

$$= \prod_{\xi \in \text{primes}(cells)} \frac{1}{1 - u^{\deg \xi}}.$$

## 12. Example: Determining the Zeta Function of a Cell Product

Let $X_1$ and $X_2$ be the 1-dimensional graphs in Example 2 of Section 6. First, we find $Z_{X_1,X_2}(u)$ using the arc-operator form in (11.4). Recalling the definitions of the $\overset{*}{T}$ and $J$ operators from Section 8, we have

$$\overset{*}{T}_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \text{ and } J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\overset{*}{T}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and } J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This gives

$$I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)u = \begin{bmatrix} 1 & -u & 0 & -u & 0 & 0 & 0 & -u \\ -u & 1 & -u & 0 & 0 & 0 & -u & 0 \\ 0 & -u & 1 & -u & 0 & -u & 0 & 0 \\ -u & 0 & -u & 1 & -u & 0 & 0 & 0 \\ 0 & 0 & 0 & -u & 1 & -u & 0 & -u \\ 0 & 0 & -u & 0 & -u & 1 & -u & 0 \\ 0 & -u & 0 & 0 & 0 & -u & 1 & -u \\ -u & 0 & 0 & 0 & -u & 0 & -u & 1 \end{bmatrix}.$$

And so we have

$$Z_{X_1,X_2}(u) = \frac{1}{\det[I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)u]} = \frac{1}{9u^8 - 28u^6 + 30u^4 - 12u^2 + 1}.$$

Second, we find $Z_{X_1,X_2}(u)$ using the vertex-operator form in (11.4). We have

$$A_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \text{ and } D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} 2 \end{bmatrix}, \text{ and } D_2 = \begin{bmatrix} 2 \end{bmatrix}.$$

This gives

$$I_0 - (A_1 \otimes A_2)u + (D_1 \otimes D_2 - I_0)u^2 = \begin{bmatrix} 1 + 3u^2 & -4u \\ -4u & 1 + 3u^2 \end{bmatrix}.$$

We also have $r_0 = \dim(C_0) = 2$, and $r_1 = \frac{1}{2}\dim(C_1) = 4$. And so we have, as expected,

$$Z_{X_1,X_2}(u) = \frac{(1-u^2)^{r_0 - r_1}}{\det[I_0 - (A_1 \otimes A_2)u + (D_1 \otimes D_2 - I_0)u^2]}$$
$$= \frac{(1-u^2)^{-2}}{9u^4 - 10u^2 + 1} = \frac{1}{9u^8 - 28u^6 + 30u^4 - 12u^2 + 1}$$

.

Finally, we consider the Euler product form in (11.4). In Section 6, we found that the cell product of $X_1$ and $X_2$ is the (labelled) surface of a torus. Evaluating $Z_{X_1,X_2}(u)$ with the Euler product form requires counting all primes on the cell product and their degrees. For instance, the following three sequences of directed diagonals all represent primes of degree 2: $\{(\alpha_3, \beta_1), (\alpha_2, \beta_1)\}, \{(\alpha_3, \beta_1), (\alpha_4, \beta_1)\}$ and $\{(\alpha_3, \beta_1), (\alpha_2, \beta_2)\}$. Counting primes in this way is not easy. In fact, it is easy to see that this cell product has infinitely many primes. For example, consider the following closed walk $w$: follow the closed walk $\{(\alpha_3, \beta_1), (\alpha_2, \beta_1)\}$ $n$ times and then follow the closed walk $\{(\alpha_3, \beta_1), (\alpha_2, \beta_2)\}$ one time. Then $w$ is a BTPC walk representing a prime ($w$ is primitive, since it does not result from traversing a *single* closed walk several times). Since $n$ can be arbitrary, there are infinitely many such primes. Thus, it is not convenient to evaluate $Z_{X_1,X_2}(u)$ here using the Euler product form. The result (11.4) guarantees that the infinite product converges and that it equals $Z_{X_1,X_2}(u)$ as evaluted by the arc-operator and vertex-operator expressions.

## 13. Interpretation in Terms of Kronecker Products

Given two graphs $X_1$ and $X_2$, with adjacency matrices $A_1$ and $A_2$, the Kronecker product (also called the tensor or graph-theoretic product) is defined to be the graph, $X_p$, uniquely determined by the adjacency matrix $A_1 \otimes A_2$. Since $X_p$ is a graph, it has an Ihara Zeta Function. R. Reeds deals closely with these functions in her paper, "Zeta Functions on Kronecker Products of Graphs" [5]. In this section, we claim that $Z_{X_1,X_2}(u)$ is precisely the Ihara Zeta function for $X_p$, which can be seen by comparing the vertex-operator form for $Z_{X_1,X_2}(u)$ in (11.4) with the vertex-operator form for $Z_{X_p}(u)$ in (4.2):

(i) $A_1 \otimes A_2$ is by definition the adjacency matrix, $A_{X_p}$, of $X_p$.

(ii) Each vertex $(v, w) \in X_p$ has degree equal to $\deg_{(X_1)}(v) \cdot \deg_{(X_2)}(w)$, where $\deg_{(X_1)}(v)$ is the degree of $v$ in $X_1$ and $\deg_{(X_2)}(w)$ is the degree of $w$ in $X_2$. Let $D_1$ and $D_2$ be the degree matrices (see Section 8) of $X_1$ and $X_2$, respectively. By (8.3), $(D_1 \otimes D_2)(v, w) = [\deg_{(X_1)}(v) \cdot \deg_{(X_2)}(w)] \cdot (v, w)$. And so, $(D_1 \otimes D_2 - I_0)$ is the "degree minus one" matrix, $Q_{X_p}$, of $X_p$.

(iii) $r_0 = \dim(C_0)$ is the number of vertices of $X_1$ times the number of vertices of $X_2$, which is precisely the number of vertices of $X_p$.

(iv) $r_1 = \frac{1}{2}\dim(C_1)$ is precisely the number of edges of $X_p$. To see this, note that if $X_i, (i = 1, 2)$ has $m_i$ edges, then it has $2m_i$ arcs. We know that the dimension of the vector space $C_1$ is the number of arcs of $X_1$ times the number of arcs of $X_2$. Therefore, $r_1 = 2m_1m_2$. Meanwhile, it is a standard result of graph theory (*e.g.* see West [6]) that the number of edges in a graph $X$ is given by the sum $\frac{1}{2}\sum_{y\in X}\deg(y)$, where the sum is over the vertices $y$ of the graph $X$. So, the number of edges of $X_p$ is equal to $\frac{1}{2}\sum_{y\in X_p}\deg(y) = \frac{1}{2}\sum_{v_i\in X_1}\sum_{w_j\in X_2}\deg(v_i)\cdot\deg(w_j) =$

$$\frac{1}{2}[\textstyle\sum_{v_i\in X_1} \deg(v_i)][\sum_{w_j\in X_2} \deg(w_j)] = \frac{1}{2}(2m_1)(2m_2) = r_1.$$

Thus, we have proven the following equality,

*Proposition.* Given two finite graphs $X_1$ and $X_2$,

$$Z_{X_1,X_2}(u) = Z_{X_p}(u), \tag{13.1}$$

where $X_p$ is the Kronecker product of $X_1$ and $X_2$.

As an example of this Proposition, consider $Z_{X_1,X_2}(u)$ from the previous Section. There, we used $X_1$ and $X_2$ from Example 2 in Section 6 and found

$$Z_{X_1,X_2}(u) = \frac{1}{9u^8 - 28u^6 + 30u^4 - 12u^2 + 1}.$$

On the other hand, the graph $X_p$ has

$$A_{X_p} = A_1 \otimes A_2 = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \text{ so } Q_{X_p} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \text{(each vertex has degree 4).}$$

Moreover, $n =$ "number of vertices of $X_p$" $= 2$, while $m =$ "number of edges of $X_p$" $= 4$ (the adjacency matrix tells us that 4 edges run between the two vertices of $X_p$). So, we have, in agreement with the Proposition,

$$Z_{X_p}(u) = \frac{(1-u^2)^{n-m}}{\det(I_n - uA_X + u^2Q_X)} = \frac{(1-u^2)^{-2}}{9u^4 - 10u^2 + 1} = \frac{1}{9u^8 - 28u^6 + 30u^4 - 12u^2 + 1}.$$

## 14. A Second Zeta Function for Cell Products

This second definition was suggested by J. Hoffman.

*Definition.* Given two finite graphs $X_1$ and $X_2$ with $T$-matrices $T_1$ and $T_2$, respectively, define a Zeta Function for the cell product as follows,

$$\tilde{Z}_{X_1,X_2}(u) = \frac{1}{\det[I - u(T_1 \otimes T_2)]}. \tag{14.1}$$

As we did with $Z_{X_1,X_2}(u)$, we take the generalized arc-operator form (14.1) to be the definition of $\tilde{Z}_{X_1,X_2}(u)$ and proceed to look for equivalent expressions. J. Hoffman suggested an Euler product form that works, but we were unable to find a generalized vertex-operator form. We discuss the Euler product form below.

*Proposition.* Let $(\gamma_i, \delta_j) \in \text{primes}(X_1) \times \text{primes}(X_2)$. Then,

(14.2)
$$\tilde{Z}_{X_1,X_2}(u) = \frac{1}{\det[I - u(T_1 \otimes T_2)]} = \prod_{(\gamma_i,\,\delta_j)} [\frac{1}{1 - u^{\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)}}]^{\mathrm{GCD}(\deg\gamma_i,\,\deg\delta_j)},$$

where the product is over all such pairs $(\gamma_i,\, \delta_j)$.

*Proof*. The proof of (14.2) mimics the proof of (11.2). Again, the power series expansions of both the RHS and LHS of (14.2) have 1 as the constant term. Therefore, to show equality between the two expressions, it suffices to show that taking $u\dfrac{d}{du}\log$ of both the LHS and RHS yields equivalent expressions.

Taking $u\dfrac{d}{du}\log$ of the RHS yields:

$$u\frac{d}{du}\log \prod_{(\gamma_i,\,\delta_j)} [\frac{1}{1 - u^{\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)}}]^{\mathrm{GCD}(\deg\gamma_i,\,\deg\delta_j)}$$

$= -u\frac{d}{du} \sum_{(\gamma_i,\,\delta_j)} \mathrm{GCD}(\deg\gamma_i,\, \deg\delta_j) \log[1 - u^{\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)}]$

$= \sum_{(\gamma_i,\,\delta_j)} \dfrac{\mathrm{GCD}(\deg\gamma_i,\, \deg\delta_j)\,\mathrm{LCM}(\deg\gamma_i,\, \deg\delta_j)\, u^{\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)}}{1 - u^{\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)}}$

$= \sum_{(\gamma_i,\,\delta_j)} \sum_{s=1}^{\infty} \mathrm{GCD}(\deg\gamma_i,\, \deg\delta_j)\,\mathrm{LCM}(\deg\gamma_i,\, \deg\delta_j)\, u^{s\,\cdot\,\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)}$

Making the substitution $n = s\,\cdot\,\mathrm{LCM}(\deg\gamma_i,\, \deg\delta_j)$ gives

$= \sum_{n=1}^{\infty} \sum_{(\gamma_i,\,\delta_j),\,\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)|n} \mathrm{GCD}(\deg\gamma_i,\, \deg\delta_j)\,\mathrm{LCM}(\deg\gamma_i,\, \deg\delta_j)\, u^n$

$= \sum_{n=1}^{\infty} (\sum_{(\gamma_i,\,\delta_j),\,\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)|n} \deg\gamma_i\,\cdot\,\deg\delta_j)\, u^n.$

Meanwhile, taking $u\dfrac{d}{du}\log$ of the LHS yields (via the standard result of linear algebra for square matrices like $(T_1 \otimes T_2)$; again, see Hashimoto [2]):

$$u\frac{d}{du}\log\frac{1}{\det(I - uT_1 \otimes T_2)}$$

$= \sum_{n=1}^{\infty} \mathrm{Tr}[(T_1 \otimes T_2)^n]\, u^n.$

To show equality between the original expressions, then, we need only show equality between $(\sum_{(\gamma_i,\,\delta_j),\,\mathrm{LCM}(\deg\gamma_i,\,\deg\delta_j)|n} \deg\gamma_i\,\cdot\,\deg\delta_j)$ and $\mathrm{Tr}[(T_1 \otimes T_2)^n]$, so that the coefficients of the derived LHS and RHS expressions are equal. Below, we use standard results about the trace of tensor products of matrices. These results can be found, for instance, on the MathWorld Website [4]. We have,

$$\mathrm{Tr}[(T_1 \otimes T_2)^n]$$

$= \mathrm{Tr}[T_1^n \otimes T_2^n]$

$= \mathrm{Tr}[T_1^n] \cdot \mathrm{Tr}[T_2^n]$; and using (3.3) we get

$= (\sum_{\gamma_i \in \mathrm{primes}(X_1),\, \deg \gamma_i | n} \deg \gamma_i) \cdot (\sum_{\delta_j \in \mathrm{primes}(X_2),\, \deg \delta_j | n} \deg \delta_j)$

$= \sum_{(\gamma_i, \delta_j),\, \deg \gamma_i | n \text{ and } \deg \delta_j | n} \deg \gamma_i \cdot \deg \delta_j$

$= \sum_{(\gamma_i, \delta_j),\, \mathrm{LCM}(\deg \gamma_i,\, \deg \delta_j) | n} \deg \gamma_i \cdot \deg \delta_j$, which completes the proof.

This zeta function for cell products, then, has the following forms,

(14.3)
$$\tilde{Z}_{X_1,X_2}(u) = \frac{1}{\det[I - u(T_1 \otimes T_2)]} = \prod_{(\gamma_i,\, \delta_j)} [\frac{1}{1 - u^{\mathrm{LCM}(\deg \gamma_i,\, \deg \delta_j)}}]^{\mathrm{GCD}(\deg \gamma_i,\, \deg \delta_j)},$$

where the product is over all pairs $(\gamma_i,\, \delta_j) \in \mathrm{primes}(X_1) \times \mathrm{primes}(X_2)$.

## 15. Example: Determining the Second Zeta Function for a Cell Product

Again, let $X_1$ and $X_2$ be the 1-dimensional graphs from Example 2 in Section 6. We use the arc-operator expression in (14.3) to determine $\tilde{Z}_{X_1,X_2}(u)$. We have, for $X_1$ and $X_2$,

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$I - u(T_1 \otimes T_2) = \begin{bmatrix} 1 & 0 & 0 & -u & 0 & 0 & 0 & 0 \\ 0 & 1 & -u & 0 & 0 & 0 & 0 & 0 \\ 0 & -u & 1 & 0 & 0 & 0 & 0 & 0 \\ -u & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -u \\ 0 & 0 & 0 & 0 & 0 & 1 & -u & 0 \\ 0 & 0 & 0 & 0 & 0 & -u & 1 & 0 \\ 0 & 0 & 0 & 0 & -u & 0 & 0 & 1 \end{bmatrix}.$$

So we have

$$\tilde{Z}_{X_1,X_2}(u) = \frac{1}{\det[I - u(T_1 \otimes T_2)]} = \frac{1}{u^8 - 4u^6 + 6u^4 - 4u^2 + 1}.$$

This is not the same as $Z_{X_1,X_2}(u)$, which we calculated in Section 12 for these same two graphs.

## 16. Conclusion

In this paper, we have proposed two possibilities for defining a zeta function for cell products of graphs. First, we suggested

$$Z_{X_1,X_2}(u) = \frac{1}{\det[I_1 - (\overset{*}{T}_1 \otimes \overset{*}{T}_2 - J_1 \otimes J_2)u]} = \prod_{\xi \in \text{primes}(cells)} \frac{1}{1 - u^{\deg \xi}}$$

$$= \frac{(1-u^2)^{r_0 - r_1}}{\det[I_0 - (A_1 \otimes A_2)u + (D_1 \otimes D_2 - I_0)u^2]} = \prod_{\gamma \in \text{primes}(X_p)} \frac{1}{1 - u^{\deg \gamma}},$$

where $X_p$ denotes the Kronecker product.

This zeta function has the nice property that it has generalized arc-operator and vertex-operator forms, as well as an Euler product form. Moreover, it gives us one idea for defining primes on cell products in a geometric way. However, one wonders if a zeta function could be defined on cell product surfaces that would not necessarily correspond to the zeta function of another graph (like the Kronecker product). Such a zeta function might be more truly "higher-dimensional" than $Z_{X_1,X_2}(u)$.

Second, we suggested

$$\tilde{Z}_{X_1,X_2}(u) = \frac{1}{\det(I - uT_1 \otimes T_2)} = \prod_{(\gamma_i,\, \delta_j)} \left[\frac{1}{1 - u^{\mathrm{LCM}(\deg \gamma_i,\, \deg \delta_j)}}\right]^{\mathrm{GCD}(\deg \gamma_i,\, \deg \delta_j)},$$

where the product is over all pairs $(\gamma_i,\, \delta_j) \in \text{primes}(X_1) \times \text{primes}(X_2)$.

This zeta function has the nice property that it has an Euler product expansion in terms of primes of the original graphs. However, this definition lacks a vertex-operator form. Further research might involve searching for such a form. Also, the relationship between $Z_{X_1,X_2}(u)$ and $\tilde{Z}_{X_1,X_2}(u)$ remains to be explored. One wonders if there are connections between these two functions, both of which seem to arise as natural generalizations of the Ihara Zeta Function.

Our hope is that the ideas presented in this paper for defining a zeta function on cell products will stimulate further ideas about using the nice properties of the Ihara Zeta Function as a model for defining zeta functions more generally on higher dimensional geometric objects.

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