[Rose-Hulman Undergraduate Mathematics Journal](https://scholar.rose-hulman.edu/rhumj)

Volume 7
Issue 1

Article 2

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Differential Geometry of Manifolds with Density

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February 1, 2006

Abstract

We describe extensions of several key concepts of differential geometry to manifolds with density, including curvature, the Gauss-Bonnet theorem and formula, geodesics, and constant curvature surfaces.

1 Introduction

Riemannian manifolds with density, such as quotients of Riemannian manifolds or Gauss space, of much interest to probabilists, merit more general study. Generalization of mean and Ricci curvature to manifolds with density has been considered by Gromov ([Gr], Section 9.4E), Bakry and Émery, and Bayle; see Bayle [Bay].

We consider smooth Riemannian manifolds with a smooth positive density $e^{\varphi(x)}$ used to weight volume and perimeter. Manifolds with density arise in physics when considering surfaces or regions with differing physical density. An example of an important two-dimensional surface with density is the Gauss plane, a Euclidean plane with volume and length weighted by $(2\pi)^{-1}e^{-r^2/2}$, where r is the distance from the origin. In general, for a manifold with density, in terms of the underlying Riemannian volume dV and perimeter dP, the new weighted volume and area are given by

$$dV_\varphi = e^\varphi dV$$

$$dP_\varphi = e^\varphi dP.$$

Following Gromov [Gr], we generalize the curvature of a curve and the mean curvature of a surface to manifolds with density. The generalizations are defined in such a manner so as to fit standard conceptions of curvature in Riemannian manifolds. Definition 3.1 states that the *curvature* κ_φ of a curve with unit normal $\mathbf{n}$ is given by

$$\kappa_\varphi = \kappa - \frac{d\varphi}{d\mathbf{n}},$$

where κ is the Riemannian curvature. Proposition 3.2 confirms that this definition satisfies the first variation formula for curvature. Similarly Definition 4.1 generalizes *mean curvature* as

$$H_\varphi = H - \frac{1}{(n-1)}\frac{d\varphi}{d\mathbf{n}},$$

where H is the Riemannian mean curvature. Proposition 4.2 confirms that the definition for H_φ satisfies the first variation formula, while Proposition 4.3 expresses H_φ in terms of the principal curvatures.

After work of Bakry and Émery and Bayle [Bay] on the Ricci curvature of manifolds with density, Definition 5.1 states that the *Gauss curvature* G_φ of a Riemannian surface with density e^φ is given by

$$G_\varphi = G - \Delta\varphi,$$

where G is the Riemannian Gauss curvature. Using this definition, we prove a generalized Gauss-Bonnet formula (Proposition 5.2) and Gauss-Bonnet theorem (Proposition 5.3). The proofs of these two results employ Stokes's theorem and the Riemannian Gauss-Bonnet formula and theorem.

Corollary 5.5 computes that the Gauss curvature of Gauss space is everywhere 2, and Proposition 5.6 uses Proposition 5.2 to relate the Euclidean area of a curve of constant curvature to its Euclidean length.

Using our curvature definitions and results, we search for geodesics in a plane with log-concave, symmetric density. Our Conjecture 2.20 says that the only embedded geodesics are a circle about the origin and lines through the origin.

Finally, we show that in $\mathbb{R}^3$ with Gaussian and certain other densities there exists a minimal cylinder and a minimal sphere (Corollaries 7.3 and 7.5). It would be interesting to find other minimal and constant-mean-curvature surfaces.

For further discussion of manifolds with density, including generalization of the volume estimate of Heintze and Karcher and the isoperimetric inequality of Levy and Gromov, see Morgan [M2].

1.1 Acknowledgements

This paper presents part of the work of the 2004 Williams College SMALL undergraduate research Geometry Group [CHSX].

The authors would like to thank our advisor Frank Morgan for his continual guidance and input with this paper and for bringing us to the Institut de Mathématiques de Jussieu Summer School on Minimal Surfaces and Variational Problems in Paris. We also thank the staff and organizing committee of the Summer School, especially Pascal Romon.

We thank the NSF for grants to the SMALL program and to Morgan. We also thank

Williams College for support.

2 Surfaces with density

Definition 2.1. We consider smooth Riemannian manifolds with a smooth positive density e^{φ} used to weight volume and perimeter. In terms of the underlying Riemannian volume dV and perimeter dP, the new weighted volume and perimeter are given by

$$dV_{\varphi} = e^{\varphi} dV$$
$$dP_{\varphi} = e^{\varphi} dP.$$

Manifolds with density arise naturally in math, physics and economics.

Example 2.2. Consider a bounded curve on the closed Euclidean half plane (boundary on the x-axis) and the surface of revolution formed by rotating that curve about x-axis. Areas and arc lengths on that surface correspond to areas and arc lengths on the half plane with a weighting of $2\pi y$.

Example 2.3. In physics, an object may have differing internal densities so in order to determine the object's mass it is necessary to integrate volume weighted with density.

Before two more examples of how manifolds with densities may arise, we introduce Gauss space and the Gauss plane, a central example of a manifold with density.

Definition 2.4. *Gauss space* G^m is R^m endowed with Gaussian density $(2\pi)^{-m/2}e^{-r^2/2}$, with r the radial distance from the origin. Specifically, the *Gauss plane* is the Euclidean plane endowed with density $(2\pi)^{-1}e^{-r^2/2}$.

Example 2.5. In government and economics it is often necessary to consider aggregate properties of groups and subgroups of people. For large groups, these aggregate properties can be determined by integrating over the members of the group, the differing individual properties (much like different densities).

Specifically, Gauss space may arise when considering groups with a number of properties which are independent random variables. The central limit theorem implies that these variables converge to Gaussian distributions and hence this group in consideration has properties distributed with Gaussian density. Gauss space provides an excellent model for this situation.

3 Curvature of Curves in Surfaces with Density

Definition 3.1. For a two-dimensional Riemannian manifold with density, following Gromov [Gr], the *curvature* κ_{φ} of a curve with unit normal $\mathbf{n}$ is given by

$$\kappa_{\varphi} = \kappa - \frac{d\varphi}{d\mathbf{n}},$$

where κ is the Riemannian curvature.

The definition of curvature is justified by the following proposition.

Proposition 3.2. *The first variation $\delta^1(v) = dL_\varphi/dt$ of the length of a smooth curve in a two-dimensional Riemannian manifold with density e^φ under a smooth variation with initial velocity v satisfies*

$$\frac{dL_\varphi}{dt} = \delta^1(v) = -\int \kappa_\varphi v ds_\varphi. \tag{1}$$

If κ_φ is constant then $\kappa_\varphi = dL_\varphi/dA_\varphi$, where A_φ denotes the weighted area on the side of the normal, and ds_φ denotes the weighted differential curve length.

Proof. The product rule and $ds_\varphi = e^\varphi ds$ give

$$\frac{d}{dt}(L_\varphi) = \frac{d}{dt}\int e^\varphi ds = \int e^\varphi \frac{d}{dt} ds + \int (\frac{d}{dt}e^\varphi) ds$$

$$= -\int e^\varphi \kappa v ds + \int e^\varphi \frac{d\varphi}{d\mathbf{n}} v ds = -\int (\kappa - \frac{d\varphi}{d\mathbf{n}}) v ds_\varphi = -\int \kappa_\varphi v ds_\varphi,$$

with the normal well-defined at all points.

Since

$$\frac{dA}{dt} = -\int v ds_\varphi, \tag{2}$$

if κ_φ is constant then $\kappa_\varphi = dL_\varphi/dA_\varphi$. □

Definition 3.3. An *isoperimetric curve* is a perimeter-minimizing curve for a given area. On a manifold with density, an isoperimetric curve minimizes weighted perimeter for a given weighted area.

Proposition 3.4. *An isoperimetric curve has constant curvature κ_φ.*

Proof. If a curve is isoperimetric, dL_φ/dA_φ must be the same for all variations. It follows from Equations (1) and (2) that κ_φ must be constant. □

Definition 3.5. A *geodesic* is an isoperimetric curve with constant curvature $\kappa_\varphi = 0$.

Proposition 3.6. *For a curve $r(\theta)$ in a two-dimensional Riemannian manifold with density $e^{\varphi(r)}$,*

$$\kappa_\varphi = \frac{r^2 + 2\dot{r}^2 - r\ddot{r}}{(r^2 + \dot{r}^2)^{3/2}} + \frac{\frac{d\varphi}{dr} r}{(r^2 + \dot{r}^2)^{1/2}} = \frac{r + \frac{d\varphi}{dr} r^2 - \ddot{r}}{r\sqrt{r^2 + \dot{r}^2}} + \frac{\dot{r}^2(r + \ddot{r})}{r(r^2 + \dot{r}^2)^{\frac{3}{2}}}. \tag{3}$$

Proof. The formula follows from evaluating the definition explicitly for a curve $r(\theta)$. □

4 Mean Curvature of Surfaces with Density

Definition 4.1. In an n-dimensional Riemannian manifold with density e^{φ}, the *mean curvature* H_{φ} of a hypersurface with unit normal $\mathbf{n}$ is given by

$$H_{\varphi} = H - \frac{1}{(n-1)} \frac{d\varphi}{d\mathbf{n}},$$

where H is the Riemannian mean curvature.

The definition of mean curvature is justified by the following proposition.

Proposition 4.2. *The first variation $\delta^1(v) = dA_{\varphi}/dt$ of the area of a smooth hypersurface in a Riemannian manifold with density e^{φ} under a smooth variation with initial velocity v satisfies*

$$\frac{dA_{\varphi}}{dt} = \delta^1(v) = -\int((n-1)H_{\varphi}v)ds_{\varphi}, \tag{4}$$

where ds_{φ} and ds denote the weighted and unweighted (respectively) differential area element of the surface.

If H_{φ} is constant then $(n-1)H_{\varphi} = dA_{\varphi}/dV_{\varphi}$.

Proof. The product rule and $dA_{\varphi} = e^{\varphi}dA$ give

$$\begin{aligned}\frac{d}{dt}(A_{\varphi}) = \frac{d}{dt}\int e^{\varphi}ds = \int e^{\varphi}\frac{d}{dt}ds + \int(\frac{d}{dt}e^{\varphi})ds = -\int e^{\varphi}((n-1)Hv)ds + \int e^{\varphi}(\frac{d\varphi}{d\mathbf{n}}v)ds \\ = -\int((n-1)(H - \frac{1}{(n-1)}\frac{d\varphi}{d\mathbf{n}})v)ds_{\varphi} = -\int((n-1)H_{\varphi}v)ds_{\varphi},\end{aligned}$$

proving (4). Since

$$\frac{dV_{\varphi}}{dt} = -\int v dA_{\varphi}, \tag{5}$$

the rest follows. □

Remark 4.3. The *principal curvatures* $\kappa_{1\varphi}, \ldots, \kappa_{(n-1)\varphi}$ of a hypersurface in $\mathbb{R}^n$ with density e^{φ} are given by

$$\kappa_{1\varphi} = \kappa_1 - \frac{d\varphi}{d\mathbf{n}}$$

$$\vdots$$

$$\kappa_{(n-1)\varphi} = \kappa_{(n-1)} - \frac{d\varphi}{d\mathbf{n}}$$

where $\kappa_1, \ldots, \kappa_{(n-1)}$ are the Riemannian principal curvatures and $\mathbf{n}$ is the normal to the hypersurface. The mean curvature is thus

$$H_{\varphi} = \frac{\kappa_1 + \ldots + \kappa_{(n-1)} - \frac{d\varphi}{d\mathbf{n}}}{(n-1)} = \frac{\kappa_{1\varphi} + \ldots + \kappa_{(n-1)\varphi}}{(n-1)} + \frac{n-2}{n-1}\frac{d\varphi}{d\mathbf{n}}. \tag{6}$$

5 Gauss Curvature and the Gauss-Bonnet Theorem

We now extend Gauss curvature and Gauss-Bonnet to surfaces with density.

Definition 5.1. The *Gauss curvature* G_φ of a Riemannian surface with density e^φ is given by

$$G_\varphi = G - \Delta\varphi$$

where G is the Riemannian Gauss curvature.

Proposition 5.2 (Generalized Gauss-Bonnet formula). *Given a piecewise-smooth curve enclosing a topological disc R in a Riemannian surface with density e^φ and inward-pointing unit normal $\mathbf{n}$, G_φ satisfies*

$$\int_R G_\varphi dA + \int_{\partial R} \kappa_\varphi ds + \sum (\pi - \alpha_i) = 2\pi, \tag{7}$$

where α_i are interior angles and the integrals are with respect to Riemannian area and arc length.

Proof.

$$\begin{aligned}\int_R G_\varphi dA + \int_{\partial R} \kappa_\varphi ds + \sum (\pi - \alpha_i) &= \int_R (G - \Delta\varphi) dA + \int_{\partial R} (\kappa - \frac{d\varphi}{d\mathbf{n}}) ds + \sum (\pi - \alpha_i) \\ &= \int_R G dA + \int_{\partial R} \kappa ds + \sum (\pi - \alpha_i) - \int_R \Delta\varphi dA - \int_{\partial R} \frac{d\varphi}{d\mathbf{n}} ds.\end{aligned}$$

By the Riemannian Gauss-Bonnet formula,

$$\int_R G dA + \int_{\partial R} \kappa ds + \sum (\pi - \alpha_i) = 2\pi.$$

By Stokes's Theorem, for inward-pointing normal,

$$\int_{\partial R} \frac{d\varphi}{d\mathbf{n}} ds = \int_{\partial R} (\nabla\varphi \cdot \mathbf{n}) ds = -\int_R \operatorname{div}(\nabla\varphi) dA = -\int_R \Delta\varphi dA.$$

Thus,

$$\int_R G dA + \int_{\partial R} \kappa ds + \sum (\pi - \alpha_i) - \int_R \Delta\varphi dA - \int_{\partial R} \frac{d\varphi}{d\mathbf{n}} ds = 2\pi.$$

□

Proposition 5.3 (Generalized Gauss-Bonnet theorem). *Given a compact, two-dimensional smooth Riemannian manifold M with density e^φ,*

$$\int_M G_\varphi dA = 2\pi\chi \tag{8}$$

where χ is the Euler characteristic of the manifold.

Proof. By the Riemannian Gauss-Bonnet theorem and Stokes's Theorem,

$$\int_M G_\varphi dA = \int_M G dA - \int_M \Delta\varphi dA = 2\pi\chi - 0 = 2\pi\chi.$$

□

Proposition 5.4. *Any radially symmetric punctured plane with density e^φ and constant Gauss curvature k must have $\varphi = -ar^2 + b\log r + c$ with a, b, c constant and $a = k/4$.*

Proof. From Proposition 5.1, $G_\varphi = G - \Delta\varphi$, and the plane has constant $G = 0$. Thus, the proof reduces to solving

$$G_\varphi = -\Delta\varphi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\varphi}{\partial r}) + \frac{\partial^2\varphi}{\partial\theta^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\varphi}{\partial r}) + 0 = k,$$

which gives $\varphi = -ar^2 + b\log r + c$. □

Corollary 5.5. *The Gauss plane $\mathcal{G}^2$ with density $e^\varphi = e^{-ar^2+c}$ has constant $G_\varphi = 4a$.*

Proposition 5.6. *In $\mathcal{G}^2$ with density e^{-ar^2+c}, any smooth, closed, embedded curve with constant curvature κ_φ must enclose Euclidean area*

$$A = \frac{\pi}{2a} - \frac{\kappa_\varphi L}{4a}, \tag{9}$$

where L is the Euclidean length of the curve.

Proof. By Corollary 5.5, $\mathcal{G}^2$ has Gauss curvature $4a$. From Proposition 5.2,

$$2\pi = \int_R G_\varphi dA + \int_{\partial R} \kappa_\varphi ds + \sum(\pi - \alpha_i) = 4aA + \kappa_\varphi L.$$

Thus, $A = \pi/2a - \kappa_\varphi L/4a$. □

Corollary 5.7. *Any smooth, closed, embedded geodesic in $\mathcal{G}^2$ must enclose Euclidean area $\pi/2a$.*

Proof. A geodesic has $\kappa_\varphi = 0$ by definition. □

An alternate application of Gauss curvature states that, in the expression for the area of a circle on a surface, the coefficient of r^4 is equal to a constant times the Gauss curvature, where r is the radius of the circle (see [M1], Section 3.7). Propositions 5.8 and 5.9 show that there is no such characterization of Gauss curvature for surfaces with density. More general versions of Proposition 5.17 are given by Z. Qian ([Q], Theorem 8) and Bayle ([Ba], (3.32/3)) with the opposite sign convention on ψ. The proofs are straightforward computations.

Proposition 5.8. *Given a plane with density e^φ, the area of a disc with Euclidean radius r is given by*

$$A = \pi r^2 e^\varphi + \frac{\pi r^4 e^\varphi}{8}(\Delta\varphi + |\nabla\varphi|^2)$$

and the perimeter given by

$$P = 2\pi r e^\varphi \left(1 + \frac{r^2}{4}(\Delta\varphi + |\nabla\varphi|^2)\right) + \dots$$

Proof. When not specified, assume that all functions are evaluated at the center of the circle. Let $\delta = e^\varphi$.

Consider δ to second order

$$\delta(r) = \delta + \delta_r r + \frac{1}{2}\delta_{rr} r^2 + \dots$$

Then the area of a disc is

$$A = \pi r^2 \delta + \int_0^{2\pi}\int_0^r \frac{1}{2}\delta_{rr} r^2 r\,dr\,d\theta = \pi r^2\delta + \frac{1}{8} r^4 \pi \Delta\delta.$$

Similarly, the perimeter is

$$P = \int_0^{2\pi} (\delta + \delta_r r + \frac{1}{2}\delta_{rr} r^2 + \dots) r\,d\theta = 2\pi r\left(\delta + \frac{\Delta\delta}{4} r^2\right) + \dots$$

The identities $\delta = e^\varphi$ and $\Delta\delta = e^\varphi(\Delta\varphi + |\nabla\varphi|^2)$ give the equations in their final forms. □

Proposition 5.9. *Given a plane with density e^φ, the area of a disc with weighted radius s is given by*

$$A(s) = \frac{\pi s^2}{e^\varphi} + \frac{\pi s^4}{e^{3\varphi}}\left(\frac{|\nabla\varphi|^2}{12} - \frac{\Delta\varphi}{24}\right) + \dots$$

The perimeter of the disc is given by

$$P(s) = 2\pi s + \frac{s^3}{e^{2\varphi}}\left(\frac{8\Delta\varphi - |\nabla\varphi|^2}{24}\right) + \dots$$

Remark 5.10. It follows that δ is determined by weighted length alone.

Proof of Proposition 5.9. Again, assume that, unless specified, functions are evaluated at the center of the circle, and take δ to second order $\delta(r) = \delta + \delta_r r + \frac{1}{2}\delta_{rr} r^2 + \dots$ Let $\delta = e^\varphi$.

By definition,

$$A = \int_0^{2\pi}\int_0^r \delta(r) r\,dr\,d\theta = \int_0^{2\pi} (\frac{1}{2}\delta r^2 + \frac{1}{3}\delta_r r^3 + \frac{1}{8}\delta_{rr} r^4 + \dots) d\theta$$

and

$$P = \int_0^{2\pi} \delta(r)\sqrt{r^2 + \dot{r}^2} d\theta,$$

where $\dot{r} = dr/d\theta$.

Since

$$s = \int_0^r \delta(r)dr = \int_0^r (\delta + \delta_r r + \frac{1}{2}\delta_{rr} r^2)dr = \delta r + \frac{1}{2}\delta_r r^2 + \frac{1}{6}\delta_{rr} r^3 + \dots,$$

solving for r in terms of s gives

$$r = \frac{1}{\delta}s - \frac{\delta_r}{2\delta^3}s^2 + \left(\frac{\delta_r^2}{2\delta^5} - \frac{\delta_{rr}}{6\delta^4}\right)s^3 + \dots$$

After substitution and simplification,

$$A(s) = \frac{\pi s^2}{\delta} + \frac{\pi s^4}{\delta^3}\left(-\frac{\Delta\delta}{24\delta} + \frac{|\nabla\delta|^2}{8\delta^2}\right) + \dots$$

and

$$P(s) = 2\pi s + s^3\left(\frac{\Delta\delta}{3\delta^3} - \frac{3|\nabla\delta|^2}{8\delta^4}\right) + \dots$$

The identities $\delta = e^\varphi$, $\nabla\delta = e^\varphi|\nabla\varphi|$, and $\Delta\delta = e^\varphi(\Delta\varphi + |\nabla\varphi|^2)$ give the equations in their final form. □

We consider (as in [ADLV]) the relationship between the center of mass for a closed curve of constant curvature and for the region it encloses.

Proposition 5.11. *Let C be closed curve of constant curvature κ in G^2, and R the region it encloses. Then the centers of mass p_C and p_R of the curve and the enclosed region satisfy:*

$$p_C = \kappa p_R.$$

In particular, if C is a geodesic, then $p_C = 0$.

Remark 5.12. The center of mass is just the average of the position vector. The result and proof hold for complete curves of finite length and generalize to complete curves of infinite length and to higher dimensions.

Proof. Under horizon translation, the first variations of length and area satisfy

$$\delta^1(L) = \int_C x,$$

$$\delta^1(A) = \int_A x,$$

and a similar result holds for vertical translation. Since $\kappa = dL/dA$, the result follows. □

6 Geodesics in Planes with Density

This section attempts to determine complete, embedded geodesics in a plane with density. Using polar coordinates, it is easily verified that a straight line in a plane with density $e^{\varphi} = e^{-ar^2+c}$ has constant curvature. For general $\varphi(r)$, straight lines are not geodesics, as is apparent from the formula for curvature; for example, for $\varphi = -r^4$, geodesics are curved lines as shown in Figure 1. All geodesics which are not lines through the origin can furthermore be parameterized in the form $r(\theta)$, where r is the polar distance and θ (mod 2π) is the polar angle.

Given such a curve $r(\theta)$, Equation (3) and $r^2 + \dot{r}^2 > 0$ together show that $r(\theta)$ is a geodesic if and only if it satisfies

$$\ddot{r} = \frac{(2 + r\frac{d\varphi}{dr})\dot{r}^2 + r^2(1 + r\frac{d\varphi}{dr})}{r}. \tag{10}$$

Alternatively, if $s = 1/r$, this equation becomes

$$\ddot{s} = \frac{\dot{s}^2}{s^2}\left(s - 2s^2 + s^3\frac{d\varphi}{ds}\right) - 1 + s\frac{d\varphi}{ds}. \tag{11}$$

Since $\ddot{r}$ is independent of θ and $\dot{r}$ occurs squared, it follows that if $\dot{r}(\theta) = 0$ then r is symmetric in the line through the origin at angle θ.

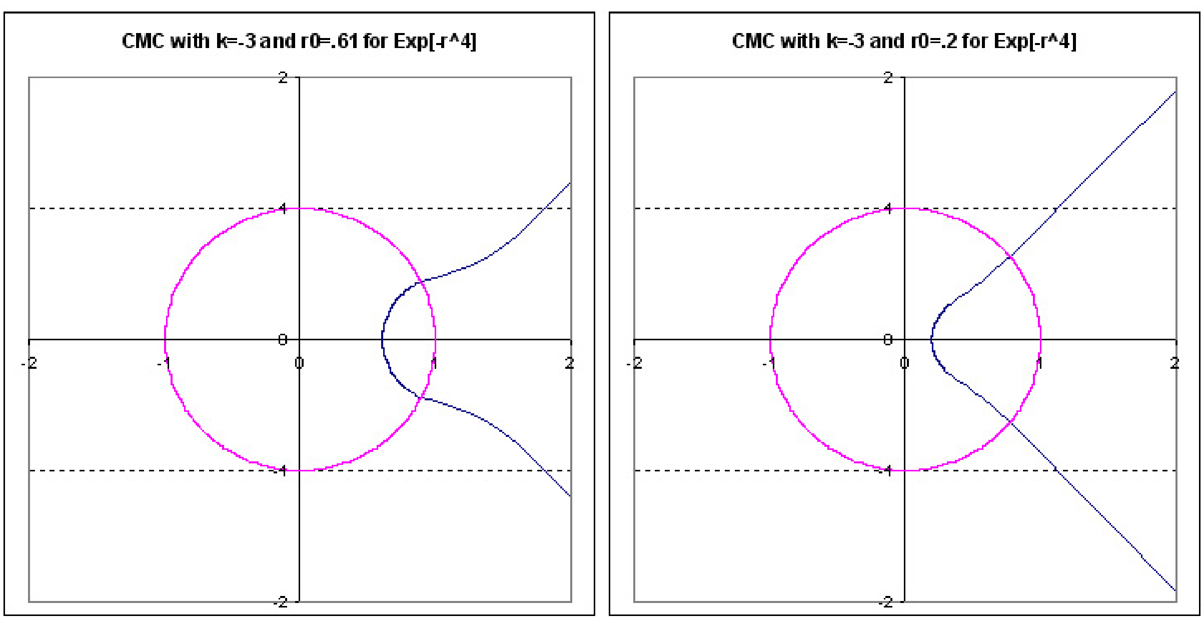

Figure 1: For general $\varphi(r)$ straight lines are not geodesics.

The centerpiece of this section is the following conjecture.

Conjecture 6.1. *Consider the plane with smooth density e^{φ}. Suppose that φ is strictly concave and rotationally symmetric. Then the only complete, embedded geodesics are a circle about the origin and lines through the origin. There do, however, exist closed, immersed geodesics that have an arbitrary number of petals (see Figure 2).*

For the rest of this section assume that φ is smooth, strictly concave, and rotationally symmetric.

To attempt a proof of this conjecture, we first show that the only embedded geodesics which are not lines through the origin are, by necessity, closed. Then we attempt to show that the only closed, embedded geodesic is a circle.

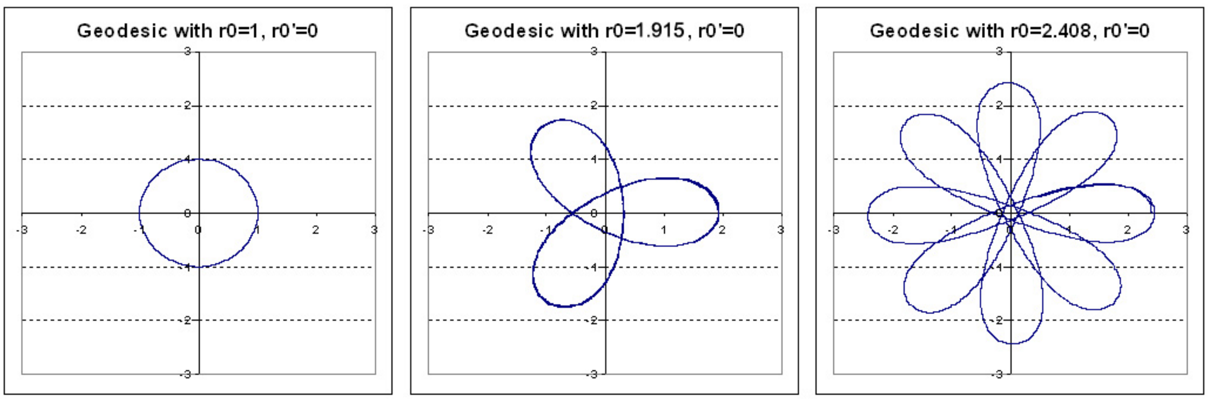

Figure 2: While the unit circle is the only embedded geodesic in the Gauss plane, besides lines through the origin, there exist petal-type closed geodesics with an arbitrary number of petals.

Lemma 6.2. *For the plane with density e^{φ}, $r\frac{d\varphi}{dr}+1$ is strictly decreasing in r and goes to $-\infty$ as r goes to ∞.*

Proof. Strict concavity of φ implies that $d\varphi/dr$ is strictly decreasing. Since $d\varphi/dr = 0$ when $r = 0$, $rd\varphi/dr$ goes to $-\infty$ as r goes to ∞.

A similar consideration gives that $s - 2s^2 + s^3\frac{d\varphi}{ds}$ goes to $-\infty$ as s goes to ∞. □

Lemma 6.3. *There exists a unique geodesic circle around the origin, with radius r_0.*

Proof. This follows from considering Equation 10 with initial conditions $\dot{r} = 0$ and $r = r_0$. The geodesic will be a circle if and only if $\ddot{r} = \dot{r} = 0$ and this occurs if and only if $1 + rd\varphi/dr = 0$. For $r = 0$, $1 + rd\varphi/dr = 1$ and so by Lemma 6.2 $1 + rd\varphi/dr = 0$ occurs uniquely — for radius r_0. □

Lemma 6.4. *Every curve $r(t)$ which satisfies Equation 10 is bounded from infinity and zero. Such a curve cannot converge to a circle of radius $r_1 \neq r_0$, or to the circle of radius r_0 (unless it is the circle with radius r_0).*

Proof. Using Equations (10) and (11), we can prove both cases simultaneously by observing that the same argument applies to r and s. Assume, to obtain a contradiction, that $r(\theta)$ and $s(\theta)$ are unbounded as θ increases and hence go to infinity. This implies that $\dot{r}$ and $\dot{s}$ are positive. For large r, Equation (10) and Lemma 6.2 imply that there exists some θ_r such that for all $\theta > \theta_r$, $\ddot{r} < -1$. Similarly, for large s, Equation (11) and Remark 6.2 imply that there exists some θ_s such that for all $\theta > \theta_s$, $\ddot{s} < -1$. Since r and s share this property, we can consider only one for the rest of the proof. This bound implies that $\dot{r}$ is bounded above, and $r(\theta)$ can only go to infinity as θ goes to infinity. But since $\ddot{r} < -1$ for all $\theta > \theta_r$, $\dot{r}(\theta) = 0$ for some $\theta > \theta_0$. Considerations above imply that $r(\theta)$ is symmetric about the radial line at this θ and therefore attains a local maximum. The same argument applies for unboundedness as θ decreases. This suffices to show that r is bounded due to symmetry.

In order to converge to a circle of radius $r_1 \neq r_0$, $\dot{r}$ and $\ddot{r}$ must both go to zero as r goes to r_0. However, as $\dot{r}$ and $\ddot{r}$ go to zero, by Lemma 6.2, Equation (10) becomes true only for r very close to r_0, which contradicts our assumption that $r_1 \neq r_0$.

Assume that $r(0) < 1$, θ is increasing, and $\dot{r}(0) > 0$. Equation (10) and Lemma 6.2 imply that for $r < r_0$, $\ddot{r} > 0$. Therefore, since $\dot{r} > 0$, r does not converge to 1 but rather overshoots it. The same argument applied to Equation (11) shows that s overshoots 1 and the curve cannot converge to the r_0-circle. □

Proposition 6.5. *All embedded geodesics, other than the r_0 circle of Lemma 6.3 or lines through the origin, are closed curves with dihedral symmetry, monotonic (and intersecting the r_0 circle) in the fundamental domain, as in Figure 3.*

Proof. Lemmas 6.2, 6.3 and 6.4 imply that any geodesic will have $\dot{r} = 0$ at least twice. A circle satisfies our conditions trivially, so we can assume that r is not a circle. Then there must exist some $\theta_1 \neq \theta_2 \pmod{2\pi}$ such that $\dot{r}(\theta_1) = \dot{r}(\theta_2) = 0$ and $r(\theta_1) \neq r(\theta_2)$. If $\theta_1 - \theta_2 = k\pi$ for some rational k then the geodesic is closed and $\dot{r}(\theta) = 0$ for $\theta = i\pi/n$ for some n and all $i \in \{0, \ldots, n-1\}$. However, if k is irrational, r must cross itself and cannot be embedded. Therefore any geodesic must exhibit dihedral symmetry.

By Lemma 5.7, any closed, embedded geodesic has Euclidean area π. Also, $\dot{r}$ can not change sign in the fundamental domain. Thus, evaluating $r(\theta) - 1$ at the endpoints of the fundamental domain must either yield 0's, or two values with opposite signs. Therefore $r(\theta)$ changes monotonically and intersects r_0. □

7 Surfaces with Density of Constant Mean Curvature

Recall that a minimal surface has zero mean curvature. Using the analogous concept of mean curvature, we extend our work from curves to surfaces with density, and we show that there exist a cylinder and a sphere which are minimal surfaces.

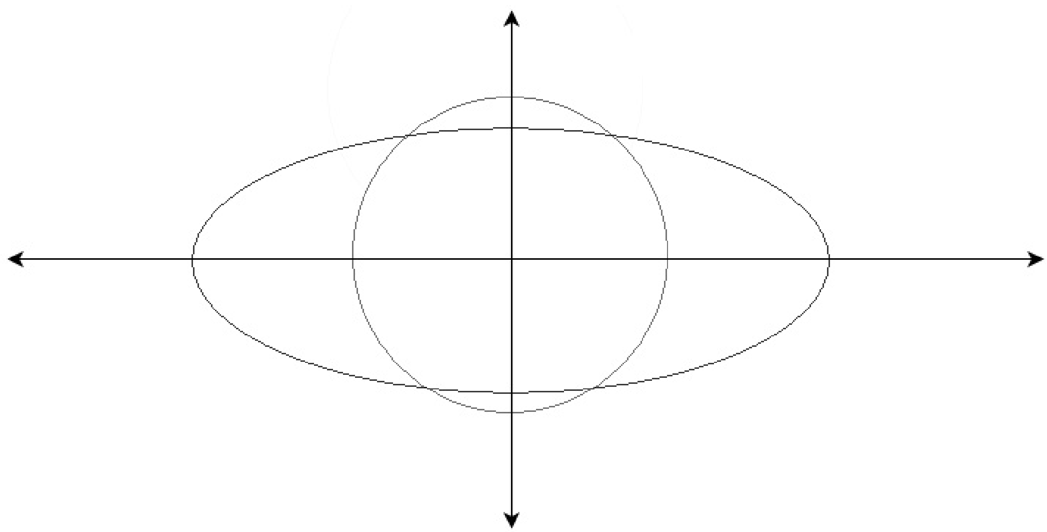

Figure 3: One candidate geodesic in Gauss space other than the circle of radius r_φ is an ellipse-shaped curve containing the origin, with r decreasing on the fundamental domain $0 \leq \theta \leq \pi/4$.

Proposition 7.1. *A hyperplane in $\mathbb{R}^n$ with density $e^\varphi = e^{-ar^2+c}$ has constant mean curvature.*

Proof. The principal Euclidean curvatures along the hyperplane are all 0. From Remark 4.3, H_φ is proportional to $d\varphi/d\mathbf{n}$, which is easily shown to be constant as well. □

Proposition 7.2. *In $\mathbb{R}^3$ with density $e^\varphi = e^{-ar^2+c}$, a cylinder centered at the origin with cross-sectional radius d has constant $H_\varphi = ad - \frac{1}{2d}$.*

Proof. Assume without loss of generality that the cylinder is rotationally symmetric about the z-axis and the normal points outward.

By Remark 4.3,

$$H_\varphi = \frac{\kappa_1 + \kappa_2 - \frac{d\varphi}{d\mathbf{n}}}{2}.$$

$\kappa_1 = 0$ is the curvature of a straight line, $\kappa_2 = \frac{-1}{d}$ is the curvature of the cross-sectional circle, and $\frac{d\varphi}{d\mathbf{n}} = \frac{d\varphi}{dr}\cos\theta = (-2ar)\frac{d}{r} = -2ad$, where θ is the angle that r makes with the xy-plane.

Thus,

$$H_\varphi = \frac{0 - \frac{1}{d} - (-2ad)}{2} = ad - \frac{1}{2d}.$$

□

Corollary 7.3. *In $\mathbb{R}^3$ with density $e^\varphi = e^{-ar^2+c}$, a cylinder with cross-sectional radius $\frac{1}{\sqrt{2a}}$ is a minimal surface.*

Proof.

$$H_\varphi = ad - \frac{1}{2d} = a(\frac{1}{\sqrt{2a}}) - \frac{1}{2(\frac{1}{\sqrt{2a}})} = \sqrt{\frac{a}{2}} - \sqrt{\frac{a}{2}} = 0.$$

□

Proposition 7.4. *In $\mathbb{R}^n$ with density $e^\varphi = e^{-ar^2+c}$, a sphere with radius d centered at the origin has constant $H_\varphi = \frac{2ad}{n-1} - \frac{1}{d}$.*

Proof. A sphere with radius d has $\kappa_i = -\frac{1}{d}$ for all $i = 1, \ldots, n-1$. Thus, by Remark 4.3,

$$H_\varphi = \frac{\kappa_1 + \ldots + \kappa_{(n-1)} - \frac{d\varphi}{d\mathbf{n}}}{(n-1)} = \kappa_i - \frac{(-2ar)}{n-1} = \frac{2ad}{n-1} - \frac{1}{d}.$$

□

Corollary 7.5. *In $\mathbb{R}^n$ with density $e^\varphi = e^{-ar^2+c}$, a sphere with radius $\sqrt{\frac{n-1}{2a}}$ is a minimal surface.*

Proof.

$$H_\varphi = \frac{2ad}{n-1} - \frac{1}{d} = \frac{2a}{n-1}\sqrt{\frac{n-1}{2a}} - \sqrt{\frac{2a}{n-1}} = 0.$$

□

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