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Thermal Imaging of Circular Inclusions within a Two-Dimensinoal Region

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1 Introduction

The ability to study the interior of an object without destroying it is an important industrial tool. One method of recent interest is *thermal imaging*. The idea is to use heat energy as a kind of "x-ray", to form an image of the interior of an object without causing damage to the object. More precisely, one applies a controlled source of heat energy to the exterior boundary of the object, then monitors the temperature of the object's boundary over time. This measured boundary temperature is influenced by the internal structure of the object. For example, an internal crack or void may block the flow of heat energy, and the heat is forced to flow around the defect. The goal is to determine the internal structure—e.g., locate cracks—from this exterior temperature data.

A simplified version of the method assumes that the problem is steady-state, that is, the applied heat flux and measured temperatures do not depend on time. We examine this version of the problem in this paper, and specifically the mathematics involved in using steady-state thermal methods to the imaging of internal circular defects or "inclusions" of a particular type described below. We should note that the same essential mathematics also governs the method of *impedance imaging*, that is, the use of applied electrical current/measured boundary voltages for imaging the interior of an object.

A similar problem for a single internal defect was studied in [4], for a rather general case of the problem. Our approach, which adapts to multiple defects, is based on a variation of the so-called "reciprocity gap" technique, which we describe below; see [1] or [2] for more information on this approach.

2 The Forward and Inverse Problems

The "forward problem" is that set of equations that govern the steady-state flow of heat through an object with one or more internal defects. For the moment we will concentrate on the case in which only one defect is present.

Let D be the object of interest, a bounded region in $\mathbb{R}^2$ with boundary ∂D. We assume, after appropriate scaling and introduction of dimensionless variables, that the thermal conductivity and diffusivity of D are both equal to one. Within D there is a circular region B (an "inclusion") of radius R centered at a point C^*; we use ∂B to refer to the boundary of B. The region B may have thermal properties (conductivity, diffusivity) that differ from those of the surrounding material $D \setminus B$, but we assume for the moment that B has the same thermal properties as the rest of D. We may think of the object D as a composite material, an object B bonded to or embedded in an object $D \setminus B$ along the curve ∂B. Refer to Figure 1:

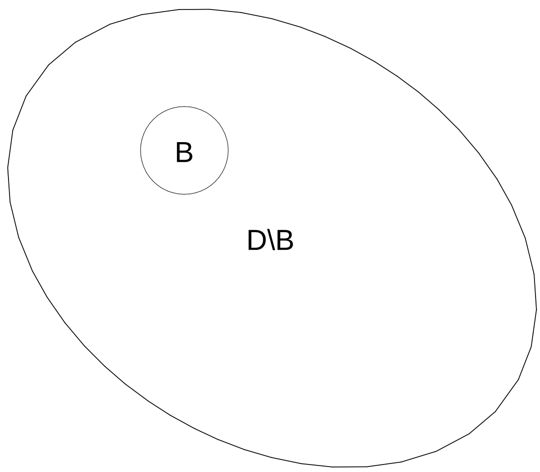

Figure 1: A region D with single inclusion B.

However, it may be the case that the interface ∂B between B and $D \setminus B$ has begun to disbond or corrode, a situation we wish to detect from measurements taken on ∂D (since the interior of the object is not accessible without causing damage to the object). If this is the case then heat will not flow freely across ∂B, but will experience a kind of "resistance" due to the presence of the corrosion or a gap. We now present a simple mathematical model of this situation.

First, we assume that a known time-independent heat flux $g(x, y)$ is applied to ∂D. The quantity $\int_{\partial D} g(x, y)\, ds$ where ds is arc length on ∂D is simply the net rate at which heat energy is pumped into D. We assume that $\int_{\partial D} g(x, y)\, ds = 0$, that is, the heat being pumped into D is balanced by heat flowing out, so that net influx of heat energy is zero. If this is the case then temperature within D will stabilize over time, and approach a steady-state temperature $u(x, y)$, which depends on (x, y) position but not time. Conservation of energy (see [5]) dictates that the function u will satisfy the steady-state heat equation or *Laplace's equation*

$$\triangle u = 0 \tag{1}$$

within $D \setminus B$ and B, where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The derivation of Laplace's equation is based on the modelling assumption that if u is the temperature in a region then $-\nabla u(x, y)$ measures the rate at which heat energy flows past (x, y) ($-\nabla u$ since heat energy flows from hot to cold).

Note, however, we do not assume that u satisfies Laplace's equation over the corroded boundary ∂B (indeed, as we model below, u may not even be continuous over ∂B).

On the outer boundary ∂D the input heat flux is modelled as (see [5], chapter 6)

$$\frac{\partial u}{\partial \mathbf{n}} = g(x, y) \text{ on } \partial D \tag{2}$$

where $\mathbf{n}$ is a unit normal outward vector field on ∂D.

We model the corroded or disbonded surface ∂B as follows: First, we assume that u need not be continuous over ∂B. For a point $(x, y) \in \partial B$ we use $u^-(x, y)$ to denote the limiting value of $u(x, y)$ from inside B, and $u^+(x, y)$ to denote the limiting value of $u(x, y)$ from outside B. We also define $[u](x, y) =$

$u^+(x,y) - u^-(x,y)$; $[u]$ is the temperature "jump" over the corroded interface ∂B. In the absence of any corrosion we would have $[u] \equiv 0$. Let $\mathbf{n}$ denote a unit normal outward vector field on ∂B. We also use $\frac{\partial u^-}{\partial \mathbf{n}}$ and $\frac{\partial u^+}{\partial \mathbf{n}}$ for the limiting values of $\nabla u \cdot \mathbf{n}$ from inside and outside ∂B, respectively. In particular, $\frac{\partial u^-}{\partial \mathbf{n}}$ is the rate at which heat energy is crossing ∂B from outside to inside B, as measured just inside ∂B. Similarly, $\frac{\partial u^-}{\partial \mathbf{n}}$ is the rate at which heat energy is crossing ∂B from outside to inside B, as measured just outside ∂B. Conservation of energy in fact forces $\frac{\partial u^-}{\partial \mathbf{n}} = \frac{\partial u^+}{\partial \mathbf{n}}$ everywhere on ∂B, so when convenient we will write simply $\frac{\partial u}{\partial \mathbf{n}}$ to refer to their common value.

The boundary condition which models corrosion on ∂B is

$$\frac{\partial u^+}{\partial \mathbf{n}} = \frac{\partial u^-}{\partial \mathbf{n}} = k[u] \text{ on } \partial B \tag{3}$$

where k is some constant, which we refer to as the "transmission constant." The constant k is related to the severity of the corrosion. For example, if $k = 0$, then ∂B is completely disbonded from D, for we have $\frac{\partial u}{\partial \mathbf{n}} = 0$, that is, no heat energy may cross ∂B. If k is large then heat flows more easily over ∂B. Equation (3) models the interface ∂B as a kind of "contact resistance", in which heat energy flows in proportion to the temperature difference ($[u]$) on each side of the interface, with constant of proportionality k.

In fact, the constant k might well depend on position too, but for simplicity we will not consider this case here (see [4] for an examination of this issue).

Equations (1)-(3) define the *forward problem* and uniquely determine u up to an additive constant. That is if u denotes any function which satisfies equations (1)-(3) then $u + c$ also satisfies the equations, for any constant c. We can nail down a unique solution by adding the condition that $\int_{\partial B} u \, ds = 0$.

The inverse problem of interest is this: suppose that we are unable to access the interior of D, but are only able to measure $u(x,y)$ on ∂D for a given $g(x,y)$. Can we determine the center, radius, and transmission constant of B?

3 The Reciprocity Gap Principle

We begin by recalling Green's Second Identity ([5], section 7.2).

Theorem 1 (Green's Second Identity) *For any pair of functions u and w that are $C^2(\overline{D})$ we have*

$$\int\int_D (u\triangle w - w\triangle u)\, dA = \int_{\partial D} \left(u\frac{\partial w}{\partial \mathbf{n}} - w\frac{\partial u}{\partial \mathbf{n}} \right) ds.$$

We can make use of Green's identity to solve the inverse problem.

Let us choose a "test function" $w(x, y)$ that is harmonic ($\triangle w = 0$) throughout D. We also recall that our temperature function $u(x, y)$ is harmonic within B and $D\backslash B$, so we also have $\triangle u = 0$ within B and $D\backslash B$. By using Green's Second Identity on B we find

$$\int_{\partial B} \left(u^- \frac{\partial w^-}{\partial \mathbf{n}} - w\frac{\partial u^-}{\partial \mathbf{n}} \right) ds \quad = \quad 0 \tag{4}$$

where all quantities are superscripted with a minus sign since the relevant values on ∂B are those obtained by approaching ∂B from the interior. By using Green's Second Identity on the region $D\backslash B$ (and note that the outward normal on $D \setminus B$ is MINUS the outward normal on B) we find

$$\int_{\partial D} \left(u^+ \frac{\partial w^+}{\partial \mathbf{n}} - w\frac{\partial u^+}{\partial \mathbf{n}} \right) ds - \int_{\partial B} \left(u^+ \frac{\partial w^+}{\partial \mathbf{n}} - w\frac{\partial u^+}{\partial \mathbf{n}} \right) ds \quad = \quad 0 \tag{5}$$

with "plus" superscripts in this case.

If we add equations (4) and (5) we find

$$\begin{aligned}\int_{\partial D} \left(u^+ \frac{\partial w^+}{\partial \mathbf{n}} - w\frac{\partial u^+}{\partial \mathbf{n}} \right) ds \quad &+ \quad \int_{\partial B} \left(u^- \frac{\partial w^-}{\partial \mathbf{n}} - w\frac{\partial u^-}{\partial \mathbf{n}} \right) ds \\ &- \quad \int_{\partial B} \left(u^+ \frac{\partial w^+}{\partial \mathbf{n}} - w\frac{\partial u^+}{\partial \mathbf{n}} \right) ds = 0\end{aligned} \tag{6}$$

We note that on ∂B we have (since w is harmonic, hence smooth through D) that $\frac{\partial w^+}{\partial \mathbf{n}} = \frac{\partial w^-}{\partial \mathbf{n}}$, $\frac{\partial u^+}{\partial \mathbf{n}} = \frac{\partial u^-}{\partial \mathbf{n}}$, and $w^+ = w^-$. Thus we have from equation

(6), after cancellations,

$$\int_{\partial D} \left(u \frac{\partial w}{\partial \mathbf{n}} - wg \right) ds = \int_{\partial B} [u] \frac{\partial w}{\partial \mathbf{n}} \, ds. \tag{7}$$

We call the left side of equation (7) the Reciprocity Gap functional, and we write it as $RG(w)$. Note that $RG(w) = \int_{\partial D} (u \frac{\partial w}{\partial \mathbf{n}} - wg) \, ds$ is computable for any w, given the input flux g and measured temperature u on ∂D. If no defect is present then $RG(w) = 0$ for all choices of w.

The functional $RG(w)$ allows us to use information from ∂D to derive information about the quantity on the right in equation (7). The general procedure is to use cleverly chosen "test functions" w to recover ∂B and/or the transmission constant.

4 Locating the Center of a Single Inclusion

We are able to use the Reciprocity Gap function to approximately locate the center of the inclusion. In what follows we will identify $\mathbb{R}^2$ with the complex plane, and write $C^* = x^* + iy^*$ when convenient. We will first consider the following Lemma.

Lemma 2 *Suppose B is an inclusion with center $C^* = x^* + iy^*$ and radius R. Let w_η be the harmonic function $w_\eta(x, y) = \frac{1}{\eta} e^{\eta(x+iy)}$ where $\eta \neq 0$ is any complex number. The Reciprocity Gap functional applied to $w_\eta(x, y)$ can be approximated as*

$$RG(w_\eta) \approx Re^{\eta C^*} \int_0^{2\pi} e^{i\theta} [u](\theta) \, d\theta + O(R^2) \int_0^{2\pi} e^{i\theta} [u](\theta) \, d\theta$$

where $[u](\theta)$ denotes $[u]$ evaluated at the point $x = x^* + R\cos(\theta), y = y^* + R\sin(\theta)$ on ∂B and $O(R^2)$ denotes a quantity bounded by AR^2 for some positive constant A (which does not depend on R).

Proof For $w_\eta(x, y) = \frac{1}{\eta} e^{\eta(x+iy)}$, we have $\nabla w_\eta = \eta w_\eta < 1, i >$. If we parameterize ∂B in polar coordinates, we find that

$$x = x^* + R\cos(\theta), \quad y = y^* + R\sin(\theta).$$

We then have

$$
\begin{aligned}
RG(w_\eta) &= \int_{\partial B} \nabla w_\eta \cdot \mathbf{n}[u](\theta)\, ds \\
&= \int_0^{2\pi} e^{\eta C^*} e^{R(\cos(\theta)+i\sin(\theta))} < 1, i > \cdot < \cos(\theta), \sin(\theta) > [u](\theta)\, ds \\
&= Re^{\eta C^*} \int_0^{2\pi} e^{R(\cos(\theta)+i\sin(\theta))} e^{i\theta} [u](\theta)\, d\theta \qquad (8)
\end{aligned}
$$

where we've used $ds = r\, d\theta$. We can approximate

$$
e^{R(\cos(\theta)+i\sin(\theta))} = 1 + O(R)
$$

and so from equation (8) we have

$$
RG(w_\eta) = Re^{\eta C^*} \int_0^{2\pi} e^{i\theta}[u](\theta)\, d\theta + O(R^2) \int_0^{2\pi} e^{i\theta}[u](\theta)\, d\theta.
$$

which proves the Lemma. ■

The Reciprocity Gap functional has been defined as a functional applied to $w_\eta(x, y)$. For notational simplicity let us now consider $RG(w_\eta)$ with $w_\eta(x, y) = \frac{1}{\eta} e^{\eta(x+iy)}$ as a function of η, and define $\phi(\eta) = RG(w_\eta)$. We also drop the higher order error term involving $O(R^2)$ (in doing so we assume that R is small). We consider the following corollary.

Corollary 3 *If the $O(R^2)$ term in Lemma 2 is dropped then the center C^* of a circular inclusion (in the form of a complex number), can approximated as*

$$
C^* = \frac{\phi'(\eta)}{\phi(\eta)}.
$$

Proof If we drop the $O(R^2)$ term in Lemma 2, we have $\phi(\eta) = RG(w) = Re^{\eta C^*} \int_0^{2\pi} e^{i\theta}[u](\theta)\, d\theta$. Differentiate $\phi(\eta)$ with respect to η to find $\phi'(\eta) = C^* \phi(\eta)$ which immediately yields the Corollary. ■

It is important to note here that we can compute $\phi'(\eta)$ from the boundary data, as

$$
\phi'(\eta) = RG\left(\frac{\partial w}{\partial \eta}\right)
$$

where $\frac{\partial w}{\partial \eta}$ can be computed explicitly.

We now have a method for locating the center of an inclusion B given boundary data on ∂D. We will illustrate below with a numerical example, after we've shown how to compute R and k.

5 Finding R and k for a Single Inclusion

We will now find the radius R of B as well as the transmission constant k, given the center C^* and the boundary data on ∂D. In what follows let $u_0(x,y)$ denote the temperature in the region D with no inclusion B, with $\frac{\partial u_0}{\partial \mathbf{n}} = g$ on ∂D; note u_0 is harmonic in D, and uniquely determined up to an additive constant.

Lemma 4 *Let $w(x,y)$ be some harmonic function on all of D. The Reciprocity Gap Function can be represented in terms of the radius, the center, and the transmission constant (R, C^*, and k, respectively) as*

$$\begin{aligned} RG(w) &= \frac{2\pi R^2}{1+2kR}\nabla u_0(C^*)\cdot\nabla w(C^*) \\ &+ \frac{\pi R^4}{1+kR}\left(\frac{\partial^2 w}{\partial x^2}(C^*)\frac{\partial^2 u_0}{\partial x^2}(C^*) + \frac{\partial^2 w}{\partial x\partial y}(C^*)\frac{\partial^2 u_0}{\partial x\partial y}(C^*)\right) + O(R^6) \end{aligned}$$

where $O(R^6)$ denotes a quantity bounded by CR^6 for some constant C.

Proof Let $v = u - u_0$. That is, v is a small correction in the "no-inclusion" temperature u_0 when there is an inclusion in the region D. The function v satisfies

$$\begin{aligned} \triangle v &= 0 \text{ in } B,\ \ D\backslash B & (9)\\ \frac{\partial v}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial D & (10)\\ \frac{\partial v^-}{\partial \mathbf{n}} &= \frac{\partial v^+}{\partial \mathbf{n}} = k[u] - \frac{\partial u_0}{\partial \mathbf{n}} \quad \text{on } \partial B. & (11) \end{aligned}$$

Because $[u_0] = 0$ we see that $[u] = [u_0] + [v] = [v]$ so that equation (11) can be written as

$$\frac{\partial v}{\partial \mathbf{n}} = k[v] - \frac{\partial u_0}{\partial \mathbf{n}} \tag{12}$$

where we use $\frac{\partial v}{\partial \mathbf{n}}$ for the common value of $\frac{\partial v^-}{\partial \mathbf{n}}$ and $\frac{\partial v^+}{\partial \mathbf{n}}$. Note that the above conditions determine v up to an arbitrary additive constant.

We use (r, θ) as polar coordinates about the center of the inclusion C^*. We will explicitly write out v in terms of u_0 to "good approximation".

Let $v_{D\backslash B}$ and v_B denote the restriction of v to $D\backslash B$ and B, respectively. We attempt an expansion of each as follows:

$$v_B(r, \theta) = \sum_{m=0}^{\infty} (c_m \cos(m\theta) + d_m \sin(m\theta)) r^m \tag{13}$$

$$v_{D\backslash B}(r, \theta) = \sum_{m=0}^{\infty} (a_m \cos(m\theta) + b_m \sin(m\theta)) r^{-m} \tag{14}$$

for constants a_m, b_m, c_m, d_m. Note that the expansion of (13) is definitely possible for suitable constants c_m, d_m (section 6.3, [5]). However, the expansion of equation (14) ignores terms of the form $r^m \cos(m\theta)$ and $r^m \sin(m\theta)$. If R is sufficiently small, however, these terms should be negligible; we will remark on this below.

Our goal is to work out the coefficients a_m, b_m, c_m, d_m as explicitly as possible in terms of u_0, then related this to $RG(w)$.

Equation (11) dictates that $\frac{\partial v_{D\backslash B}}{\partial r} = \frac{\partial v_B}{\partial r}$ on $r = R$, for $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial r}$ because the outward normal vector for the circle is in the radial direction. Thus we need

$$\begin{aligned} \sum_{m=1}^{\infty} m(c_m \cos(m\theta) &+ d_m \sin(m\theta)) R^{m-1} \\ &= \sum_{m=1}^{\infty} -m(a_m \cos(m\theta) + b_m \sin(m\theta)) R^{-m-1}. \end{aligned}$$

By matching terms we find that

$$a_m = -R^{2m} c_m \tag{15}$$

and

$$b_m = -R^{2m} d_m. \tag{16}$$

Let u_0 have Fourier expansion

$$u_0 = \sum_{m=0}^{\infty} (e_m \cos(m\theta) + f_m \sin(m\theta)) r^m \tag{17}$$

(note that e_0 can be chosen arbitrarily, so we'll take $e_0 = 0$). From equation (12) we have for $r = R$ that

$$\sum_{m=1}^{\infty} m(c_m \cos(m\theta) + d_m \sin(m\theta))R^{m-1} - k[\cos(m\theta)(a_m R^{-m} - c_m R^m)$$
$$+ \sin(m\theta)(b_m R^{-m} - d_m R^m)] = -\sum_{m=1}^{\infty} m(e_m \cos(m\theta) + f_m \sin(m\theta))R^{m-1}.$$

Matching the cosine terms above yields

$$mc_m R^{m-1} - k(a_m R^{-m} - c_m R^m) \quad = \quad -me_m R^{m-1}$$

so that with equation (15) we have

$$c_m = \frac{-me_m}{m + 2kR}. \tag{18}$$

The same reasoning with equation (16) shows that $d_m = \frac{-mf_m}{m+2kR}$. Thus we have

$$\begin{aligned} [u](R, \theta) &= [v](R, \theta) \\ &= \sum_{m=1}^{\infty} \cos(m\theta)(a_m R^{-m} - c_m R^m) + \sin(m\theta)(b_m R^{-m} - d_m R^m) \\ &= 2 \sum_{m=1}^{\infty} \left(\frac{e_m \cos(m\theta) + f_m \sin(m\theta)}{m + 2kR} \right) mR^m. \end{aligned} \tag{19}$$

This gives us $[u](R, \theta)$ in terms of the Fourier coefficients of u_0.

Let us perform a similar computation for any harmonic test function w. We expand

$$w = \sum_{m=0}^{\infty} (g_m \cos(m\theta) + h_m \sin(m\theta)) r^m \tag{20}$$

for coefficients g_m, h_m, so that when we evaluate the derivative $\frac{\partial w}{\partial \mathbf{n}}$ for $r = R$ we find

$$\frac{\partial w}{\partial r} = \sum_{m=1}^{\infty} m(g_m \cos(m\theta) + h_m \sin(m\theta))R^{m-1}. \tag{21}$$

Now recall the Reciprocity gap functional $RG(w) = \int_{\partial B}[u]\frac{\partial w}{\partial \mathbf{n}}\, ds$. From the expansions (19) and (21) we obtain (after integrating term by term and using orthogonality)

$$RG(w) = \frac{2\pi R^2}{1+2kR}(e_1 g_1 + f_1 h_1) + \frac{4\pi R^4}{1+kR}(e_2 g_2 + f_2 h_2) + O(R^6). \tag{22}$$

By using $\frac{\partial}{\partial r} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \theta} = -\frac{\sin(\theta)}{r}\frac{\partial}{\partial x} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial y}$ it's not hard to see from the expansions (17) and (20) that

$$\begin{array}{cc} \nabla w(C^*) =< g_1, h_1 > & \nabla u_0(C^*) =< e_1, f_1 > \\ e_2 = \frac{1}{2}\frac{\partial^2 u_0}{\partial x^2}(C^*) & f_2 = \frac{1}{2}\frac{\partial^2 u_0}{\partial x \partial y}(C^*) \\ g_2 = \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(C^*) & h_2 = \frac{1}{2}\frac{\partial^2 w}{\partial x \partial y}(C^*). \end{array}$$

When we substitute the previous results into equation (22) we find the Reciprocity Gap function to be

$$\begin{aligned} RG(w) &= \frac{2\pi R^2}{1+2kR}\nabla u_0(C^*)\cdot\nabla w(C^*) \\ &+ \frac{\pi R^4}{1+kR}\left(\frac{\partial^2 w}{\partial x^2}(C^*)\frac{\partial^2 u_0}{\partial x^2}(C^*) + \frac{\partial^2 w}{\partial x \partial y}(C^*)\frac{\partial^2 u_0}{\partial x \partial y}(C^*)\right) + O(R^6). \end{aligned}$$

This proves the Lemma. ■

If we choose the harmonic function $w(x,y) = \frac{1}{\eta}e^{\eta(x+iy)}$ and choose two distinct values for η, we then have a system of two equations, $RG(w(\eta_1))$ and $RG(w(\eta_2))$, with two unknowns, R and k. Therefore we can solve for R and k. We have the found the center, radius, and transmission constant of the inclusion B.

6 A Numerical Example

Using several computer programs, we will now give a numerical example of our algorithm. We assume that D is the unit disk in two dimensions. Using a C program developed by Dr. Kurt Bryan, we are able to input the center, radius, and transmission constant of an inclusion, as well as an input heat flux, and the program outputs the values of $u(x,y)$ at n points along ∂D. For the following example we use an inclusion centered at $(0.3, 0.4)$ with a radius of 0.15 and a transmission constant of 0.9. We will set the heat flux to be $g = \sin(2\theta)$ and use the program to compute 100 values of $u(x,y)$ evenly spaced around ∂D.

This "measured" data is loaded into a Maple notebook. We use a harmonic function $w_\eta(x,y) = \frac{1}{\eta}e^{\eta(x+iy)}$ for various choices of η and numerically evaluate $RG(w_\eta)$ using the trapezoidal rule. As mentioned above, we define $\phi(\eta) = RG(w_\eta)$, and recall that we can calculate $\phi'(\eta)$. In this example with $\eta = 1$ we find the center reconstructed according Corollary 3 at $(0.315, 0.420)$. The choice of η has little effect on the estimate of the center.

Once we've located the center we can estimate the radius and transmission coefficient. In what follows we will take, for the moment, our estimate of the center as the true value $(0.3, 0.4)$ (rather than the slightly erroneous estimate above) and calculate the radius and transmission constant of the inclusion. We choose two values of η and calculate $\phi(\eta)$ for each η. For our example, we choose $\eta_1 = 1 + i$ and $\eta_2 = 1 - i$. We obtain $\phi(\eta_1) = 0.009 + 0.050i$ and $\phi(\eta_2) = 0.086 + 0.080i$. We thus obtain the following system of two equations with two unknowns, R and k:

$$\begin{aligned}
0.009 + 0.050i &= \frac{2\pi R^2}{1+2kR}(0.4+0.3i)\ e^{(1+i)(0.3+0.4i)} \\
&+ \frac{\pi R^4}{1+kR}(i(1+i)e^{(1+i)(0.3+0.4i)}) \\
0.086 + 0.080i &= \frac{2\pi R^2}{1+2kR}(0.4+0.3i)\cdot e^{(1-i)(0.3+0.4i)} \\
&+ \frac{\pi R^4}{1+kR}(i(1-i)e^{(1-i)(0.3+0.4i)}).
\end{aligned}$$

We then solve the above equations with Newton's method to find $R = 0.1505$ and $k = 0.8527$.

However, if we use the slightly inaccurate value for C^* we obtain very poor results—large negative values for k, highly erroneous values for R. The computation is quite unstable with respect to the simultaneous estimates of k and R. If, however, we regard k as known (even approximately) we can recover R with good stability. In the present case using $k = 1$ in the equation $\phi(1 + i) = c$ (with $C^* = 0.315 + 0.42i$ as recovered, and where c is computed from the boundary data) yields $R \approx 0.1504$, while $k = 0.1$ yields $R \approx 0.133$. Alternatively, we can recover k stably if R is considered known. The precise stability of the problem of recovering both k and R simultaneously is a topic for further study.

7 Multiple Inclusions

We will now consider the case of multiple circular inclusions within a two-dimensional region. As with the one inclusion case, we will consider a two-dimensional region as D with boundary ∂D. Within D we assume that there exist N circular inclusions which we will denote by B_j, $1 \leq j \leq N$; here N itself may also be considered unknown. Each B_j has a radius of R_j and a transmission constant of k_j. We assume $\triangle u = 0$ in $D \setminus \cup_{j=1}^{N} B_j$ and in each B_j, and that equation (2) holds on ∂D. We also assume that equation (3) holds on each ∂B_j, with k replaced by k_j.

We will use a slightly different approach to locating the inclusions within D. Most importantly (and unfortunately), we do not have a method for finding R_j and k_j simultaneously; we need R_j in order to find k_j, or we need k_j in order to find R_j.

We first recall our Reciprocity Gap test function $w_\eta(x, y) = \frac{1}{\eta} e^{\eta(x+iy)}$, and set $\phi(\eta) = RG(w_\eta)$. Also recall that in the case of a single inclusion B, from Lemma 2 we can write, to good approximation, $\phi(\eta) = Je^{\eta C^*}$ where $J = \int_{\partial B} e^{i\theta}[u](\theta)\, ds$ where $ds = R\, d\theta$. For multiple inclusions a very similar argument shows that

$$\phi(\eta) = \sum_{j=1}^{N} J_j e^{\eta C_j^*}. \tag{23}$$

with

$$J_j = \int_{\partial B_j} e^{i\theta}[u](\theta)\, ds + O(R^2). \tag{24}$$

We may not know the exact number N of inclusions in our region since the interior cannot be accessed, but below we outline a procedure for finding N.

7.1 Locating N Centers

Because $\phi(\eta)$ is of the form (23), it must satisfy a constant-coefficient linear ODE of the form

$$c_M\phi^{(M)}(\eta) + c_{M-1}\phi^{(M-1)}(\eta) + \ldots + c_1\phi'(\eta) + c_0\phi(\eta) = 0 \tag{25}$$

for certain constants c_j and $M \geq N$. The reason for this is that the general solution to an ODE like (25) is a linear combination of exponentials, $e^{p\eta}$, where $p \in \mathbb{C}$ is a root of the characteristic polynomial $x^M + \sum_{j=1}^{M} c_j x^j$. If the c_j are chosen so that the C_j^* are among the roots (possible if $M \geq N$) then $\phi(\eta)$ as defined by equation (23) will satisfy the ODE (25).

Recall that we can use our boundary data to compute $\phi(\eta)$ and its derivatives of any order. If we choose $M \geq N$ distinct values of η then equation (25) yields M linear equations with M unknowns c_j, which we can solve. As shown in [3] the rank of the resulting matrix gives the number of exponential terms in ϕ (which is the number of inclusions), provided we use a value of M which exceeds N, the true number of inclusions.

On N is determined we can then evaluate (25) for N distinct values of η and solve the resulting linear system of N equations in N unknowns for the c_j. Given the coefficients c_j, we can solve for the roots of $p(x)$ (the characteristic equation for the ODE) where

$$p(x) = x^N + \sum_{j=1}^{N} c_j x^j.$$

The roots of $p(x)$ are the centers of the inclusions. An example is given below.

Note that once we have recovered the centers C_j^*, we can use equation (23) to evaluate $\phi(\eta_k)$ for N distinct values of η_k and thereby recover the J_j by solving a system of linear equations.

7.2 The Jump Integral

As we stated earlier, we will present a method for finding either R_n or k_n, given that the other is known. The central result is the following lemma.

Lemma 5 *Let J be defined by equation (24) for the case $j = 1$ with a single inclusion ($N = 1$). Then J can be rewritten in terms of the center, radius, and transmission constant of the inclusion as*

$$|J| = \frac{2\pi R^2 \,|\nabla u_0(C^*)|}{1 + 2kR} + O(R^2).$$

Proof The proof of Lemma 5 closely mimics the proof of Lemma 4. We again use $u_0(x, y)$ for the harmonic function which has Neumann data g and set $v = u - u_0$. With $w_\eta(x, y) = \frac{1}{\eta} e^{\eta(x+iy)}$ we obtain $\nabla w_\eta =< 1, i > e^{\eta(x+iy)}$. Lemma 4 states that

$$RG(w_\eta) = \frac{2\pi R^2}{1 + 2kR} \nabla u_0(C^*) \cdot \nabla w_\eta(C^*) + O(R^4)$$

Taking the magnitude of the Reciprocity Gap functional and using the above function $w_\eta(x, y)$ yields

$$\begin{aligned} |RG(w_\eta)| &= \frac{2\pi R^2}{1 + 2kR} |\nabla u_0(C^*)| \, e^{\eta C^*} + O(R^4) && (26) \\ &= J e^{\eta C^*} + O(R^2) && (27) \end{aligned}$$

where we make use of (23) and (24). Comparing the right sides of (26) and (27) shows

$$|J| = \frac{2\pi R^2}{1 + 2kR} |\nabla u_0(C^*)| + O(R^2)$$

as asserted. ∎

For multiple inclusions, we also expect the integral J_j as defined by equation (24) to be (to leading order in R_j) of the form

$$|J_j| = \frac{2\pi R_j^2 \left|\nabla u_0(C_j^*)\right|}{1 + 2k_j R_j}. \tag{28}$$

We now have the integrals J_j in terms of C^*, R_j, and k_j. We recall our multiple inclusion Reciprocity Gap function, equation (23), and consider it as a function of η. We can calculate $\phi(\eta)$ and we now have C_j^*. Therefore if we choose N distinct values of η, we have N equations of N unknowns (those unknowns being J_j). Thus we can solve for each value of J_j.

Once we have the J_j and C_j^* we can use equation (28) to solve for R_j if we are given k_j or vice-versa.

8 A Numerical Example of Multiple Inclusions

We will now present a numerical example that illustrates how to locate N inclusions within a region. To solve the forward problem we use a similar program to the one we used to demonstrate our single inclusion case. We will give the program the center, the radius, and transmission constant for each inclusion B_j within D, where D is again the unit disk. For our example we will use two inclusions within D. Our first inclusion B_1 will be centered at $(0.4, 0.6)$ with a radius of $R_1 = 0.15$ and a transmission constant of $k_1 = 0.9$. Our second inclusion B_2 will be centered at $(-0.3, 0.7)$ with a radius of $R_2 = 0.1$ and a transmission constant of $k_2 = 0.75$. We again use input flux $g(\theta) = \sin(2\theta)$ (so the harmonic function with the boundary data is $u_0(x, y) = xy$).

Let us first consider the case in which we know the actual number of inclusions. We use a C program to find the value of $u(x, y)$ at a given number of points (50 to 100) along ∂D. We feed these values into a Matlab notebook, we are able to calculate $\phi(\eta)$ for any given η as well as the derivatives of $\phi(\eta)$. We will choose two values for η, $\eta_1 = 1$ and $\eta_2 = -1$, and have Matlab calculate the first two derivatives of $\phi(\eta_1)$ and $\phi(\eta_2)$. We use these values to solve the system of equations obtained from (25), specifically

$$c_2\phi''(\eta_1) + c_1\phi'(\eta_1) + c_0\phi(\eta_1) = 0$$
$$c_2\phi''(\eta_2) + c_1\phi'(\eta_2) + c_0\phi(\eta_2) = 0$$

where without loss of generality we may take $c_0 = 1$. Once we have values for c_2 and c_1 we can solve for the roots of the quadratic equation

$$c_2m^2 + c_1m + c_0 = 0. \tag{29}$$

In the present case we find $c_2 = -0.5596 + 0.1048i, c_1 = -0.1087 - 1.3224i$ (recall $c_0 = 1$). The roots of the characteristic equation (29) are $-0.3027 + 0.7072i$ and $0.4114 + 0.6152i$, quite close to the correct center values.

Once we have located the centers of the inclusions, we are easily able to calculate J_1 and J_2 by evaluating $\phi(\eta)$ for two values of η. In this case we take $\eta = 1$ and $\eta = -1$ and find $J_1 = 0.0706+0.0484i, J_2 = 0.0401-0.0167i$. From this we can calculate R or k, if we are given the other. Let us assume that we know $k_1 = 0.9$ and $k_2 = 0.75$. From equation (28) we calculate $R_1 = 0.1533$ and $R_2 = 0.1018$ (note we know that $\nabla u_0 =< y, x >$). Figure 2 below shows the accuracy of the reconstructions. Alternatively, we can consider R_1 and R_2 known and so estimate that $k_1 = 0.74$ and $k_2 = 0.565$, not quite as accurate as the radius estimation.

If we use equation (25) with a guess of $M = 5$ (and five distinct values for η; we use the fifth roots of unity) we find that the resulting linear system for the c_j has rank two. Specifically, a singular value decomposition of the five by five matrix obtained from equation (25) yields two non-zero singular values 0.4665 and 0.0663; the remaining values are less than 10^{-5}. This indicates that only two inclusions are present, and we can proceed as above.

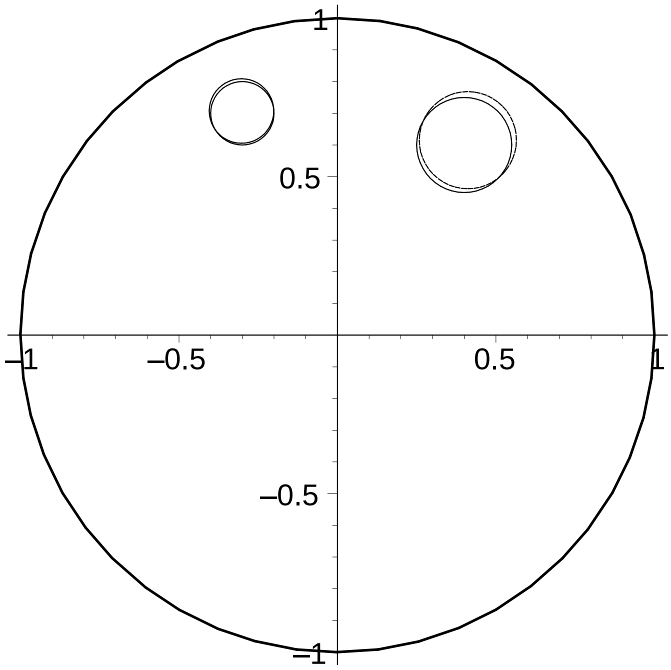

Figure 2: Reconstruction of two inclusions.

9 Conclusion and Future Work

We used the reciprocity gap approach with carefully chosen test functions to reduce the problem of identifying the centers of one or more inclusions to that of identifying the coefficients in a function ϕ which is a sum of exponentials. This is easily done by noting that such functions satisfy very simple ODE's. The radii of the inclusions are obtained by deriving a relation between the "jump" coefficients J_n which appear in ϕ and the radii. This relation involves the transmission coefficient for each inclusions, which we must consider known. However, in the case of a single inclusion a more careful analysis shows that we can recover both the radius and transmission coefficient. The k value was found to be more sensitive to noise than was R.

Future work should be done to stabilize estimates of k.

It would be desirable to find a way to get both R and k simultaneously for the multiple inclusion case since there are more real world applications for that scenario. We would also like to find an algorithm for single and multiple inclusions in $\mathbb{R}^3$. Another case to consider would be a case where the material on the outside of the inclusion boundary is different from the material on the inside of the inclusion boundary. This would mean that $\frac{\partial u^+}{\partial r} = \alpha \frac{\partial u^-}{\partial r}$ for some constant α, that is, the material on the outside of the inclusion has a different thermal conductivity than the material on the inside of the inclusion.

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