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FRACTALS AND THE WEIERSTRASS-MANDELBROT FUNCTION

Anthony Zaleski

Abstract. The Weierstrass-Mandelbrot (W-M) function was first used as an example of a real function which is continuous everywhere but differentiable nowhere. Later, its graph became a common example of a fractal curve. Here, we first review some basic ideas from measure theory and fractal geometry, focusing on the Hausdorff, box counting, packing, and similarity dimensions. Then we apply these to the W-M function. We show how to compute the box-counting dimension of its graph, and discuss previous attempts at proving the not yet completely resolved conjecture of the equality of its Hausdorff and box counting dimensions. We also consider a surface generalization of the W-M function, compute its box dimension, and discuss its Hausdorff dimension.

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1 Introduction

Loosely speaking, a fractal is a set which is in some way irregular and fundamentally different from the usual lines, balls, and smooth curves that we are accustomed to. While “normal” sets of the plane behave predictably when viewed on decreasing scales (e.g., a smooth curve looks like a line when viewed up close), fractals may reveal hidden detail when viewed on increasingly fine scales. Such intricate structure can be used to model forms both occurring in nature and produced by industrial means; for example, both coastlines and machined surfaces exhibit fractal detail (see Edgar [5] and Blackmore [3], and more popular treatments such as Barnsley [2] and Mandelbrot [10]).

But how do we rigorously define the level of “irregularity” of a set? The key tool needed here is the fractal dimension. It is defined to equal the normal topological dimension for familiar sets: a straight line has fractal dimension one; a cube has fractal dimension three. But as its name implies, this new dimension can also attain fractional values. For example, as we shall see, the graph of a continuous function of one variable (of topological dimension one) can be so “jagged” that its fractal dimension exceeds unity. Hence, the fractal dimension is a finer measure of the complexity of a set. Although there is no single strict definition of a fractal, the famous mathematician Benoit Mandelbrot suggested that the name be applied to a set with non-integer fractal dimension (Edgar [5], Falconer [7]).

This paper aims to accomplish two things. First, we wish to give the reader an understandable introduction to the various fractal dimensions which are commonly used. Along the way, we will take time to go over some interesting elementary examples. Secondly, we wish to discuss the famous open problem of finding the Hausdorff dimension of a specific fractal function.

In Section 2 we give an introduction to fractal dimension. First we define the similarity dimension, which is easy to compute but only applicable to certain self-similar fractals. We compute this dimension for the Cantor set and Sierpiński Carpet. Then we introduce the more versatile box counting dimension and a two-dimensional analogue of the Cantor set. Finally, we define the Hausdorff dimension, which is more abstract but has nicer properties than the box counting dimension. To underline the difference between the box counting and Hausdorff dimensions, we shall provide an example of a fractal with a Hausdorff dimension different from its box dimension. It should be noted that, while many texts (cf. [6]) clearly state that these dimensions need not be equal, we have not yet found a specific example showing this, besides ours, which we thus assume is fairly novel.

In discussing these simple fractal examples, our approach will be fairly intuitive, although we shall always try to keep a rigorous foundation by citing references where proofs are omitted. Hence we avoid a lengthy introduction to Lebesgue and fractal measures by instead referring to the intuitive concepts of length, area and volume and then extending them to fractal measures. Our goal is to motivate and prepare the reader for the subsequent discussion of the Weierstrass-Mandelbrot function and its surface generalization, while remaining concise and accessible.

After this expository material, we shall be ready to look into the main problem this paper addresses: namely, finding the Hausdorff dimension of the Weierstrass-Mandelbrot function (or “W-M function” for short). The original Weierstrass functions were defined by German mathematician Karl Weierstrass in 1872 for reasons other than their fractal properties [5].
Namely, they served as a counterexample to the long-held belief that a continuous function in the x-y plane could only fail to be differentiable at a set of isolated points. Amazingly, Weierstrass’ functions are continuous everywhere and differentiable nowhere. Later on Mandelbrot investigated the fractal properties of the graph of Weierstrass’ functions; hence the longer hyphenated name.

In Section 3 we define the Weierstrass-Mandelbrot functions of a single variable and present some graphs. We briefly discuss some of its well-known properties, referring the reader to [5] for more information. Then we move on to its fractal properties, presenting a derivation of its box dimension adapted from [7]. This proof will be covered in its entirety, but, as usual, we shall try to point out the underlying geometric intuition which leads to the steps taken.

Then we discuss the long-standing open problem of finding the Hausdorff dimensions of the W-M functions. We note several of the (partly successful) attempts to solve this problem, including some very recent ones [1, 4, 8].

In Section 4 we introduce a surface generalization of the W-M function that is a special case of the family of functions used to analyze machined surfaces covered by Blackmore and Zhou in [3]. As before, we shall show some plots of the surface, derive its box dimension, and state an analogous conjecture about its Hausdorff dimension.

Finally, in Section 5 we give a brief summary and give our thoughts on possible future work. Section 6 lists references, including both introductory texts on fractals and research papers related to the Weierstrass-Mandelbrot function.

2 Fractal Dimension

In this section we introduce the reader to commonly used fractal dimensions. First, we discuss the similarity dimension, applicable to the class of self-similar sets—sets which consist of scaled copies of themselves. Then we look into the more versatile box counting dimension, and finally the more abstractly defined Hausdorff and packing dimensions.

We shall for the most part limit our discussion to sets in the real line \( \mathbb{R} \) and the plane, \( \mathbb{R}^2 \) although these definitions can be extended to general metric spaces. Also, the distance between two points \( x \) and \( y \) in a set will be denoted \( d(x, y) \). We shall use the following standard mathematical notation: iff means if and only if; \( \forall \) denotes for all; \( \exists \) stands for there exists; s.t. is an abbreviation for such that; \( \Rightarrow \) denotes implies, \( \Leftrightarrow \) stands for equivalent, and rhs and lhs mean right hand side and left hand side, respectively.

**Definition 2.1.** A mapping \( s : A \to s(A) \) is a similarity iff there exists an \( r > 0 \) s.t. \( d(s(x), s(y)) = rd(x, y) \) for all \( x, y \in A \). In this case, \( r \) is called the ratio of the similarity; if \( r < 1 \), then \( s \) is a contraction.

Geometrically, a similarity is a transformation that preserves the shape of an object, but not necessarily the size. Similarities map squares to squares and rectangles to rectangles. It can be shown that a mapping is a similarity iff it can be expressed as a composition of scaling, translation, rotation, and reflection transformations [7].
Example 2.2. Define a mapping in $\mathbb{R}^2$ by $s(x, y) = (x/2, -y/2)$. Then $s$ is a contracting similarity with ratio $r = 1/2$, since

$$d(s(x), s(y)) = d((x_1, x_2), (y_1, y_2)) = \frac{1}{2} \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} = \frac{1}{2} d(x, y) \forall x, y \in \mathbb{R}^2.$$ 

The mapping $s$ affects a set by reflecting it across the $x$-axis and scaling it by $1/2$. For example, $s$ takes the unit square onto $s([0,1]^2) = [0,1/2] \times [-1/2,0]$. □

The similarity dimension is defined for self-similar sets, or sets which may be expressed as a union of similar copies of themselves. We motivate the definition as follows.

As stated before, we want our dimension—call it $\dim$—to agree with the expected integer for simple sets. For example, we want $\dim_s [0,1] = 1$. Now the interval $[0,1]$ is obviously self-similar: it can be divided into two scaled copies of itself, $[0,1/2]$ and $[1/2,1]$. The two similarities relating the whole to its smaller parts both have ratio $1/2$. Note $(1/2)^1 + (1/2)^1 = 1$.

Similarly, we can express the unit square as a union of four quarters:

$$[0,1]^2 = [0,1/2] \times [0,1/2] \cup [1/2,1] \times [0,1/2] \cup [0,1/2] \times [1/2,1] \cup [1/2,1] \times [1/2,1].$$

Here, we have four similarities, each of ratio $1/2$. Note $4(1/2)^2 = 1$.

Finally we may divide a unit cube into 27 smaller cubes, each of side $1/3$. Adding the cubes of the similarity ratios, we once again find $27(1/3)^3 = 1$.

There is a pattern here: each time we sum the similarity ratios, raised to a certain exponent, we get 1. This exponent turns out to be the (usual) dimension of the set. Now say we are given some arbitrary set. If we can break it into smaller similar copies of itself, with similarity ratios $\{r_i\}$, there is a unique $s$ satisfying $\sum (r_i)^s = 1$. We define this exponent (which is not necessarily an integer) to be the similarity dimension:
**Definition 2.3.** Suppose a set $F$ is *self-similar*; i.e., suppose $F = \bigcup s_i(F)$, where $\{s_i\}$ is a finite set of contracting similarities with respective ratios $\{r_i\}$. Then the similarity dimension of $F$ is 

$$\dim_s F := s, \quad \text{where } s \in \mathbb{R} \text{ satisfies } \sum r_i^s = 1.$$  

One major drawback to the similarity dimension is that it is not unique. For example, note $[0,1] = [0,\frac{2}{3}] \cup [\frac{1}{3},1]$. Here, the ratios are 2/3; solving $2(2/3)^s = 1$, by taking the natural log of each side, we obtain

$$s = -\frac{\log 2}{\log(2/3)} = \frac{\log 2}{\log 3 - \log 2} \approx 1.7 > 1.$$  

This shows that to get reasonable results for the similarity dimension, we should find similarity relations with minimum overlap.

We now apply our definition to what is probably the best known example of a fractal on the real line.

**Example 2.4: The Cantor Set.** Define $C_0 := [0,1]$, and for $k \in \mathbb{N}$, define $C_k$ to be what remains after removing the middle thirds (excluding endpoints) from the intervals comprising $C_{k-1}$. For example, $C_1 := [0,1/3] \cup [2/3,1]$.

We want to define the Cantor set $C$ as the limit of the set $\{C_i\}$. Since the $C_i$ are obviously decreasing ($C_0 \supset C_1 \supset \ldots$), it makes sense to define their limit as an intersection:

$$C := \cap C_i.$$  

Thus, the Cantor set consists of the points common to *all* of the $C_i$. For example, it is easy to see that the endpoints of any $C_i$ are contained in $C$. And since the lengths of any intervals in some $C_i$ limits to zero, we see that $C$ does not contain any intervals; it is very “fine” or “dust-like” and completely or totally disconnected in mathematical parlance.

To get a better understanding of $C$, we use *base 3 (ternary) notation* (see [6]).

**Definition 2.5.** The base 3 representation of a number $x \in [0,1]$ is $x = (a_1 a_2 \ldots)_3$, where the $a_k \in \{0,1,2\}$ are chosen so $\sum_{k=1}^\infty 3^{-k} a_k = x$.

Note this definition is not unique for certain numbers; for example, $(.1)_3 = 1/3 = 2(1/9 + 1/27 + \ldots) = (.0222\ldots)_3$.

**Lemma 2.6.** $C_i = \{x \in [0,1] : x = (a_1 a_2 \ldots)_3, a_k \neq 1 \forall k \leq i\}$
Proof by induction. The equality obviously holds for $i = 0$. Now assume it holds for some $i$—that is, assume $C_i$ consists of all numbers in $[0, 1]$ not having 1 in the first $i$ digits of their ternary expansions. Then $C_i = \bigcup_{n_1, a_2, \ldots, a_i \in \{0, 2\}} ((a_1 a_2 \ldots a_i) + [0, 1/3^i]),$ a union of intervals. (Note $[0, 1/3^i] = [0, (0...0222...)]$, where 2 is the $(i + 1)$th digit.) The middle thirds of these intervals are:

$$(a_1 a_2 \ldots a_i) + [1/3^{i+1}, 2/3^{i+1}] = (a_1 a_2 \ldots a_i) + [(0...01), (0...02)]$$

$$= (a_1 a_2 \ldots a_i) + [(0..., 01), (0..., 02)]_{i+1}.$$ 

So when we remove these, we are removing all numbers with 0 or 2 in the first $i$ digits of their expansions, and 1 in the $(i + 1)$th digit. What remains is $C_{i+1}$, the set of all numbers not having 1 in their first $i + 1$ digits.

\end{proof}

Theorem 2.7. $C$ consists of all numbers in $[0, 1]$ having ternary expansions without 1’s.

Proof. Let $E := \{x \mid x = (a_1 a_2 \ldots a_i)_{3}, a_k \neq 1 \forall k\}$. We want to show $C = E$.

First, we show $C \subseteq E$. Suppose $x \in C$. Then $x \in C_i$, so by the lemma, $x$ cannot have a 1 in the first $i$ digits of its ternary expansion. Since this holds for all $i$, we see none of the digits can be 1. Thus $x \in E$.

To show $E \subseteq C$, suppose $x \in E$. Then $x \in C_i$ for all $i$ by the lemma, so $x \in C = \cap C_i$.

From this it is clear that $C$ is uncountable. In fact, it has the cardinality (\#) of the continuum:

Remark 2.8. \#$C = \#R = 2^{\aleph_0}$.

Proof. Suppose $x \in C$ has ternary expansion $a_1 a_2 \ldots$. Since each $a_i$ is either 0 or 2, we have an associated binary expansion

$$\varphi(x) = \varphi\left(\sum_{k=1}^{\infty} a_k 3^{-k}\right) := \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k} = (b_1 b_2 \ldots)_2,$$

where each $b_i := a_i / 2$ is either 0 or 1.

The mapping $\varphi : C \to [0, 1]$ is injective (one-to-one) by the uniqueness of the binary expansion. It is surjective (onto) by the previous theorem and the fact that any $x \in [0, 1]$ has a binary expansion $b_1 b_2 \ldots$, where each $b_i$ is either 0 or 1. So $\varphi$ is a bijection, and \#$C = \#[0, 1] = \#R$.

Now, by considering the lengths of each $C_i$, one finds that $C$ has length (i.e., one-dimensional Lebesgue measure) zero; further, its topological dimension is also zero [6]. Yet $C$ contains significantly “more” than a zero-dimensional countable set, as our remark illustrates. We would like a better characterization of its dimension. Here is where the similarity dimension becomes useful.
**Theorem 2.9.** \( \dim_s C = \log 2 / \log 3 \).

**Proof.** We assert that the Cantor set satisfies the similarity relation \( C = C / 3 \cup (C / 3 + 2 / 3) \). To see why this holds, note \( C / 3 \) is the set of all numbers \((.0a_1a_2...)_3, a_i \neq 1\), and \( C / 3 + 2 / 3 = C / 3 + (.2)_3 \) is the set of all numbers \((.2a_1a_2...)_3, a_i \neq 1\). It is now clear that the union of these is \( C \), the set of all numbers \((.a_1a_2...)_3, a_i \neq 1\).

We have expressed \( C \) in terms of two similarities of ratio 1/3. To find \( \dim_s C = s \), we solve

\[
2 \left( \frac{1}{3} \right)^s = 1 \iff \log 2 - s \log 3 = 0 \iff s = \frac{\log 2}{\log 3} \approx .63.
\]

As expected, this value lies between 0 and 1. \( \square \)

Next we examine an interesting set in the plane.

**Example 2.10: The Sierpiński Carpet.** We construct this fractal using a method similar to that of the previous example. Now, instead of starting with an interval, we start with a square, \( S_0 := [0,1]^2 \). To get \( S_1 \), we divide \( S_0 \) into 9 smaller squares, and remove the center one, but spare the boundary to ensure we have a closed set. Note \( S_1 \) consists of 8 copies of \( S_0 \), all scaled by 1/3. Thus, to get \( S_2 \), we can apply the previous procedure to each of the squares of side 1/3. In general, to get \( S_i \) from \( S_{i+1} \) we partition each square comprising \( S_i \) into 9 smaller congruent ones, and remove the centers. The Sierpiński Carpet is the limit of \( \{S_i\} \), i.e., \( S := \cap S_i \).

To compute the similarity dimension of \( S \), note that \( \forall i, S_{i+1} = \bigcup_{k=1}^8 s_k(S_i) \), where the \( s_i \) are the 8 similarities of ratio 1/3 which map the unit square into the different pieces of \( S_1 \). Taking the limit of each side, we obtain \( S = \bigcup_{k=1}^8 s_k(S) \) (for a more rigorous explanation, see [6]). So to find \( s = \dim_s S \), we just solve

\[
8(1/3)^s = 1 \iff \log 8 - s \log 3 = 0 \iff s = \frac{\log 8}{\log 3} \approx 1.90.
\]

To motivate the definition of the box-counting dimension, a more generally applicable definition of fractal dimension, we go back to a previous example.
Before, we noted that \([0,1]\) can be split into 2 congruent parts similar to the whole. If we break it into three equal parts, each will have length \(1/3\), and in general, we can split it into \(N\) equal parts of length \(1/N\).

As before, we can extend this to a unit square, which can be split into 4 squares of side \(1/2\), 16 squares of side \(1/4\), and in general, \(N^2\) squares of side \(1/N\).

Finally, a cube can be split into \(N^3\) smaller congruent cubes of side \(1/N\).

In each example, if we let \(N_\delta\) be the minimum number of boxes of side \(\delta = 1/N\) which can cover the set, we obtained the relation \(N_\delta \sim \delta^{-d}\), where \(d\) is the dimension. The higher the dimension of the object, the more tiny boxes it takes to fill it.

The advantage here is that \(N_\delta\) can be computed for any set: just cover it with a grid of boxes with side \(\delta\), and count how many intersect the set. Once we do this, we need to find how \(N_\delta\) is related to \(\delta\) – in particular, if it behaves like a power of \(\delta\). To do this, note that \(N_\delta \sim \delta^{-d} \iff \log \frac{N_\delta}{\log \delta} = d\).

Now since our set will not necessarily be as simple as a line or square, we do not expect the above to exactly hold for some \(d\). However, we expect it to hold in the limit of small \(\delta\). But since the limit may not exist, we instead use upper (limit supremum) and lower (limit infimum) limits as follows.

**Definition 2.11.** Let \(E \subset \mathbb{R}^n\). Define \(N_\delta\) to be the minimum number of \(\delta\)-boxes (\(n\)-dimensional rectangles of equal side \(\delta\)) that cover \(E\). Then the **upper and lower box dimensions** of \(E\) are respectively

\[
\overline{\dim}_\delta E := \overline{\lim}_{\delta \downarrow 0} \left( -\frac{\log N_\delta}{\log \delta} \right),
\]

\[
\underline{\dim}_\delta E := \underline{\lim}_{\delta \downarrow 0} \left( -\frac{\log N_\delta}{\log \delta} \right).
\]

If these are the same [i.e., if the limit \(\lim_{\delta \downarrow 0} (-\log N_\delta / \log \delta)\) exists], the common value is the **box dimension** \(\dim_\delta E\).

There are a few more technicalities here. Notice that we have \(\delta\) going to 0 from the plus side. This is because it does not make sense to talk of a box with a side of negative length. In fact, because we are interested in the limit of small \(\delta\), we need only compute \(N_\delta\) for a decreasing sequence of \(\delta\)-values that converges to 0 [7].
Also, our $N_\delta$ is defined slightly differently than it was previously—before, we said $N_\delta$ could be determined by immersing $E$ in a grid and counting how many boxes it intersects. Actually, we need not worry; it can be shown that both definitions of $N_\delta$ produce the same box dimensions, since in the limit, they approach each other [7]. In fact, there are many other ways to compute $N_\delta$; see [7].

For simple fractals like the (self-similar) ones we have discussed so far, the box dimension is easy to calculate, and gives the same value as the similarity dimension:

**Remark 2.12.** $\dim_b C = \dim_s C$ and $\dim_b S = \dim_s S$.

*Proof.* $C_0$ is covered by exactly one interval of length 1. This also covers $C$. So $N_1 = 1$. Similarly, by looking at $C_1$, we find that two intervals of length 1/3 are needed to cover $C$; hence $N_{1/3} = 2$. Continuing, we find that $N_{(1/3)^n} = 2^n$. We do not need to find any more $N$’s since $\delta_n := (1/3)^n \downarrow 0$. Now, in this case it is obvious that $N_\delta$ obeys a power law, but we will use the limit definition:

$$\lim_{\delta \downarrow 0} \left( - \frac{\log N_\delta}{\log \delta} \right) = \lim_{n \to \infty} \left( - \frac{\log N_{\delta_n}}{\log \delta_n} \right) = \lim_{n \to \infty} \left( - \frac{n \log 2}{-n \log 3} \right) = \frac{\log 2}{\log 3} = \dim_s C.$$  

In an analogous way, we can show that the Sierpiński Carpet is covered by $8^n$ squares of side $1/3^n$. Thus

$$\dim_b S = \lim_{n \to \infty} \left( - \frac{n \log 8}{-n \log 3} \right) = \frac{\log 8}{\log 3} = \dim_s S. \qed$$

Now let us look at a non-self-similar example where the box dimension is nevertheless still applicable. In our calculations, we shall let $N_{A(\delta)}$ denote the number of $\delta$-boxes required to cover a set $A$.

**Example 2.13: A Picture Frame with Cantor Cross-Sections.** Here we find the box dimension of $E := \bigcup \{s : s$ is a square centered at $(0,0)$ and $s \cap (C + 1) \neq \emptyset\}$.

Let $\delta = 1/3^n$, and $N_{E(\delta)}$ be the number of $\delta$-boxes it takes to cover $E$. Then, for small $\delta$, $N_{E(\delta)} \sim 4N_{F(\delta)}$, where $F := \{(x, y) \in E : x \geq 0, |y| \leq x\}$. Now

$$(C+1) \times [-1, 1] \subset F \subset (C+1) \times [-2, 2],$$

so
\begin{align*}
\frac{2}{\delta} N_{C(\delta)} \leq N_{F(\delta)} \leq \frac{4}{\delta} N_{C(\delta)}.
\end{align*}

(Think of stacking boxes in the \(y\)-direction.) Thus, for small \(\delta\), \(N_{E(\delta)} \sim 4N_{F(\delta)}\) gives
\begin{align*}
\frac{8}{\delta} N_{C(\delta)} \leq N_{E(\delta)} \leq \frac{16}{\delta} N_{C(\delta)} \Rightarrow \\
\frac{\delta}{16N_{C(\delta)}} \leq \frac{1}{N_{E(\delta)}} \leq \frac{\delta}{8N_{C(\delta)}} \Rightarrow
\end{align*}

\begin{align*}
\log \delta - \log 16 - \log N_{C(\delta)} \leq -\log N_{E(\delta)} \leq \log \delta - \log 8 - \log N_{C(\delta)} \Rightarrow \\
1 - \frac{\log 16}{\log \delta} - \frac{\log N_{C(\delta)}}{\log \delta} \leq \frac{\log N_{E(\delta)}}{\log \delta} \leq 1 - \frac{\log 8}{\log \delta} - \frac{\log N_{C(\delta)}}{\log \delta},
\end{align*}

\(\delta \downarrow 0 \Rightarrow\)

\begin{align*}
1 + \dim_b C \leq \dim_b E \leq 1 + \dim_b C \Rightarrow \quad \dim_b E = 1 + \frac{\log 2}{\log 3}.
\end{align*}

This example gives rise to two questions about the box dimension. The first comes from the observation that we were able to break \(E\) into four pieces, each having the same dimension as \(E\). So it is reasonable to ask: What is the box dimension of a union? From this example, we might conjecture that a union of sets of the same dimension will also have the same dimension. Also, note that \(\dim_b C \times [-1,1] = \dim_b C + 1 = \dim_b C + \dim_b [-1,1]\). Is this always the case? That is, is the dimension of a product the sum of the dimensions? Actually, in many cases, both of our guesses are true. For more on this, see [7].

Each of our previous definitions of fractal dimension involved analyzing the complexity of a set on smaller and smaller scales. The Hausdorff dimension does the same; but it is arrived at in a very different, somewhat more abstract, way. It is constructed using measures, which are generalized concepts of length, area, and volume which apply to a wide variety of metric spaces. Thus, the Hausdorff dimension is defined for a broad class of sets; and it also turns out that it has nicer theoretical properties than the simpler dimensions described above. However, due to its rather complicated abstract definition, the Hausdorff dimension is usually difficult to compute for just about all except the simplest sets.

To see why a measure theoretic approach is plausible, consider the usual \(n\)-dimensional Lebesgue measure, essentially obtained by minimizing the sum of the volumes of a countable cover of \(n\)-dimensional rectangles (for more on measures, see Royden [12]). If we denote \(n\)-dimensional Lebesgue measure by \(L^n\), then \(L^1\) is the length of a set on the real line; \(L^2\) is the area of a set in the plane; and \(L^3\) is the volume of a set in three-space (\(\mathbb{R}^3\)).
How do we know what \( n \) to choose when finding \( L^n \) for a certain set? For example, let \( E := [0,1]^2 \), and say we want to find the Lebesgue measure. Then \( L^1 \) is the minimum sum of the lengths of line segments (1-dimensional rectangles) covering \( E \). But this is infinite; one-dimensional rectangles are too “fine” to create a countable covering of \( E \). So we try \( n = 2 \). Of course, this works; \( E \) itself is a covering of two-dimensional rectangles, and we find \( L^2 (E) = \text{Area}(E) = 1 \). If \( n \) is greater than 2, it is easy to see that \( L^n (E) = 0 \); for example, \( L^3 (E) = 0 \) since a rectangle has zero volume.

In short, there is a unique value of \( n \) for which \( L^n (E) \) is nonzero and finite. In the case just considered, the critical value of \( n \) was the dimension.

If we want to generalize this, we must first create a \( d \)-dimensional measure, where \( d \) can be non-integer. As with the Lebesgue measure, we cover the set with many small sets of known measure, and then minimize the sum of their measures. The two main differences are: (1) the small sets are \( n \)-dimensional balls which need not be of the same diameter; and (2) their individual measures are defined to be their diameters raised to the power \( d \).

**Definition 2.14.** A \( \delta \)-covering of a set \( E \) is a countable covering of balls with diameter less than or equal to \( \delta \) - i.e., a countable collection of balls \( \{B_i\} \) s.t. \( E \subset \bigcup B_i \) and \( (\text{diam} B_i) \leq \delta \) for all \( i \).

**Definition 2.15.** For a set \( E \subset \mathbb{R}^n \) define

\[
H^d_\delta (E) := \inf \left\{ \sum (\text{diam} B_i)^d : \{B_i\} \subset \mathbb{R}^n, \text{is a } \delta \text{-cover of } E \right\}.
\]

The \( d \)-dimensional Hausdorff measure of \( E \) is

\[
H^d (E) := \lim_{\delta \to 0^+} H^d_\delta (E) = \sup \{H^d_\delta (E) : \delta > 0\}.
\]

At first, it may be hard to geometrically picture what is going on here. Let us break it down into parts.

\( H^d_\delta \) is the minimum sum of measures of \( n \)-balls forming a \( \delta \)-cover of \( E \). To give a measure accurate on small scales, we want \( \delta \) to tend to zero. For example, picture an infinitely long spiral contained in a ball of diameter 1 = \( H^1_1 \). Since we do not want the 1-measure to be one, we must sum the diameters of tiny balls which cover the spiral; in the limit, this gives us the length.

Now since \( H^d_\delta \) is a minimum of all coverings with balls of diameter \( \leq \delta \), it must increase as \( \delta \) decreases, since this means we have less sizes of balls to choose from to create a minimal cover. That explains the equality \( \lim_{\delta \to 0} H^d_\delta (E) = \sup \{H^d_\delta (E) : \delta > 0\} \). Since the supremum of a set in \( \mathbb{R} \) is always a real number (or infinity), we know \( H^d (E) \in \mathbb{R} \cup \{\infty\} \) is always well-defined.
It turns out that $H^1$ just gives us the length of a set. $H^2$ is similar to the area measure, but differs by a constant multiple. For example, $H^2(B_1) = \pi \neq \mathbb{L}^2(B_1) = \pi$. In fact, the (integer) Hausdorff measure is always equal to the Lebesgue measure, up to a constant multiple\(^1\).

Now, from our discussion of the Lebesgue measure, we would expect $H^d(E)$ to be nonzero and finite only for some critical value of $d$, which we could naturally define to be the dimension. In fact, this is always the case; for a proof, see [7].

**Definition/Theorem 2.16.** Let $E$ be a set in a metric space. Then there exists a $d \in \mathbb{R}^+ \cup \{\infty\}$ such that

$$H^d(E) = \begin{cases} \infty, & t < d \\ c < \infty, & t = d \\ 0, & t > d \end{cases}$$

This $d$ is called the **Hausdorff dimension** of $E$, and we write $\dim_H E = d$.

It can be shown that we always have $\dim_H (E) \leq \dim_B (E)$ [7]. In most cases, we actually have equality. But there are borderline examples where by choosing small coverings of balls of different sizes, we can show $H^{\dim_B} = 0$, and hence $\dim_H (E) < \dim_B (E)$. This happens because the box counting dimension only considers covers with balls of equal size.

First, let us look at an example where equality holds.

**Example 2.17.** Show $\dim_B C = \dim_H C = \log 2 / \log 3 = s$, where $C$ is the Cantor set.

For a cover, choose the intervals in $C_i$. This gives us $H^s \leq 2^i (1/3^i)^{\log 2/\log 3} = 2^i (1/3^i)^{\log 2} = 1$.

Now, let $\{U_i\}$ be any finite cover of closed intervals (by compactness, we can always choose a finite subcover of an open cover, and then expand the intervals slightly to get a countable closed cover). The sum of the measures of these intervals can be made arbitrarily close to that of the original cover.) Now, for each $U_i$, there exists a $k_i$ s.t. $I \subset U_i$, where $I$ is one of the intervals of $C_k$. Let $k = \min\{k_i\}$. Then each interval of $C_k$ is contained in some $U_i$. So

---

\(^1\) Some authors even insert a multiplicative constant in the definition of Hausdorff measure, so that the two measures are exactly the same for integer values of $d$. 

\[
\sum (\text{diam} U_i)^\varepsilon \geq \sum \text{diam} I_i = 1.
\]

Hence, \( H^\varepsilon (C) = 1 \), which is finite and nonzero. Thus \( \dim_H C = s = \log 2 / \log 3 \).

Next, we look at an example where \( \dim_H (E) < \dim_b (E) \).

**Example 2.18: A Set with Non-equal Box and Hausdorff Dimensions.** Consider the rationals, \( \mathbb{Q} \). Since \( \mathbb{Q} \) is countable, we can easily show \( \dim_H \mathbb{Q} = 0 \). Given \( s, \varepsilon > 0 \), we can choose a countable cover \( \{ U_i : i = 1, 2, ... \} \) of intervals centered at points in \( \mathbb{Q} \), such that \( (\text{diam} U_i)^\varepsilon = \varepsilon / 2^i \), and so \( \sum (\text{diam} U_i)^\varepsilon = \varepsilon \). As \( \varepsilon \downarrow 0 \), \( \delta := \max \{ \text{diam} U_i \} \rightarrow 0 \), and \( \sum (\text{diam} U_i)^\varepsilon \rightarrow 0 \).

Hence, \( H^\varepsilon (\mathbb{Q}) = 0 \). This holds for all \( s > 0 \), so we find \( \dim_H \mathbb{Q} = 0 \).

However, given \( \delta = 1/n \), we shall always need at least \( n \) boxes (intervals) of length \( \delta \) to cover \( \mathbb{Q} \) (this follows from the density of the rationals: since the intervals have nonzero length, to cover \( \mathbb{Q} \) they must actually cover \( \mathbb{R} \)). So \( \dim_b \mathbb{Q} = 1 \).

This example shows how much more efficient it is to allow varying sizes for the elements of the cover. But it is trivial because it is not a fractal (both dimensions are integers). Can we find another example having non-integer box and Hausdorff dimensions? The answer is yes. We shall give the idea behind constructing such an example, and then present a specific one.

From the last example, it is easy to see that intersecting a set with \( \mathbb{Q} \) changes the Hausdorff dimension to 0 but leaves the box dimension the same. Let \( C \subset \mathbb{R} \) be some fractal, and \( D = C \cap \mathbb{Q} \). Then \( D \) has fractional box dimension, but we need to make its Hausdorff dimension—which is zero now—fractional. So we choose another fractal \( E \) satisfying \( 0 < \dim_b E = \dim_H E < \dim_b D \), and form \( F = D \cup E \). We now have \( 0 < \dim_H F = \dim_b E < \dim_b D = \dim_b F \) by the properties of the union operator (see [6]).

**Example 2.19: A Fractal with Non-equal Box and Hausdorff Dimensions.** Let \( C \) be the *middle fifths Cantor set*, obtained by starting with \([0, 1]\) and removing the middle fifth, then removing the middle fifths of the two remaining intervals, and so on. Since \( C \) consists of two copies of itself scaled by \( 4/5 > 1/3 \), one would expect \( C \) to have dimension greater than that of the regular middle thirds Cantor set. In fact, using the methods we covered, it can be shown that \( \dim_b C = (\log 2) / (\log 5 - \log 2) > \log 2 / \log 3 = \dim_H E \),

where \( E \) is the middle thirds Cantor set. Using \( C \) and \( E \) in the construction above gives us our example.
For completeness, we look at one more common definition of fractal dimension. The packing dimension is similar to the Hausdorff dimension, but uses a measure involving maximal “packings” by disjoint balls.

**Definition 2.20.** For a set \( E \subseteq \mathbb{R}^n \) define

\[
P^s_\varepsilon(E) := \sup \left\{ \sum (\text{diam } B_i)^s : \{B_i\} \text{ is a collection of disjoint balls of radius } \leq \varepsilon \text{ with centers in } E \right\}.
\]

Let

\[
P^s(E) := \lim_{\varepsilon \downarrow 0} P^s_\varepsilon(E) = \inf\{P^s_\varepsilon(E) : \varepsilon > 0\}.
\]

The \( s \)-dimensional packing measure of \( E \) is

\[
P^s(E) := \inf \left\{ \sum P^s(E_i) : E \subseteq \bigcup E_i \right\}.
\]

Note the extra step here. We need the last part because it can be shown that \( P^s \) does not satisfy the properties required for a measure [12]. For example, \( P^s([0,1] \cap \mathbb{Q}) = 1 \), while the measure of a countable union of sets of measure 0 should be 0. That is why we break \( E \) into a countable number of pieces, sum the measures of those, and take the infimum, to get the final packing measure.

The packing dimension now comes naturally:

**Definition 2.21.** For a set \( E \subseteq \mathbb{R}^n \), the **packing dimension** is

\[
\dim_p E = \inf \left\{ s : P^s(E) = 0 \right\}.
\]

### 3 The Weierstrass-Mandelbrot Function

Now that we have covered the basics of fractal geometry, we may apply them to an interesting example: the graph of the Weierstrass-Mandelbrot (W-M) function. In this section define the original W-M function, show how to compute the box dimension of its graph, and finally look at previous attempts to find its currently unknown Hausdorff dimension.
The W-M function was first introduced by the German mathematician Karl Weierstrass in 1872 as a counterexample to the belief that a function could fail to be differentiable only on a set of isolated points [5].

The W-M function is defined on the interval [0,1]. It is continuous everywhere, yet differentiable nowhere. Hence, while its graph is connected, it looks “jagged” when viewed on arbitrarily small scales. This fine detail suggests that it is a fractal; and as we shall see, it is. Mandelbrot was the first to point out the fractal nature of Weierstrass’ function, which accounts for its longer hyphenated name.

The function is constructed as an infinite series of oscillating terms which increase in frequency (to ensure the slope is not well-defined) and decrease in amplitude (to ensure continuity).

There are actually an infinite number of W-M functions, since our definition depends on two parameters:

**Definition 3.1.** Fix $\lambda > 1$ and $1 < s < 2$. The associated W-M function $f : [0,1] \rightarrow \mathbb{R}$ is

$$f(x) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k x).$$

While we shall only focus on the fractal properties of this function, the proofs of its continuity and nowhere-differentiability are interesting, and may be found in [5].

To see what this function looks like, we can plot some partial sums. The following pages show plots for varying parameter values.
First 200 terms, with $\lambda = 1.500$ and $s = 1.200$
First 200 terms, with lambda= 1.500 and s= 1.500
First 200 terms, with \( \lambda = 1.500 \) and \( s = 1.800 \)
In these cases, it is clear that the complexity of the graph increased as $s$ was increased. This hints that the $s$ parameter has something to do with the dimension of the graph. In fact, in many cases, we shall show that $s$ is the box dimension of the graph. To do this, we use a computation adapted from [7].

First, we must introduce the notion of Hölder continuity. Recall that regular continuity means $f(x) \to f(y)$ as $x \to y$. Hölder continuity occurs when $f(x) \to f(y)$ at least as fast as some positive power of $(x - y)$. We also cover what could be called a “reverse” Hölder condition.

**Lemma 3.2.** Suppose a continuous function $f : [0,1] \to \mathbb{R}$ satisfies the following Hölder conditions of exponent $s$, where $s \in [1,2)$:

(i) *Hölder condition:* $\exists c, \delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq c |x - y|^{2-s}$. 

(ii) *Reverse Hölder condition:* $\exists c, \delta_0 > 0$ s.t. $\forall x \in [0,1], \delta \in (0, \delta_0], \exists y$ s.t. $|x - y| < \delta$ and $|f(x) - f(y)| \geq c\delta^{2-s}$. [Note this $c$ need not be the same as $c$ from (i).]

Then $\text{graph}(f) := \{(x, y) : y = f(x)\}$ has box dimension $s$.

**Proof.** In (i), divide the interval $[0,1]$ into sub-intervals of length $\delta$. Do the same for the $y$-axis, thus constructing a $\delta$-grid over $\text{graph}(f)$. Let $I$ be one of the subintervals of $[0,1]$. Applying (i) and the extreme value theorem, we find that $|\sup(I) - \inf(I)| \leq c\delta^{2-s}$. Hence, the part of the graph over $I$ may be covered by a rectangle of width $\delta$ and height $c\delta^{2-s} = c\delta^{1-s}$ boxes of side $\delta$. Since there are about $1/\delta$ subintervals in $[0,1]$, $N_\delta$, the total number of grid elements intersecting $\text{graph}(f)$, will not exceed $c\delta^{1-s}(1/\delta) = c\delta^{-s}$. This means $\overline{\dim}_b \text{graph}(f) \leq s$.

Using the same grid and $I$ in (ii), we find that $|\sup(I) - \inf(I)| \geq c\delta^{2-s}$ when $\delta$ is small enough. Thus, $N_\delta \geq c\delta^{-s}$, and so $\underline{\dim}_b \text{graph}(f) \geq s$.

Combining the inequalities from (i) and (ii), we get

$$s \leq \underline{\dim}_b \text{graph}(f) \leq \dim_b \text{graph}(f) \leq \overline{\dim}_b \text{graph}(f) \leq s,$$

which gives us the result. \(\square\)

**Theorem 3.3.** For $\lambda$ sufficiently large, the W-M function satisfies $\dim_b \text{graph}(f) = s$.

**Proof.** First, we want to show $f$ satisfies the Hölder condition (i) in the above Lemma 3.2. That is, we want to show $|f(x + h) - f(x)| \leq ch^{2-s}$ for $h$ sufficiently small.

Suppose we are given some $h \in (0,1)$. We want to find a partial sum of $f$, show this satisfies the condition above, and then bound the tail of the sum, and use this to show $f$ also satisfies the
condition. The number of terms we need to sum should depend on \( h \); for smaller values of \( h \), we need more terms to get a good approximation. Let us sum the first \( N \) terms of \( f \), where \( N \) satisfies

\[
\lambda^{-(N+1)} \leq h < \lambda^{-N} \tag{3.1}
\]

The remaining terms may be separated:

\[
| f(x+h) - f(x) | = \\
\sum_{k=1}^{N} \lambda^{(r-2)k} \sin(\lambda^k (x+h)) + \sum_{k=N+1}^{\infty} \lambda^{(r-2)k} \sin(\lambda^k (x+h)) - \sum_{k=1}^{N} \lambda^{(r-2)k} \sin(\lambda^k x) - \sum_{k=N+1}^{\infty} \lambda^{(r-2)k} \sin(\lambda^k x) \leq \\
\sum_{k=1}^{N} \lambda^{(r-2)k} \left[ \sin(\lambda^k (x+h)) - \sin(\lambda^k x) \right] + \sum_{k=N+1}^{\infty} \left[ \lambda^{(r-2)k} \sin(\lambda^k (x+h)) - \sin(\lambda^k x) \right] \leq \\
\sum_{k=1}^{N} \lambda^{(r-2)k} \left| \sin(\lambda^k (x+h)) - \sin(\lambda^k x) \right| + \sum_{k=N+1}^{\infty} \lambda^{(r-2)k} \left| \sin(\lambda^k (x+h)) - \sin(\lambda^k x) \right|,
\]

where we have used the triangle inequality. We can use the mean-value theorem (M-VT) to bound the first absolute value so for some \( t \) between \( x \) and \( x+h \),

\[
| \sin(\lambda^k (x+h)) - \sin(\lambda^k x) | = | \cos(\lambda^t) \lambda^k h | \leq \lambda^k h.
\]

Also, applying the triangle inequality, we see the second absolute value is clearly bounded by 2. The resulting sums can be evaluated with the geometric series formula (here is where we assume \( \lambda \) is sufficiently large):

\[
| f(x+h) - f(x) | \leq \sum_{k=1}^{N} \lambda^{(r-1)k} h + \sum_{k=N+1}^{\infty} \lambda^{(r-2)k} = h \frac{\lambda^{(r-1)(N+1)} - \lambda^{N-1}}{\lambda^{r-1} - 1} + \frac{\lambda^{(r-2)(N+1)}}{1 - \lambda^{r+2}} \leq \\
\frac{h\lambda^{(r-1)(N+1)}}{\lambda^{r-1} - 1} + \frac{\lambda^{(r-2)(N+1)}}{1 - \lambda^{r+2}} \quad \text{(since } \lambda \gg 1) \]

\[
= \frac{h\lambda^{N(r-1)}}{1 - \lambda^{r+2}} + \frac{\lambda^{(r-2)(N+1)}}{1 - \lambda^{r+2}}. \tag{3.2}
\]

We want to pull out an \( h^{r+s} \) and show the above is \( \leq ch^{2-s} \), where \( c \) does not depend on \( h \). Let us look at the first term. The only factors dependent on \( h \) are \( h \) and \( \lambda^{N(r-1)} \) [since \( N \) depends on \( h \) by (3.1)]. We would like to bound the second factor in terms of \( h^{r-s} \) so that this would combine with the \( h \) to yield \( h^{2-s} \). We now use (3.1), our condition on \( N \). We raise (3.1) to the 1- \( s \) power, and switch the order of the inequalities, since this is negative, to obtain

\[
\lambda^{-(N+1)(r-s)} \geq h^{r-s} > \lambda^{N(r-1)} \Rightarrow \lambda^{N(r-s)} < h^{1-s}. \]

Thus, the first term on the right-hand side (rhs) of (3.2) is \( \leq h^{r-s} / (1 - \lambda^{r-s}) \).

To bound the second term, raise (3.1) to the 2- \( s \) power to obtain

\[
\lambda^{-(N+1)(2-s)} \leq h^{2-s} \Leftrightarrow \lambda^{(N+1)(2-s)} \leq h^{2-s}.
\]
If we use this and the previous result in (3.2), we finally get
\[ |f(x+h) - f(x)| \leq h^{2-s} \left( \frac{1}{1 - \lambda^{1-s}} + \frac{2}{1 - \lambda^{2-s}} \right). \]

We still need to show (ii) of Lemma 3.2. This is a bit trickier, since we need to bound an absolute value from below. This time we split each sum into three parts:
\[
f(x+h) - f(x) = \sum_{k=1}^{N-1} \lambda^{(s-2)k} \sin(\lambda^k (x+h)) + \lambda^{(s-2)N} \sin(\lambda^N (x+h)) + \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k (x+h))
\]
\[
- \sum_{k=1}^{N-1} \lambda^{(s-2)k} \sin(\lambda^k x) - \lambda^{(s-2)N} \sin(\lambda^N x) - \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k x)
\]
\[
\implies |f(x+h) - f(x) - \lambda^{(s-2)N} \left[ \sin(\lambda^N (x+h)) - \sin(\lambda^N x) \right]| = \sum_{k=1}^{N-1} \lambda^{(s-2)k} \left[ \sin(\lambda^k (x+h)) - \sin(\lambda^k x) \right] + \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} \left[ \sin(\lambda^k (x+h)) - \sin(\lambda^k x) \right].
\]

Now, as before, we use the mean-value theorem on the first sum on the rhs and the triangle inequality on the second:
\[
|f(x+h) - f(x) - \lambda^{(s-2)N} \left[ \sin(\lambda^N (x+h)) - \sin(\lambda^N x) \right]| \leq \sum_{k=1}^{N-1} \lambda^{(s-2)k} \lambda^k h + \sum_{k=N+1}^{\infty} 2\lambda^{(s-2)k}
\]
\[
= h \frac{\lambda^{N(s-1)} - \lambda^{s-1}}{1 - \lambda^{s-2}} + \frac{2\lambda^{(s-2)(N+1)}}{1 - \lambda^{s-2}} \leq h \frac{\lambda^{N(s-1)} - \lambda^{s-1}}{1 - \lambda^{s-2}} + \frac{2\lambda^{(s-2)(N+1)}}{1 - \lambda^{s-2}} = \frac{h\lambda^{N(s-1)-s+1}}{1 - \lambda^{s-2}} + \frac{2\lambda^{(s-2)(N+1)}}{1 - \lambda^{s-2}}.
\]

where we have used \( h < \lambda^{-N} \) from (3.1). Now since \( 1 < s < 2 \), the exponents in the denominators of the last expression are both positive; hence, the part in parentheses \( \to 0 \) as \( \lambda \to \infty \). Thus, for sufficiently large \( \lambda \):
\[
|f(x+h) - f(x) - \lambda^{(s-2)N} \left[ \sin(\lambda^N (x+h)) - \sin(\lambda^N x) \right]| \leq \lambda^{(s-2)N} \left( \frac{1}{20} \right) \forall N. \tag{3.3}
\]

Remember that \( \lambda \) is a fixed parameter of the function, and \( N \) and \( h \) are related by (3.1). We have shown that (3.3) holds when both (3.1) is satisfied and \( \lambda \) is sufficiently large, which we are assuming is true. In (ii) from Lemma 3.2, choose and fix \( 0 < \delta_0 < 1/\lambda \). Say we are given some \( 0 < \delta \leq \delta_0, x \in [0,1] \); then we need to find \( 0 < h \leq \delta \) s.t. \( |f(x+h) - f(x)| \geq c\delta^{2-s} \). Choose \( N \) to satisfy
\[
\lambda^{-N} \leq \delta < \lambda^{-(N-1)} \tag{3.4}
\]
We assert that we can find an $h$ satisfying (3.1) and
\[
\left| \sin \lambda^N (x + h) - \sin \lambda^N x \right| = \left| \sin(\lambda^N x + \lambda^N h) - \sin \lambda^N x \right| > \frac{1}{10}. \tag{3.5}
\]
This is because (3.1) $\implies 1/\lambda \leq \lambda^N h < 1$, and we know the sine function must vary by more than one tenth over an interval of length $1-1/\lambda$, for $\lambda$ large. For this choice of $h$, (3.3) and the triangle inequality imply that
\[
\left| f(x+h) - f(x) \right| - \lambda^{(r-2)N} \left[ \left| \sin(\lambda^k (x + h)) - \sin(\lambda^N x) \right| \right] < \frac{\lambda^{(r-2)N}}{20} \implies
\]
\[
-\frac{\lambda^{(r-2)N}}{20} < \left| f(x+h) - f(x) \right| < \lambda^{(r-2)N} \left[ \left| \sin(\lambda^k (x + h)) - \sin(\lambda^N x) \right| \right] < \frac{\lambda^{(r-2)N}}{20} \implies
\]
\[
\left| f(x+h) - f(x) \right| > \frac{\lambda^{(r-2)N}}{20} + \lambda^{(r-2)N} \left[ \left| \sin(\lambda^k (x + h)) - \sin(\lambda^N x) \right| \right] > \lambda^{(r-2)N} \left( \frac{1}{10} - \frac{1}{20} \right)
\]
where (3.5) has been used and (3.4) has been used in the last step. Now by Lemma 3.2, we have the desired result, namely $\dim_h graph(f) = s$. \hfill \square

Since we know the Hausdorff dimension of a set is always less than or equal to the box dimension [7], Theorem 3.3 immediately implies $\dim_h graph(f) \leq s$ for $\lambda$ sufficiently large. Can we replace the “$\leq$” with an “$=$” sign? While this is conjectured to be the case, as far as we know, this has yet to be proved for all values of $\lambda$. As stated before, the fact that balls of arbitrarily small size may be used in the Hausdorff measure makes it hard to prove $H'(graph(f)) > 0$ implying $\dim_h graph(f) \geq s$. There might be highly specialized coverings producing a Hausdorff s-measure of 0. If this were so, we would have a strict inequality: $\dim_h graph(f) < s$.

Next, we shall briefly cover some previous attempts to compute the Hausdorff dimension, and give some of our own ideas.

One approach to the problem is using measures on $graph(f)$. We want to show $H'(graph(f)) > 0$. Suppose we can come up with a measure $\mu$ s.t. $\mu(B) = \ldots$
\(\mu(B \cap \text{graph}(f)) \leq |B|\) for any ball \(B\), and \(\mu(\text{graph}(f)) = M > 0\). Then additive properties of measures tell us \(\sum |B_i| \geq \sum \mu(B_i) \geq M > 0\) for all countable covers \(\{B_i\}\). Therefore, we have \(H'(\text{graph}(f)) > 0\) and \(\dim_H \text{graph}(f) \geq t\).

While to our knowledge, the above method has not been used with \(t = s\), Mauldin and Williams were able to get close [11]. In the end, they concluded that \(s - C/\log \lambda \leq \dim_H \text{graph}(f) \leq s\), where \(C > 0\) is some constant. Note when \(\lambda\) is large, we know the Hausdorff dimension is at least very close to \(s\).

In their paper, Mauldin and Williams actually prove the result for a more general class of functions of the form

\[f_\lambda(x) := \sum_{n=0}^{\infty} \lambda^{(s-2)k} \Phi(\lambda^n x + \theta_n),\]

where \(\Phi\) must satisfy certain continuity properties (similar to Hölder conditions), and the constants \(\{\theta_n\}\) comprise the phase sequence.

In Hunt’s paper [9], it is shown that the graph of \(f_\lambda\) with \(\Phi = \cos(x)\) has Hausdorff dimension \(s\) for almost every sequence of phases. In other words, if we randomly pick a sequence of phase shifts, the probability that the resulting graph has Hausdorff dimension \(s\) is one. Unfortunately, Hunt’s paper does not say for which phase sequences this happens!

However, we believe that Hunt’s result and some others in the literature can be combined to provide the main ingredient for a relatively short proof of the equality of the Hausdorff and box counting dimensions for the graph of the W-M function. In particular, these results should enable one to construct a sequence of functions \(\{f_n\}\) converging to the W-M function \(f\) such that:

(i) The set \(\text{graph}(f_n)\) also converges to \(\text{graph}(f)\) in a certain sense; (ii) \(\dim_H \text{graph}(f_n)\) converges to \(\dim_H \text{graph}(f)\); and most importantly (iii) \(\dim_H \text{graph}(f_n)\) converges to \(\dim_b \text{graph}(f)\). It is also likely that such a limit-based argument can also be adapted to prove analogous results for generalizations and higher dimensional extensions of the W-M function (see the next section).

In a recent (2011) paper, Carvalho [4] modifies Hunt’s definition of \(f_\lambda\) to create a sparse version in which most of the amplitudes and frequencies are skipped. In the new function, only terms corresponding to the indices \(n = \gamma^i, i = 1, 2, 3, \ldots\) are kept in the series for \(f_\lambda\); this makes for a series that converges faster. Moreover, unlike Hunt’s result, Carvalho’s is deterministic. That is, it precisely states the Hausdorff dimensions of the sparse W-M functions for any sequence of phases. But of course, removing terms from a sequence generally changes what it converges to. Thus, Carvalho’s result does not allow us to make a similar conclusion about the original W-M functions.

In another 2011 paper (published 2012), Barański [1] shows that the Hausdorff and box dimensions are equal for another type of modification of \(f_\lambda\):
\[ f_\lambda(x) = \sum_{n=0}^{\infty} \lambda^{(s-2)k} \Phi(b_n x + \theta_n), \] where \( b_{n+1}/b_n \to \infty. \)

The only difference is in the frequencies of the terms in the series, which now form a rapidly increasing sequence. Again, Barański’s result does not apply to the original W-M function, for which \( b_{n+1}/b_n = \lambda \not\to \infty. \)

Finally, we note one more very recent result which was called to our attention during the revision stage of this paper. In an article published online February 2012, Fu [8] proves—with a very long and detailed argument—that (for the original W-M functions) the Hausdorff dimension equals the box dimension for large values of \( \lambda. \) While this seems to be the farthest advance to date in resolving the conjecture, finding the Hausdorff dimension of the W-M function for all parameter values (especially in a more efficient manner than in the current literature) still appears to be an open problem. However, the flurry of new results suggests that a complete resolution of equal fractal dimension conjecture for the W-M function may be close at hand.

### 4 A Surface Generalization of the W-M Function

Our last topic is a fractal surface similar to the W-M function in the plane. Here, we define this function, and as before, we find the box dimension of its graph. We also state a conjecture about its Hausdorff dimension. Our definition and derivation were adapted from the even more general model of Blackmore and Zhou [3], which was originally formulated to model rough surfaces with (approximate) self-similarity. For example, when one views a machined metal surface on a small enough level, it will no longer look smooth; rather, a microscope will reveal many sharp hills and valleys strewn across the surface. And, to a certain extent, we can zoom in closer and still get a similar picture: the bumps have smaller bumps attached. Of course, looking even closer changes the picture again; it might look smooth once more. Nevertheless, there is a certain range of scale where the physical surface is well approximated by an infinitely self-similar mathematical surface.

When we constructed the planar W-M function, we started with an oscillating function (the sine), and then kept adding appropriately scaled copies of the same function. In the surface case, we are working in \( \mathbb{R}^3, \) so we need to come up with an oscillatory function of two variables. Let us pick \( \varphi(x, y) := \sin x \cos y : \)
Now, we want to add increasingly small “bumps” to this surface to make it fractal. It turns out we can do this in a way very similar to that of Section 3:

**Definition 4.1.** On the unit square $I^2$, define the $W$-$M$ surface generalization to be the graph of

$$\Phi(x, y) := \sum_{n=1}^{\infty} \beta^{(s-3)n} \varphi(\beta^n x, \beta^n y) := \sum_{n=1}^{\infty} \beta^{(s-3)n} \sin(\beta^n x) \cos(\beta^n y).$$

Note an important difference here is in the exponent of $\beta$: there is an $s - 3$ instead of $s - 2$. As before, we can plot some partial sums to get a good picture. As the following plots show, the graph again gets more “jagged” for larger values of $s$. The last picture shows the contours of the surface, which we expect have interesting (possibly fractal) properties in themselves.
First 50 terms, with beta = 1.500 and s = 2.100
We shall compute the box dimension of this fractal surface using a method analogous to that of the previous section. We start by defining Hölder conditions for a function of two variables.

**Lemma 4.2.** Suppose a continuous function \( \Phi: I^2 \rightarrow \mathbb{R} \) satisfies the following generalized Hölder conditions of exponent \( s \), for some \( s \in [2,3) \):

(i) **Hölder condition:** \( \exists c, \delta > 0 \) s.t. \( 0 < \|h\| < \delta, x \in I^2 \Rightarrow |\Phi(x+h) - \Phi(x)| \leq c \|h\|^{3-s} \).

(ii) **Reverse Hölder condition:** \( \exists c, \delta_0 > 0 \) s.t. \( \forall x, \delta < \delta_0, \exists h \) s.t. \( 0 < \|h\| < \delta \) and \( |\Phi(x+h) - \Phi(x)| \geq c \|h\|^{3-s} \).

Then \( \text{graph}(\Phi) := \{(x, y, z) : z = \Phi(x, y)\} \) has box dimension \( s \).

**Proof.**

(1) First, we show condition (i) implies \( \overline{\dim}_b \text{graph}(\Phi) \leq s \). Pick \( h \) s.t. \( h\sqrt{2} < \delta \) and construct an \( h \)-grid over \( I^2 = [0,1]^2 \) in the \( x \)-\( y \) plane. The grid will have \( 1/h^2 \) squares. If two points \( x, y \) lie in the same square, then \( \|x-y\| \leq h\sqrt{2} \), the length of the diagonal. Hence, using (i), we find that \( |\Phi(x) - \Phi(y)| \leq c \left( h\sqrt{2} \right)^{3-s} \). This gives us a bound on how much \( \Phi \) varies over each of the squares. Hence, we can cover \( \text{graph}(\Phi) \) with \( 1/h^2 \) three-dimensional rectangles, each sitting over one of the \( x \)-\( y \) squares and having height \( c \left( h\sqrt{2} \right)^{3-s} \). And each rectangle can be covered by a stack of no more than \( c \left( h\sqrt{2} \right)^{3-s}/h \) cubes of side \( h \). Finally, to cover the whole graph, we need \( \left[ c \left( h\sqrt{2} \right)^{3-s}/h \right]/h^2 = Ch^{-s} \) cubes of side \( h \), where \( C \) is a constant independent of \( h \). So we know \( N_h \), the least number of cubes needed to cover \( \text{graph}(\Phi) \), satisfies \( N_h \leq Ch^{-s} \). Hence,

\[
\overline{\dim}_b \text{graph}(\Phi) = \lim_{h \downarrow 0} \frac{-\log N_h}{\log h} \leq \lim_{h \downarrow 0} \frac{s \log h}{\log h} = s.
\]

(2) In a similar way, (ii) implies that there is a \( C' \) such that \( N_\delta \geq C' \delta^{-s} \) for \( \delta < \delta_0 \). So we also have \( \overline{\dim}_b \text{graph}(\Phi) \geq s \). Combining this with (1) gives \( \dim_b \text{graph}(\Phi) = s \).

**Theorem 4.3.** For \( \beta \) sufficiently large, \( \dim_b \text{graph}(\Phi) = s \).

**Proof** (adapted from [3]).
Let \( \bar{x} = (x, y) \in \mathbb{R} \). Given \( \bar{h} = (h_1, h_2) \) satisfying \( 0 \leq \| \bar{h} \| < 1 \), choose a positive integer \( N \) and a constant \( c \) s.t. \( c \beta^{-N+1} \leq \| \bar{h} \| < c \beta^{-N} \). (Remember \( \beta \) is assumed to be large.) Now, as in the proof of Theorem 3.3, we break up the following sum:

\[
|\Phi(\bar{x} + \bar{h}) - \Phi(\bar{x})| = \left| \sum_{n=1}^{\infty} \beta^{(s-3)n} \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sum_{n=1}^{\infty} \beta^{(s-3)n} \sin(\beta^n x) \cos(\beta^n y) \right|
\]

\[
\leq \sum_{n=1}^{N} \beta^{(s-3)n} \left| \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sin(\beta^n x) \cos(\beta^n y) \right|
\]

\[
+ \sum_{n=N+1}^{\infty} \beta^{(s-3)n} \left| \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sin(\beta^n x) \cos(\beta^n y) \right|.
\]

(4.1)

As before, we want to use the M-VT to bound the first absolute value in (4.1). But here, we have a function being evaluated at two points, \( \bar{x} = (x, y) \) and \( \bar{x} + \bar{h} \), in the plane. So we use the 1-dimensional M-VT on the function \( \varphi := \sin(\beta^n x) \cos(\beta^n y) \) restricted to the line segment \( L \) joining \( \bar{x} \) and \( \bar{x} + \bar{h} \). The 1-dimensional derivative evaluated at some point in \([a, b]\) becomes a directional derivative evaluated at some point on \( L \); \( b - a \), the interval length, becomes \( \| \bar{h} \| \). We find that

\[
\varphi(\bar{x} + \bar{h}) - \varphi(\bar{x}) = \| \bar{h} \| D_h \varphi(\bar{x} + c\bar{h}) = \| \bar{h} \| (\varphi_{x}, \varphi_{y})(\bar{x} + c\bar{h})\| \bar{h} \|
\]

for some \( c \in [0, 1] \). Since \( |\varphi_{x, y}| \leq \beta^n \) everywhere, we compute that

\[
\left| \varphi(\bar{x} + \bar{h}) - \varphi(\bar{x}) \right| \leq \| \varphi_{x, y} \| \| \bar{h} \| = \| \bar{h} \| \sqrt{\varphi_{x}^2 + \varphi_{y}^2} \leq \| \bar{h} \| \sqrt{\beta^{2n} + \beta^{2n}} = \sqrt{2} \beta^n \| \bar{h} \|.
\]

Also, the second absolute value in (4.1) is obviously bounded by 2. So we obtain

\[
|\Phi(\bar{x} + \bar{h}) - \Phi(\bar{x})| \leq \sqrt{2} \sum_{n=1}^{N} \beta^{(s-2)n} \| \bar{h} \| + \sum_{n=N+1}^{\infty} 2 \beta^{(s-3)n}
\]

\[
= \sqrt{2} \| \bar{h} \| \left( \beta^{(s-2)} - \beta^{(s-2)(N+1)} \right) + 2 \beta^{(s-3)(N+1)}
\]

\[
= \sqrt{2} \| \bar{h} \| \left( \beta^{(s-2)(N+1)} - \beta^{(s-2)} \right) + 2 \beta^{(s-3)(N+1)}
\]

(4.2)

(Note the signs! Each term should be positive.)

Now we want to bound (4.2) by \( C \| \bar{h} \|^{s-2} \), where \( C \) is independent of \( \bar{h} \). We know \( c \beta^{-(N+1)} \leq \| \bar{h} \| \) by our choice of \( N \). Thus \( c \beta^{-(N+1)(2-s)} \leq \| \bar{h} \|^{2-s} \). So we want to pull out
\[ \beta^{-(N+1)(2-s)} \] from the first quotient in (4.2). Also, \( c \beta^{-(N+1)(3-s)} \leq \| h \|^{(3-s)} \). Thus, we want to extract
\( c \beta^{-(N+1)(3-s)} \leq \| h \|^{(3-s)} \) from the second term in (4.2). We get
\[
\left| \Phi(x + \bar{h}) - \Phi(x) \right|
\leq \sqrt{2} \| \bar{h} \| \beta^{-(N+1)(2-s)} \frac{\beta^{(x-2)(N+1)(2-s)}}{1 - \beta^{(2-s)}} + \beta^{(x-3)(N+1)} \frac{2}{1 - \beta^{(x-3)}}
\leq c_1 \| \bar{h} \|^{(3-s)} \beta^{2-s} + c_2 \| \bar{h} \|^{(3-s)}
= C \| \bar{h} \|^{(3-s)},
\]
where all the \( c \)'s are independent of \( \bar{h} \).

We still need to verify the second Hölder condition. Not surprisingly, we do this by splitting \( \Phi(x + \bar{h}) - \Phi(x) \) into three parts and bringing the \( N \)th term to the left:
\[
\Phi(x + \bar{h}) - \Phi(x) = \sum_{n=1}^{N-1} \beta^{(x-3)n} \left\{ \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sin(\beta^n x) \cos(\beta^n y) \right\} + \\
\beta^{(x-3)N} \left\{ \sin \left[ \beta^N (x + h_1) \right] \cos \left[ \beta^N (y + h_2) \right] - \sin(\beta^N x) \cos(\beta^N y) \right\} + \\
\sum_{n=N+1}^{\infty} \beta^{(x-3)n} \left\{ \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sin(\beta^n x) \cos(\beta^n y) \right\}
\Rightarrow
\left| \Phi(x + \bar{h}) - \Phi(x) - \beta^{(x-3)N} \left\{ \sin \left[ \beta^N (x + h_1) \right] \cos \left[ \beta^N (y + h_2) \right] - \sin(\beta^N x) \cos(\beta^N y) \right\} \right|
\leq \sum_{n=1}^{N-1} \beta^{(x-3)n} \left\{ \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sin(\beta^n x) \cos(\beta^n y) \right\} + \\
\sum_{n=N+1}^{\infty} \beta^{(x-3)n} \left\{ \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sin(\beta^n x) \cos(\beta^n y) \right\}
\leq \sum_{n=1}^{N-1} \beta^{(x-3)n} \left\{ \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sin(\beta^n x) \cos(\beta^n y) \right\} + \\
\sum_{n=N+1}^{\infty} \beta^{(x-3)n} \left\{ \sin \left[ \beta^n (x + h_1) \right] \cos \left[ \beta^n (y + h_2) \right] - \sin(\beta^n x) \cos(\beta^n y) \right\}.
\]
To bound the term in braces on the rhs, we use the mean-value theorem as before; the second expression in braces is clearly bounded by 2:
Now, by comparing this with what we got in the 2-dimensional proof, you might guess what comes next. We want to pull out $\beta^{(s-3)N}$ and be left with something that converges to 0 for large $\beta$:

$$\left| \Phi(x + \bar{h}) - \Phi(x) - \beta^{(s-3)N} \left\{ \sin[\beta^N (x + h_1)] \cos[\beta^N (y + h_2)] - \sin(\beta^N x) \cos(\beta^N y) \right\} \right|$$

\[
\leq \sqrt{2} \left\| \bar{h} \right\| \left( \frac{\beta^{s-2}}{1 - \beta^{s-2}} - \beta^{(s-2)N} \right) + 2 \beta^{(s-3)(N+1)}
\]

\[
= \beta^{(s-3)N} \left[ \sqrt{2} \left\| \bar{h} \right\| \left( \frac{\beta^N - \beta^{s-2+(s-3)N}}{\beta^{s-2} - 1} \right) + 2 \beta^{s-3} \right]
\]

\[
< \beta^{(s-3)N} \left( \sqrt{2} \left\| \bar{h} \right\| \frac{\beta^N}{\beta^{s-2} - 1} + 2 \frac{1}{\beta^{s-3} - 1} \right)
\]

\[
= \beta^{(s-3)N} \left( \frac{\sqrt{2} c \beta^{-N}}{\beta^{s-2} - 1} + \frac{2}{\beta^{s-3} - 1} \right), \tag{4.3}
\]

where we have used the fact that $N$ was chosen s.t. $\left\| \bar{h} \right\| < c \beta^{-N}$. Now, indeed, we see that the last term in parentheses goes to 0 as $\beta \to \infty$. (Remember that $2 < s < 3$).

Let $\delta_0 = 1 / \beta$, and say we are given a $0 < \delta < \delta_0$. We need to find an $\bar{h}$ s.t. $\left\| \bar{h} \right\| < \delta$ and $| \Phi(x + \bar{h}) - \Phi(x) | \geq K \left\| \bar{h} \right\|^{-s}$, where $K$ is some constant independent of $\delta$. Assume $\beta$ is so large that
\[
\left( \frac{\sqrt{2c}}{\beta^{r-2}-1} + \frac{2}{\beta^{3r-1}} \right) < \frac{1}{20}.
\]

Then from (4.3) we obtain
\[
\left| \Phi(x + h) - \Phi(x) - \beta^{(x-3)N}\left\{ \sin[\beta^N (x + h_z)] \cos[\beta^N (y + h_z)] - \sin(\beta^N x) \cos(\beta^N y) \right\} \right|
\]
\[
= \left| \Phi(x + h) - \Phi(x) - \beta^{(x-3)N} F \right|
\]
\[
< \frac{\beta^{(x-3)N}}{20} \Rightarrow
\]
\[
-\frac{\beta^{(x-3)N}}{20} < \left| \Phi(x + h) - \Phi(x) - \beta^{(x-3)N} F \right| < \frac{\beta^{(x-3)N}}{20}
\]
\[
\Rightarrow
\]
\[
\left| \Phi(x + h) - \Phi(x) \right| > \beta^{(x-3)N} F - \frac{\beta^{(x-3)N}}{20} = \beta^{(x-3)N} \left\{ |F| - \frac{1}{20} \right\} = \left( \beta^{-1} \right)^{(x-3)N} \left\{ |F| - \frac{1}{20} \right\} > \delta^{(x-3)N} \left\{ |F| - \frac{1}{20} \right\},
\]

where we have used the condition $\delta < \delta_0 = \beta^{-1}$. This holds for all $h, N$ satisfying $c\beta^{-(N+1)} \leq \|h\| < c\beta^{-N}$. Now, if we can find an $h$ s.t. $\delta > \|h\|$ and $\{ |F| - 1/20 \}$ is a nonzero constant, we are done. First, pick $N$ s.t. $c\beta^{-N} \leq \delta < c\beta^{-(N+1)}$. We assert that there is an $h$ s.t. $c\beta^{-(N+1)} \leq \|h\| < c\beta^{-N}$ and $|F| > \frac{c}{10}$; the argument (cf. [3]) for this is similar to that used in the 2-dimensional case for a W-M function, but more complicated.

This completes the proof. \(\square\)

As before, we expect the Hausdorff dimension to equal the box dimension, although this has never been shown, or even (to our knowledge) considered in depth. Thus we have the following conjecture, analogous to the one made for the W-M function of one variable:

**Conjecture 4.4.** \(\dim_H graph(\Phi) = \dim_B graph(\Phi) = s\).

It appears that it may be possible to extend, by fairly straightforward modification, the result of Fu [8] in order to verify this conjecture for the case when $\beta$ is sufficiently large.
5 Concluding Remarks

In this paper we have given a brief and fairly intuitive introduction to the concept of fractal dimension in order to prepare the reader for our main results—namely, the fractal with unequal box and Hausdorff dimensions and our discussion of W-M functions of both one and two variables. Readers interested in gaining a more thorough knowledge of fractal geometry may consult the references cited in the paper.

As mentioned above, it is still an open question whether the Hausdorff dimension of the W-M function is always the conjectured value. In future research, we hope to gain further insight into this long standing problem, and perhaps even resolve it once and for all.

6 References


