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Formulas for Fibonacci-Like Sequences Produced by Pascal-Like Triangles

Hiroshi Matsui
Kwansei Gakuin High School, Japan

Toshiyuki Yamauchi
Kwansei Gakuin High School, Japan

FORMULAS FOR FIBONACCI-LIKE SEQUENCES PRODUCED BY PASCAL-LIKE TRIANGLES

Hiroshi Matsui and Toshiyuki Yamauchi
Kwansei Gakuin High School, Nishinomiya City JAPAN
e-mail: miyadera1272000@yahoo.co.jp

ABSTRACT

In this paper we are going to present three formulas to express Fibonacci-like sequences with the Fibonacci sequence.
In reference [1] we constructed Pascal-like triangles using probabilities of a game, and these Pascal-like triangles can be considered generalizations of the well known Pascal's triangle. Using these triangles, we can make Fibonacci-like sequences.
These formulas can reveal very interesting relationships between Fibonacci-like sequences and the Fibonacci sequence, and we can expect a rich possibility of the research from these Fibonacci-like sequences.

1 MATHEMATICAL BACKGROUND OF PASCAL-LIKE TRIANGLES

First we are going to present the game by which we can make Pascal-like triangles.

Definition 1.1. Let p, n, m be fixed natural numbers such that $m \leq n$. We have p players who are seated in a circle. The game begins with the first player. Proceeding in order, a box is passed from hand to hand. The box contains n numbered cards. The numbers on the cards in the box are assigned in numerical order, from 1 to n. Any number x, such that $x \leq m$, is printed on a red card. When a player receives the box, he draws a card from the box. The player who draws a red card loses the game. The players cannot see the card when they draw, hence they draw at random. Once a card is taken from the box, that card will not be returned to the box. Therefore, when the first player draws the first card, the number of the card can be any number 1, 2, ..., n, while the second player can draw any number 1,2,..., n except the number of the card that the first player drew, and the game continues until one of the players gets a red card and loses the game.

Definition 1.2. For natural numbers p, n, m, v such that $m \leq n$ and $v \leq p$ we define $U(p, n, m, v) = \sum_{z=0}^{t-1} {}_{n-v-pz}C_{m-1}$, where $t = \lfloor \frac{n-m+p-v+1}{p} \rfloor$.

Theorem 1. For any natural numbers n, m, p, v such that $m \leq n$ and $v \leq p$ we have the following (a) and (b).
(a) $U(p, n, m, v)$ is the number of possible cases in which the vth player becomes the loser of the game in Definition 1.1 with p players, n cards and m red cards.
(b) $U(p, n, m, v) + U(p, n, m + 1, v) = U(p, n + 1, m + 1, v)$.

(a) We represent n cards as $\{a_1, a_2, \cdots, a_n\}$, and these cards must be picked up in this order. Therefore the first card to be drawn is a_1, and the number of the card can be 1, 2, $\cdots$,n. The last one is a_n. The game ends with the yth card if the yth card is red and other $m - 1$ red cards are in the $(y+1)$th, $\cdots$, the nth places. There are ${}_{n-y}C_{m-1}$ ways to put red cards into places this way, and ${}_{n-y}C_{m-1}$ is the number of the possible cases that the game ends with the yth card.

The number of cards to be drawn is $n - m + 1$ at most, since we have m red cards. The vth player draws the vth, the $(v + p)$th, the $(v + 2p)$th,. . . , the $(v+(t-1)p)$th card, where t is the biggest natural number that satisfies $v + (t - 1)p \leq n - m + 1$, and hence
$t = \lfloor (n - m + p - v + 1)/p \rfloor$.
Therefore by Definition 1.2 $U(p, n, m, v)$ is the number of the possible cases in which the vth player becomes the loser of the game.

(b) By Definition 1.2 we have

$$U(p,n,m,v) = \sum_{z=0}^{t_1-1} {}_{n-v-pz}C_{m-1}, \tag{1.1}$$

$$U(p,n,m+1,v) = \sum_{z=0}^{t_2-1} {}_{n-v-pz}C_{m} \tag{1.2}$$

and

$$U(p,n+1,m+1,v) = \sum_{z=0}^{t_3-1} {}_{n+1-v-pz}C_{m}, \tag{1.3}$$

where

$$t_1 = \lfloor (n-m+p-v+1)/p \rfloor, \tag{1.4}$$

$$t_2 = \lfloor (n-(m+1)+p-v+1)/p \rfloor \tag{1.5}$$

and

$$t_3 = \lfloor (n+1-(m+1)+p-v+1)/p \rfloor. \tag{1.6}$$

It is clear that $t_2 \leq t_1 = t_3$. and hence we have for $z = 0, 1, \cdots, t_2$

$${}_{n-v-pz}C_{m-1} + {}_{n-v-pz}C_{m} = {}_{n+1-v-pz}C_{m}. \tag{1.7}$$

By (1.4) and (1.5) $t_1 = t_2$ or $t_1 = t_2 + 1$. We are going to deal with these two cases separately.
(i) If $t_1 = t_2$, by (1.1), (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7) we have
$U(p,n,m,v) + U(p,n,m+1,v) = U(p,n+1,m+1,v)$.
(ii) We suppose that $t_1 = t_2 + 1$, then we have only to prove that the t_1th term of (1.1) is equal to the t_1th term of (1.3), since (1.2) does not have the t_1th term.
By the fact that $t_1 = t_2 + 1$, we know that $n - m + p - v + 1$ is a multiple of p, and hence we have
$n - m + p - v + 1 = pt_1$.
From this we have
$n - v - p(t_1 - 1) = m - 1$

and
$n+1-v-p(t_3-1)=m$,
which imply that

$$\begin{cases} {}_{n-v-p(t_1-1)}C_{m-1} = {}_{m-1}C_{m-1} = 1, \\ {}_{n+1-v-p(t_3-1)}C_m = {}_mC_m = 1. \end{cases} \tag{1.8}$$

By (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7) and (1.8) we have $U(p,n,m,v)+U(p,n,m+1,v) = U(p,n+1,m+1,v)$. □

Remark 1.1. The authors have already proved Theorem 1 in reference [1]. They present the proof of Theorem 1 here, since the proof is important for the reader to understand the structure of $U(p,n,m,v)$.

Definition 1.3. Let $G(p,n,m,v)=\frac{U(p,n,m,v)}{{}_nC_m}$.

Since ${}_nC_m$ is the total number of the possible cases of choosing m red cards from n cards, by Theorem 1 it is easy to see that $G(p,n,m,v)$ is the probability that the vth player becomes the loser in the game.

Example 1.1. We are going to calculate $U(3,8,3,1)$ and $G(3,8,3,1)$.
Three persons are going to play the game of Definition 1.1 with the condition that $n=8$ and $m=3$. Then we have 8 cards and two of them are red cards.

We represent 8 cards as $\{a_1, a_2, a_3, \cdots, a_8\}$, and we assume that these cards are to be picked up according to this order. Therefore the first card to be picked up is a_1, and the number of a_1 can be 1, 2, $\cdots$,8. The last one to be picked up is a_8.
The 1st player can draw the 1st, the 4 rd, the 7th cards.
The 1st player becomes a loser with the 1st card if the 1st card is red and other 2 red cards are in the 2nd, the 3rd, the 4th,$\cdots$, the 8th places. There are ${}_7C_2$ ways to put red cards into places this way.
The 1st player becomes a loser with the 4th card if the 4th card is red and other 2 red cards are in the 5th, the 6th, the 7th and the 8th places. There

are ${}_4C_2$ ways to do this.
The 1st player cannot become a loser with the 7th card since it is impossible to put other 2 red cards in the 8th place.
The total number of the possible cases is
$U(3, 8, 3, 1) = {}_7C_2 + {}_4C_2 = 21+6 = 27$.
It is clear that
$G(3, 8, 3, 1)=\frac{U(3,8,3,1)}{{}_8C_3} = \frac{27}{56}$.

2 SEQUENCES PRODUCED BY PASCAL-LIKE TRIANGLES

By Theorem 1 $U(p, n, m, v)$ has a property that reminds us of ${}_nC_m$, and hence the list $\{U(p, n, m, v),\ m \leq n$ and $n = 1,2,3,\cdots\}$ forms a triangle that is very similar to the Pascal's triangle. It will be natural to call this a Pascal-like triangle. We are going to observe this fact in Example 2.1.

Example 2.1. The list $\{\ U(3, n, m, 1),\ m \leq n$ and $n = 1,2,3,\cdots,7\}$.
It is sufficient to calculate the values of $U(3, n, 1, 1)$ and $U(3, n, n, 1)$ for n = 1, 2, ...,7, since by Theorem 1(b) the value of $U(3, n, k, 1)$ with $1 < k < n$ can be obtained by these values.
Since $t = \lfloor\frac{n-1+3-1+1}{3}\rfloor = \lfloor\frac{n+2}{3}\rfloor$,
$U(3, n, 1, 1) = \sum_{z=0}^{t-1} {}_{n-1-3z}C_0 = t = \lfloor\frac{n+2}{3}\rfloor$.
Since $t = \lfloor\frac{n-n+3-1+1}{3}\rfloor = 1$,
$U(3, n, n, 1) = \sum_{z=0}^{1-1} {}_{n-1-3z}C_{n-1} = 1$.

$Figure 1$

1
1,1
1,2,1
2,3,3,1
2,5,6,4,1
2,7,11,10,5,1
3,9,18,21,15,6,1

It is interesting to see the differences and similarities between this triangle and the Pascal's triangle.

It is well known that the numbers on diagonals of the Pascal's triangle add to the Fibonacci sequence, but the numbers on diagonals of a Pascal-like triangle add to a Fibonacci-like sequence. We are going to illustrate this fact in the following Example 2.2.

Example 2.2. If you use Figure 1, then you get the sequence $b_1 = 1, b_2 = 1, b_3 = 1+1 = 2, b_4 = 2+2 = 4, b_5 = 2+3+1 = 6, b_6 = 2+5+3 = 10, b_7 = 3+7+6+1 = 17, \cdots$, and the rule of this sequence is

$$b_n = b_{n-1} + \begin{cases} b_{n-2}+1, & \text{if } n = 1 \ (\bmod 3). \\ b_{n-2}, & \text{if } n = 0 \ (\bmod 3). \end{cases}$$

It is natural to call this sequence a Fibonacci-like sequence, because it has a rule that is very much like that of the Fibonacci sequence.

We are going to generalize the result of Example 2.2 to define a sequence $B_p(n)$.

Definition 2.1. Let p be a fixed natural number. We define $B_p(1) = B_p(2) = 1$ and for natural number n

$$B_p(n) = B_p(n-1) + \begin{cases} B_p(n-2)+1, & \text{if } n = 1 \ (\bmod p). \\ B_p(n-2), & \text{if } n = 0 \ (\bmod p). \end{cases}$$

In [4] we constructed $B_p(n)$ by using $U(p, n, m, 1)$, but in this paper we defined it without using the result of [4].

3 THE RELATIONSHIPS BETWEEN THE SEQUENCE $B_p(n)$ AND THE FIBONACCI SEQUENCE F_n

There are some very interesting relationships between the sequence $B_p(n)$ and F_n.

Example 3.1. Here we compare the Fibonacci sequence F_n and $B_p(n)$.
(1) F_n is $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...\}$.
(2) $B_2(n)$ is $\{1, 1, 3, 4, 8, 12, 21, 33, 55, 88, 144, 232, 377, 609, 987, ...\}$.
(3) $B_3(n)$ is $\{1, 1, 2, 4, 6, 10, 17, 27, 44, 72, 116, 188, 305, 493, 798, ...\}$.

It is well known that we can express $B_2(n)$, $B_3(n)$ with the Fibonacci sequence.
$B_2(n) = F_{n+1} - \frac{(1+(-1)^n)}{2}$. See A004695 of [5].
$B_3(n) = \lfloor(\frac{F_{n+2}}{2})\rfloor$. See A052952 of [5].
The authors presented a formula to express $B_4(n)$ with the Fibonacci sequence F_n in Reference [3].
The authors are going to present formulas for $B_p(n)$ when $p > 4$ in Theorem 2, 3 and 4.

Theorem 2. Let $p = 4q$ for a natural number q. Here we denote $B_p(n)$ by $f(n)$, and we define $F_0 = f(0) = 0$.
Then $f(n)$ satisfies the following equations.

$$\begin{cases} f(4qt) = \frac{F_{2qt}F_{2qt+2q}}{F_{2q}}, \\ f(4qt+1) = \frac{F_{2qt+1}F_{2qt+2q}}{F_{2q}}, \\ f(4qt+2) = \frac{F_{2qt+2}F_{2qt+2q}}{F_{2q}}, \\ \vdots \\ f(4qt+4q-1) = \frac{F_{2qt+4q-1}F_{2qt+2q}}{F_{2q}}. \end{cases} \tag{3.1}$$

Proof. Since $f(1) = f(2) = 1$, the second and the third equations of (3.1) is valid for $t = 0$. Therefore it is sufficient to prove that the left terms of the equations of (3.1) satisfy the condition of Definition 2.1.

By using (3.1) for $t = m$ and $k = 0, 1, \cdots, 4q - 3$ we have

$$\begin{aligned} & f(4qm + k) + f(4qm + k + 1) \\ = & \frac{F_{2qm+k}F_{2qm+2q} + F_{2qm+k+1}F_{2qm+2q}}{F_{2q}} \\ = & \frac{F_{2qm+k+2}F_{2qm+2q}}{F_{2q}} = f(4qm + k + 2). \end{aligned} \tag{3.2}$$

By using the $(4q - 1)$th and $4q$th equations of (3.1) for $t = m$ we have

$$\begin{aligned} & f(4qm + 4q - 2) + f(4qm + 4q - 1) \\ = & \frac{F_{2qm+4q-2}F_{2qm+2q} + F_{2qm+4q-1}F_{2qm+2q}}{F_{2q}} \\ = & \frac{F_{2qm+4q}F_{2qm+2q}}{F_{2q}} = f(4qm + 4q), \end{aligned} \tag{3.3}$$

where the last equation is derived from the first equation of (3.1) for $t = m+1$. By using the $(4q + h)$th equation of (3.1) for $t = m$ and the first equation of (3.1) for $t = m + 1$ we have

$$\begin{aligned} & f(4qm + 4q - 1) + f(4qm + 4q) \\ = & \frac{F_{2qm+4q-1}F_{2qm+2q} + F_{2qm+2q}F_{2qm+4q}}{F_{2q}} \\ = & \frac{F_{2qm+2q}F_{2qm+4q+1}}{F_{2q}}. \end{aligned} \tag{3.4}$$

By using the second equation of (3.1) for $t = m + 1$ we have

$$f(4qm + 4q + 1) = \frac{F_{2qm+2q+1}F_{2qm+4q}}{F_{2q}}. \tag{3.5}$$

We are going to compare (3.4) and (3.5). It is well known that for any natural numbers u and v with $u > v$

$$(-1)^v F_{u-v} = F_{v+1}F_u - F_vF_{u+1}$$

, and hence if we let $u = 2qm + 4q$ and $v = 2qm + 2q$, then we have

$$F_{2qm+2q}F_{2qm+4q+1} + F_{2q} = F_{2qm+4q}F_{2qm+2q+1}. \tag{3.6}$$

Therefore by (3.4),(3.5),(3.6)

$$f(4qm+4q-1)+f(4qm+4q)+1=f(4qm+4q+1). \qquad (3.7)$$

By (3.2),(3.3),(3.7) the left terms of (3.1) satisfy the condition of Definition 2.1. □

Theorem 2 presents a formula for a natural number p such that p is a multiple of 4, and this is a generalization of the formula in [3]. This formula is quite simple, but the following formulas are for an arbitrary natural number p and the formulas are a little bit complicated.

Theorem 3. Let p be an even number. Here we denote $B_p(n)$ by $f(n)$, and we define $F_0 = f(0) = 0$.
Then $f(n)$ satisfies the following equations.

$$\begin{cases} f(pt) = \frac{F_{pt+p}-F_{pt}-F_0-F_p}{2F_{p+1}-F_p-2}, \\ f(pt+1) = \frac{F_{pt+p+1}-F_{pt+1}-F_1+F_{p-1}}{2F_{p+1}-F_p-2}, \\ f(pt+2) = \frac{F_{pt+p+2}-F_{pt+2}-F_2-F_{p-2}}{2F_{p+1}-F_p-2}, \\ \quad \vdots \qquad\qquad \vdots \\ f(pt+k) = \frac{F_{pt+p+k}-F_{pt+k}-F_k-(-1)^k F_{p-k}}{2F_{p+1}-F_p-2}, \\ \quad \vdots \qquad\qquad \vdots \\ f(pt+p-3) = \frac{F_{pt+p+p-3}-F_{pt+p-3}-F_{p-3}+F_3}{2F_{p+1}-F_p-2}, \\ f(pt+p-2) = \frac{F_{pt+p+p-2}-F_{pt+p-2}-F_{p-2}-F_2}{2F_{p+1}-F_p-2}, \\ f(pt+p-1) = \frac{F_{pt+p+p-1}-F_{pt+p-1}-F_{p-1}+F_1}{2F_{p+1}-F_p-2}. \end{cases} \qquad (3.8)$$

Proof. Since $f(1) = f(2) = 1$, the second and the third equations of (3.8) is valid for $t = 0$. Therefore it is sufficient to prove that the left terms of the equations of (3.8) satisfy the condition of Definition 2.1.

By using (3.8) for $t = m$ and $k = 0, 1, \cdots, p-3$ we have

$$\begin{aligned}
&f(pm+k) + f(pm+k+1) \\
&= \frac{F_{pm+p+k} - F_{pm+k} - F_k - (-1)^k F_{p-k}}{2F_{p+1} - F_p - 2} \\
&+ \frac{F_{pm+p+k+1} - F_{pm+k+1} - F_{k+1} - (-1)^{k+1} F_{p-k-1}}{2F_{p+1} - F_p - 2} \\
&= \frac{F_{pm+p+k+2} - F_{pm+k+2} - F_{k+2} - (-1)^{k+2} F_{p-k-2}}{2F_{p+1} - F_p - 2} \\
&= f(pm+k+2).
\end{aligned} \tag{3.9}$$

By using the $(p-1)$th and the pth equations of (3.8) for $t = m$ we have

$$\begin{aligned}
&f(pm+p-2) + f(pm+p-1) \\
&= \frac{F_{pm+p+p-2} - F_{pm+p-2} - F_{p-2} - F_2}{2F_{p+1} - F_p - 2} + \frac{F_{pm+p+p-1} - F_{pm+p-1} - F_{p-1} + F_1}{2F_{p+1} - F_p - 2} \\
&= \frac{F_{pm+p+p} - F_{pm+p} - F_p - F_0}{2F_{p+1} - F_p - 2} \\
&= \frac{F_{p(m+1)+p} - F_{p(m+1)} - F_0 - F_p}{2F_{p+1} - F_p - 2} \\
&= f(p(m+1)),
\end{aligned} \tag{3.10}$$

, where the last equation is derived from the first equation of (3.8) for $t = m+1$.

By using the pth equation of (3.8) for $t = m$ and the first equation of (3.8) for $t = m + 1$ we have

$$
\begin{aligned}
& f(pm + p - 1) + f(p(m + 1)) + 1 \\
= & \frac{F_{pm+p+p-1} - F_{pm+p-1} - F_{p-1} + F_1}{2F_{p+1} - F_p - 2} + \frac{F_{p(m+1)+p} - F_{p(m+1)} - F_0 - F_p}{2F_{p+1} - F_p - 2} + 1 \\
= & \frac{F_{pm+p+p+1} - F_{pm+p+1} - F_{p+1} + F_1}{2F_{p+1} - F_p - 2} + 1 \\
= & \frac{F_{pm+p+p+1} - F_{pm+p+1} + F_{p+1} + F_1 - F_p - 2}{2F_{p+1} - F_p - 2} \\
= & \frac{F_{p(m+1)+p+1} - F_{p(m+1)+1} - F_1 + F_{p-1}}{2F_{p+1} - F_p - 2} \\
= & f(p(m + 1) + 1), \qquad (3.11)
\end{aligned}
$$

, where the last equation is derived from the second equation of (3.8) for $t = m + 1$.

By (3.9),(3.10) and (3.11) the left terms of (3.8) satisfy the condition of Definition 2.1. $\square$

Theorem 4. Let p be an odd number. Here we denote $B_p(n)$ by $f(n)$, and we define $F_0 = f(0) = 0$.
Then $f(n)$ satisfies the following equations.

$$
\begin{cases}
f(pt) = \frac{F_{pt+p} + F_{pt} - F_0 - F_p}{2F_{p+1} - F_p}, \\
f(pt + 1) = \frac{F_{pt+p+1} + F_{pt+1} - F_1 + F_{p-1}}{2F_{p+1} - F_p}, \\
f(pt + 2) = \frac{F_{pt+p+2} + F_{pt+2} - F_2 - F_{p-2}}{2F_{p+1} - F_p}, \\
\quad \vdots \qquad\qquad \vdots \\
f(pt + k) = \frac{F_{pt+p+k} + F_{pt+k} - F_k - (-1)^k F_{p-k}}{2F_{p+1} - F_p}, \\
\quad \vdots \qquad\qquad \vdots \\
f(pt + p - 3) = \frac{F_{pt+p+p-3} + F_{pt+p-3} - F_{p-3} - F_3}{2F_{p+1} - F_p}, \\
f(pt + p - 2) = \frac{F_{pt+p+p-2} + F_{pt+p-2} - F_{p-2} + F_2}{2F_{p+1} - F_p}, \\
f(pt + p - 1) = \frac{F_{pt+p+p-1} + F_{pt+p-1} - F_{p-1} - F_1}{2F_{p+1} - F_p}.
\end{cases}
\qquad (3.12)
$$

Proof. Since $f(1) = f(2) = 1$, the second and the third equations of (3.12) is valid for $t = 0$. Therefore it is sufficient to prove that the left side of the equations in (3.12) satisfy the condition of Definition 2.1. By using (3.12) for $t = m$ and $k = 0, 1, \cdots, p-3$

$$\begin{aligned} & f(pm+k) + f(pm+k+1) \\ &= \frac{F_{pm+p+k} + F_{pm+k} - F_k - (-1)^k F_{p-k}}{2F_{p+1} - F_p} \\ &+ \frac{F_{pm+p+k+1} + F_{pm+k+1} - F_{k+1} - (-1)^{k+1} F_{p-k-1}}{2F_{p+1} - F_p} \\ &= \frac{F_{pm+p+k+2} + F_{pm+k+2} - F_{k+2} - (-1)^{k+2} F_{p-k-2}}{2F_{p+1} - F_p} \\ &= f(pm+k+2). \end{aligned} \tag{3.13}$$

By using the $(p-1)$th and the pth equations of (3.12) for $t = m$ we have

$$\begin{aligned} & f(pm+p-2) + f(pm+p-1) \\ &= \frac{F_{pm+p+p-2} + F_{pm+p-2} - F_{p-2} + F_2}{2F_{p+1} - F_p} + \frac{F_{pm+p+p-1} + F_{pm+p-1} - F_{p-1} - F_1}{2F_{p+1} - F_p} \\ &= \frac{F_{pm+p+p} + F_{pm+p} - F_p - F_0}{2F_{p+1} - F_p} \\ &= f(p(m+1)), \end{aligned} \tag{3.14}$$

where the last equation is derived from the first equation of (3.12) for $t = m + 1$.

By using the pth equation of (3.12) for $t = m$ and the first equation of (3.12) for $t = m + 1$

$$
\begin{aligned}
& f(pm + p - 1) + f(p(m+1)) + 1 \\
= & \frac{F_{pm+p+p-1} + F_{pm+p-1} - F_{p-1} - F_1}{2F_{p+1} - F_p} + \frac{F_{p(m+1)+p} + F_{p(m+1)} - F_0 - F_p}{2F_{p+1} - F_p} + 1 \\
= & \frac{F_{pm+p+p+1} + F_{pm+p+1} - F_{p+1} - F_1}{2F_{p+1} - F_p} + 1 \\
= & \frac{F_{pm+p+p+1} + F_{pm+p+1} + F_{p+1} - F_1 - F_p}{2F_{p+1} - F_p} \\
= & \frac{F_{p(m+1)+p+1} + F_{p(m+1)+1} - F_1 + F_{p-1}}{2F_{p+1} - F_p} \\
= & f(p(m+1) + 1), \qquad (3.15)
\end{aligned}
$$

, where the last equation is derived from the second equation of (3.12) for $t = m + 1$.

By (3.12),(3.14) and (3.15) the left terms of (3.12) satisfy the condition of Definition 2.1. □

Theorem 2, Theorem 3 and Theorem 4 show that there are very close relationships between Fibonacci-like sequences and the Fibonacci sequence. Therefore we can expect a wide range of applications for these Fibonacci-like sequences.

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