Least-Perimeter Partitions of the Sphere

Conor Quinn
Williams College, conor.b.quinn@gmail.com
Abstract. We consider generalizations of the honeycomb problem to the sphere $S^2$ and seek the perimeter-minimizing partition into $n$ regions of equal area. We provide a new proof of Masters' result that three great semicircles meeting at the poles at 120 degrees minimize perimeter among partitions into three equal areas. We also treat the case of four equal areas, and we prove under various hypotheses that the tetrahedral arrangement of four equilateral triangles meeting at 120 degrees minimizes perimeter among partitions into four equal areas.

1. Introduction

The Honeycomb problem seeks the least-perimeter way to partition the plane into unit areas. Long conjectured, the Honeycomb Theorem, which states that regular hexagons provide this least-perimeter partition, was proved by Hales in 1999 [7], see also [15] Ch. 15. The analogous conjecture for the sphere would be that a partition into $n$ congruent regular $m$-gons minimizes perimeter among partitions into $n$ equal areas. There exist only five such partitions: two disks meeting at a great circle, three digons meeting at antipodal points at 120 degrees, four equilateral triangles in a tetrahedral arrangement, six regular 4-gons in a cubical arrangement, and twelve regular pentagons in a dodecahedral arrangement. The octahedral arrangement of eight equilateral triangles is not perimeter minimizing, because it has vertices of degree four, which can profitably decomposed into two vertices of degree three (see Theorem 2.1). A similar argument excludes the icosahedral partition. In fact, Lamarle [11] and Heppes [9] proved that there exist only ten nets of geodesics
meeting in threes at 120 degrees on the sphere as in Figure 1 (see also [15]). These are all probably perimeter minimizing:

**Conjecture 1.1.** All ten spherical geodesic nets meeting in threes at 120 degrees minimize perimeter for prescribed areas.

![Figure 1](image)

**Figure 1.** The ten nets of spherical geodesics meeting at 120 degrees, all conjectured to minimize perimeter for fixed areas. Picture originally from Almgren and Taylor [1], copyright 1976 Scientific American.

Fejes Tóth proved each monohedral case (composed of congruent regular \( m \)-gons) of Conjecture 1.1 under the assumption that every region was convex ([21], Chapter VII, Section 31). Hales [8] generalized his planar methods to prove the \( n = 12 \) case of the conjecture, that the dodecahedral arrangement of twelve pentagons (see Figure 1(e)) provides the perimeter-minimizing partition into twelve equal areas. Hales’ proof is difficult and computational (see Section 6). We consider the \( n = 3 \) and \( n = 4 \) cases and seek a simpler, geometric proof. These \( n \) Honeycomb problems are special cases of the more general \( n-1 \) soap bubble problem, which seek the least-perimeter enclosure of some set of areas \( A_1, \ldots, A_{n-1} \). In 1994, Masters [12] proved, using tools of soap bubble theory and a computer argument, the double bubble theorem on the sphere, which states that two digons meeting at 120 degrees provide the least-perimeter enclosure of any two areas on the sphere (or equivalently the least-perimeter partition into three prescribed areas). The great difficulty in solving any of these problems is that soap bubbles can enclose area with disconnected regions. For example, the first region \( R_1 \) might have two 5-gon components and one 4-gon component, each enclosing area \( A_1/3 \). Much of the proof of Proposition
3.1 and the partial proof of Conjecture 5.1 is spent trying to prove that each region is connected.

1.1. **Section 3: n=3.** Our Proposition 3.1 provides a new elementary proof of the equal-area case of Masters’ result, as well as a nearly complete proof of the nonequal-area result, failing to eliminate the possibility that each region consists of two 4-gons (see Remark 3.2). Our proof of Proposition 3.1 depends on Lemma 2.21, a generalization of a result of Wichiramala [[18], Prop. 3.1], which limits the number of convex components in a stable nongeodesic $k$-bubble to $k$. Lemma 2.10 eliminates digons from any least-perimeter soap bubble except the standard double bubble (which consists of just two digons). The proof of Proposition 3.1 considers the possible composition of the high pressure region. Lemma 2.21 limits its number of components to two. Each component, since convex, has at most five edges, by Gauss-Bonnet. Since the other region and the exterior must alternate around it, it must have an even number of edges. Thus the high pressure region consists of one or two 4-gons, if nonstandard. We eliminate the one 4-gon case by showing that in such a bubble, the other region or the exterior must be composed of two digons. It is in eliminating the two 4-gon case that we must use the hypothesis that both regions and the exterior have equal area, since in this case, equal area implies that the whole bubble is geodesic. In order to exclude this and other geodesic soap bubbles, the proof utilizes an instability argument, which uses the second variation formula for soap bubbles, Proposition 5, to show that certain soap bubbles can be deformed with negative second variation of perimeter. For the two 4-gon case, we show that a deformation which shrinks one 4-gon and expands the other at unit rate preserves area but decreases perimeter to second order.

1.2. **Section 4: The Double Bubble in RP$^2$.** Conjecture 4.1 conjectures that the standard double bubble enclosing two areas, each smaller than the exterior, minimizes perimeter in RP$^2$.

1.3. **Section 5: n=4.** Section 5 treats the equal-area $n = 4$ case, Conjecture 5.1, that four equilateral triangles meeting at 120 degrees in a tetrahedral arrangement (see Figure 1(c)) provide a least-perimeter partition of the sphere into four equal areas. We prove in Theorem 5.2 that the result follows from any of five conditions: (1) the high pressure region is connected, (2) the low pressure region contains a triangle, (3) the high pressure region has the same pressure as any other region, (4) the partition contains a geodesic $m$-gon for odd $m$, or (5) the partition is geodesic. The proof of this result first shows that conditions (1)-(4) imply condition (5). We then refer to Proposition 2.20, which proves that perimeter-minimizing geodesic triple bubbles must be standard by excluding the other nine possible nets of spherical geodesics. Many of these cases are simply excluded by showing that there is no way to partition the components into four regions of equal area. The four 5-gon, four 4-gon case can be excluded by relabeling the components to make two edges superfluous.

The rest of section 5 contains a partial proof that the high pressure region must be connected. We first prove that there exist sixteen possible decompositions of the high pressure region (Proposition 5.9) by using Gauss-Bonnet to generate upper bounds on the area of convex $m$-gons for each $m = 3, 4, 5$. Lemmas 5.15 and 5.16 then exclude two of these decompositions by showing that when components...
attain their maximum area, they are geodesic. Each of these decompositions contains a pentagon with area equal to the upper bound; thus it is geodesic. Since all three regions are needed to surround a pentagon, all regions have the same pressure, and the whole partition is geodesic. We then use Theorem 5.2 to show that no partition containing a geodesic pentagon is perimeter minimizing. Lemma 5.14 excludes four cases by showing that each case in which the high pressure region contains two or more triangles is unstable. A result of Bezdek and Naszódi [4] shows that two such triangles are congruent (see Lemma 5.7). We then deform the partition by shrinking and expanding two of these triangles at unit rate, which initially decreases perimeter to second order.

1.4. **Open Questions.** Many cases of Conjecture 1.1 remain open. Only the cases for \( n = 2, 3, \) and 12 have been proven. The next case to consider after proving the \( n = 4 \) case is the final monohedral (each component congruent) conjecture, the \( n = 6 \) case (Figure 1(g)), which conjectures that the cubical arrangement of six 4-gons minimizes perimeter among partitions into six equal areas.

Since the component bounds of Lemma 2.21 do not apply to geodesic soap bubbles, they require special attention. Conjecture 2.23 says that any geodesic perimeter-minimizing soap bubble has connected regions. Since there are just ten geodesic nets, we need only analyze how their (at most twelve) components can be partitioned into \( n \) regions, \( n \leq 11. \) We prove the cases where \( n = 3 \) and \( n = 4 \) in (Propositions 2.19 and 2.20).

We also leave open a number of questions in the specific \( n = 3 \) and \( n = 4 \) cases. The proof of the general area case of Proposition 3.1 (see Remark 3.2) fails to exclude the case in which the high pressure region is composed of two 4-gons. We also conjecture that the standard double bubble is in fact the unique stable double bubble (Conjecture 3.3) and give the beginning of a proof, though many cases remain untreated. Much like the proof of Proposition 3.1, we proceed by cases on the composition of the high pressure region. Unfortunately, Lemma 2.10 does not exclude digons from merely stable soap bubbles, so we must consider more cases in the proof of Conjecture 3.3. We treat the cases where the high pressure region is composed of (1) one digon, (2) one 4-gon, and (3) two digons. We leave open the cases where the high pressure region is composed of (4) one digon and one 4-gon, and (5) two 4-gons. A proof of the analogous result in \( \mathbb{R}^2 \) appears in ([18], Theorem 3.2).

For \( n = 4, \) the main conjecture, Conjecture 5.1, remains open, though we prove in Theorem 5.2 that a number of conditions that, if verified, would each imply the result. Indeed, we prove that the perimeter-minimizing partition of \( S^2 \) is tetrahedral if (1) the high pressure region is connected, (2) the low pressure region contains a triangle, (3) the high pressure region has the same pressure as any other region, (4) the partition contains a geodesic \( m \)-gon for odd \( m, \) or (5) the partition is geodesic. However, an approach to verifying most of these conditions was not forthcoming. For instance, though we do prove in Proposition 5.9 that there exist just sixteen possible decompositions of the high pressure region, Lemmas 5.10 and 5.11 allow many more possible decompositions of the low pressure region. The approach to proving Conjecture 5.1 that we begin attempts to prove that in a perimeter-minimizing partition of the sphere into four equal areas the high pressure region is connected. We exclude six of fifteen disconnected cases for the composition of the high pressure region in Lemmas 5.14-5.16. The logical next cases to consider
are the final three cases in which the high pressure region has just two components: cases 2, 3, and 4. Another separate approach would prove that the perimeter-minimizing partition of the sphere must be geodesic by means of some new method.

1.5. The Triangular Isoperimetric Inequality. Finally, Section 6 gives a brief introduction to the methods of Hales’ proof, a second possible approach to the proof of the $n = 4$ (and $n = 6$) Honeycomb theorem.

1.6. Appendix. We conclude with a partial converse to Proposition 2.6 by Quinn Maurmann.

1.7. Acknowledgments. The author would like to thank his undergraduate advisor Professor Frank Morgan for his patience and helpful comments throughout the process. Some of this research began in the 2006 NSF Williams College SMALL undergraduate research program [5], and the author thanks Williams and the National Science Foundation for their support of the SMALL program. He thanks Colin Carroll, Adam Jacob, and Robin Walters for their help during the summer; he also thanks Rohan Mehra for his patience as a sounding board for even the author’s worst ideas. He thanks Quinn Maurmann for the wonderful appendix. Finally, the author sincerely thanks the Williams mathematics department for the opportunity to research at this level as an undergraduate.

2. Soap Bubbles and Partitions

Let $M$ be a smooth, compact Riemannian surface of area $A$. For given $n$, we consider two related problems:

**The Soap Bubble Problem.** Given $n - 1$ positive numbers $A_i$ with sum less than $A$, find the least-perimeter way to enclose and separate regions of area $A_i$.

**The Honeycomb Problem.** Find the least-perimeter way to partition $M$ into $n$ regions of equal area.

We note that the Honeycomb Problem is equivalent to the Soap Bubble Problem with every $A_i = A/(n-1)$, so that the exterior of the soap bubble has the same area as each bubble and can be considered the $n$th region in the Honeycomb problem. In this section, we will prove some results about soap bubbles in general.

**Theorem 2.1.** Existence and Regularity ([14], Thm. 2.3 and Cor. 3.3) Given a smooth compact Riemannian surface $M$ and positive areas $A_i$ summing to the total area of $M$, there is a least-perimeter partition of $M$ into regions of area $A_i$. It is given by finitely many constant-curvature curves meeting in threes at finitely many points.

**Definition 2.2.** A soap bubble or $n$-bubble consists of $n$ disjoint regions (not necessarily connected) of given areas, bounded and separated by finitely many smooth curves meeting in threes at finitely many points.

Propositions 2.5 and 2.6 will use the first variation formula, equation 1, to show that any candidate least-perimeter soap bubble will consist of constant-curvature curves meeting in threes at 120 degrees, where the sum of the three curvatures around any vertex is zero. The following is the $S^2$ case of the first variation formula ([10], Lemma 3.1).
Proposition 2.3. Consider a soap bubble in $\mathbb{S}^2$ and a smooth variation vector field $u$ for which $dA/dt = 0$ on each region at time zero. Any such $u$ is the initial velocity of smooth area-preserving flows. Let $u_{ij}$ be the component of $u$ normal to the interface between $R_i$ and $R_j$, $\kappa_{ij}$ be the curvature of the interface between $R_i$ and $R_j$, and let $T_i(p)$ be the unit tangent vector to the $i$th incident curve at a point $p$. Then the first variation of perimeter equals

\begin{equation}
-\sum_{0<i<j} \int \kappa_{ij} u_{ij} - \sum_p u \cdot (T_1(p) + T_2(p) + T_3(p))
\end{equation}

Definition 2.4. A soap bubble is in equilibrium if its first variation vanishes.

Proposition 2.5. For a soap bubble in equilibrium, each region has a pressure, defined up to addition of a constant, so that the sum of the curvatures crossed by a path from the exterior to the interior of that region is its pressure.

By convention, the pressure of the exterior or the region of least pressure (on compact surfaces) is often taken to be zero.

Proof. We let the pressure of the exterior be zero and then show that any path to a certain region yields the same pressure for that region. It is equivalent to show that the sum of the curvatures crossed by any path which begins and ends in the same region is zero. Suppose there exists a path $\gamma$ which begins and ends in $R_1$ and that the sum of the curvatures crossed by the path is nonzero. If the sum of the curvatures crossed by a path is negative, then the sum of the curvatures crossed by the same path with opposite orientation is positive. Thus without loss of generality, we will consider a path so that the sum of the curvatures crossed is positive. We can choose small balls $B_1, \ldots, B_n$ on the interfaces crossed by $\gamma$ so that $\sum \gamma \kappa_i > 0$, where $\kappa_i$ is the curvature at any point in ball $B_i$. Consider a nontrivial deformation supported inside $B_1, \ldots, B_n$ with initial normal velocity $u$ such that $\int_{B_i} u = \int_{B_j} u$, so that area is preserved. We compute the initial first variation of perimeter for this deformation according to equation 1. Since $u$ vanishes on vertices, the second term vanishes. Choose constants $a_i$ so that $k_i \geq a_i$ with $\sum a_i > 0$ since $\sum \gamma \kappa_i > 0$. Then, since $\int_{B_i} u = \int_{B_j} u$ is some positive number, the $\sum_{0<i<k} a_i u_{ik} \geq \sum_{0<i<k} \int \kappa_{ik} u_{ik}$ term is positive, so the initial first variation is negative, a contradiction. □

For convenience, we will order the regions of a $k$-bubble so that $R_1$ has pressure greater than or equal to $R_2$, and $R_2$ has pressure greater than or equal to $R_3$, and so on, so that $\kappa_{ij} \geq 0$ for any $0 < i \leq j$.

Proposition 2.6. For a soap bubble in equilibrium, the interface between any two regions is of constant curvature, and the curves meet at 120 degrees, where the sum of the three curvatures at any vertex is zero.

Proof. It follows from Proposition 2.5 that the interface between any two regions is of constant curvature and that the sum of the three curvatures about a vertex is zero.

Suppose there exists a vertex $p$ about which the curves do not meet at 120 degrees. Take any nontrivial area-preserving deformation defined only in a small ball about this vertex, so that the $\int \kappa_{ij} u_{ij}$ term is smaller in absolute value than $u \cdot (T_1(p) + T_2(p) + T_3(p))$. Since the second term is nonzero and the first term is smaller, even if it is positive, the initial first variation will be negative, a contradiction. □
We include, in the appendix, a partial proof of the converse to Proposition 2.6 by Quinn Maurmann.

**Lemma 2.7.** The boundary of any component of a region of a perimeter-minimizing soap bubble of the sphere is connected. Furthermore, the partition itself is connected.

*Proof.* Suppose there exists a component of the partition with disconnected boundary (e.g., an annulus). A component of the boundary can be slid, maintaining perimeter and area, until it intersects another boundary component, which would create a valence four vertex, contradicting Proposition 2.6. A similar argument shows that the whole partition must be connected. □

The following corollary allows us to apply Gauss-Bonnet to any component of a perimeter-minimizing partition.

**Corollary 2.8.** Any component in a perimeter-minimizing partition of $\mathbb{R}^2$, $S^2$ or $H^2$ is topologically a disk.

*Proof.* Since the boundary of any component is connected and cannot cross itself, every component is topologically a disk. □

**Definition 2.9.** In this paper, an *m*-gon will have constant-curvature edges; a *geodesic m*-gon will have geodesic edges. We will refer to a 2-gon as a *digon*, and a 3-gon as a *triangle*.

**Lemma 2.10.** In a perimeter-minimizing partition of the sphere into $n > 2$ regions of prescribed area, each component must have at least two edges. Furthermore, if a perimeter-minimizing partition of the sphere contains a digon, then it is the standard 3-digon partition.

*Proof.* Suppose there is a 1-gon in a minimal partition. Then it must be entirely on the interior of another region, contradicting Lemma 2.7.

Suppose there exists a partition of the sphere containing a digon. Then, unless the partition is the standard double bubble, there are vertices in the partition, which are not on the digon, along the edges which extend from the vertices of the digon. The digon can be slid toward one such vertex without altering the area of the two regions it bounds or the total length of the partition (see Figure 2). Slide the digon as such until one of its vertices coincides with another vertex in the partition, but this new vertex will have valence four, contradicting Proposition 2.6. □

![Figure 2](image-url)
Lemma 2.11. A perimeter-minimizing partition of the sphere into \( n > 2 \) prescribed areas does not contain a set of components whose union is a digon, with distinct incident edges.

Proof. If there exists such a digon, it can be slid as in the proof of Lemma 2.10 until it bumps into something else, contradicting regularity. \( \square \)

Lemma 2.12. In \( \mathbb{R}^2, \mathbb{S}^2, \) and \( \mathbb{H}^2, \) for given algebraic area (so that regions which overlap themselves count the overlapped areas with multiplicity), a circle minimizes perimeter among oriented curves.

Proof. Let \( R \) be a region with integer multiplicity. As in [[15], Figure 10.1.1], decompose \( R \) as a sum of nested regions of multiplicity one. Each region of area \( A_i \) has at least as much perimeter as the perimeter \( P(\tilde{A}_i) \) of a circle enclosing area \( A_i. \) By the strict concavity of \( P, \) a single circle is uniquely best. \( \square \)

Lemma 2.13. In \( \mathbb{S}^2, \mathbb{H}^2, \) and \( \mathbb{R}^2, \) there exists a circular \( n \)-gon (an \( n \)-gon whose vertices lie on the same circle) of prescribed edge lengths. In \( \mathbb{S}^2, \) we assume that the sum of the edge lengths is less than \( 2\pi, \) and it follows that the \( n \)-gon lies in a hemisphere.

Proof. Let \( l_1, \ldots, l_n \) be a set of \( n \) edge lengths, ordered so that \( l_1 < l_2 < \ldots < l_n. \) Choose some large circle with diameter longer than \( l_n. \) Connect \( n \) points on this circle so that the chords between them have lengths \( l_1, \ldots, l_{n-1}, \) and the \( n \)th distance is longer than \( l_n. \) Now shrink the circle, adjusting the \( n-1 \) short chords so that the lengths stay the same, until the \( n \)th chord reaches its correct length, so that the circle as it shrinks arrives at the correct length of \( l_n \) before or at the point where the diameter is \( l_n. \) \( \square \)

Lemma 2.14. Among geodesic triangles in \( \mathbb{S}^2 \) of given base length and perimeter less than \( 2\pi, \) an isosceles triangle uniquely maximizes the smaller enclosed area. Among geodesic triangles in \( \mathbb{H}^2 \) of given base length and perimeter, an isosceles triangle uniquely maximizes area enclosed, where triangles in \( \mathbb{H}^2 \) have area less than \( \pi. \)

Proof. For \( \mathbb{S}^2, \) L’Huillier’s Theorem [[22] or [19]] states that for any geodesic triangle of side lengths \( a, b, \) and \( c \) enclosing area \( A, \)

\[
\tan^2(A/4) = \tan\left(\frac{a+b+c}{4}\right)\tan\left(\frac{b+c-a}{4}\right)\tan\left(\frac{a+c-b}{4}\right)\tan\left(\frac{a+b-c}{4}\right).
\]

Note that both the smaller area \( A \) and the larger area \( 4\pi - A \) yield the same \( \tan^2(A/4), \) and that both sides of the equation diverge as the triangle approaches a great circle, with area and perimeter \( 2\pi. \) Also note that the asserted (embedded) isosceles triangle exists because the prescribed perimeter is less than \( 2\pi. \)

Fix base length \( a \) and the sum of the two other legs \( b + c, \) which fixes perimeter. Using the identities \( \sin(u)\sin(v) = 1/2[\cos(u-v) - \cos(u+v)] \) and \( \cos(u)\cos(v) = 1/2[\cos(u-v) + \cos(u+v)], \) we can see that \( \tan^2(A/4) \) is proportional to

\[
\tan\left(\frac{a+c-b}{4}\right)\tan\left(\frac{a+b-c}{4}\right) = \frac{\sin\left(\frac{a+c-b}{4}\right)\sin\left(\frac{a+b-c}{4}\right)}{\cos\left(\frac{a+c-b}{4}\right)\cos\left(\frac{a+b-c}{4}\right)} \cdot \frac{\cos\left(\frac{c-b}{2}\right) - \cos\left(\frac{c}{2}\right)}{\cos\left(\frac{c-b}{2}\right) + \cos\left(\frac{c}{2}\right)}.
\]
Thus area is maximized when \( c - b = 0 \).

Therefore in \( S^2 \), isosceles triangles maximize area among triangles with fixed base length and perimeter.

In \( H^2 \), the area of a triangle with side lengths \( a, b, c \) is given by the hyperbolic Heron’s formula \([13], \text{Ch. 8, equation 27}\) or \([20]\):

\[
\tan(A/2) = \sqrt{1 - \alpha^2 - \beta^2 - \gamma^2 + 2\alpha\beta\gamma},
\]

where \( \alpha, \beta, \) and \( \gamma \) are the hyperbolic cosines of the three side lengths \( a, b, \) and \( c \). Note that the right side of the equation is always positive, which implies that the area of a hyperbolic triangle is always less than \( \pi \). Consideration of large hyperbolic triangles shows that the angle measures approach zero as the triangle gets larger, and a simple calculation using Gauss-Bonnet shows that such a triangle will have area approaching \( \pi \) from below.

Again, we fix \( a \) and \( b + c \), which fixes perimeter, and now we will show that area is maximized when \( b = c \). Heron’s formula for this case simplifies to

\[
\tan^2(A/2) = \frac{-l - \cosh^2(b) - \cosh^2(c) + 2k\cosh(b)\cosh(c))}{(m + \cosh(b) + \cosh(c))^2}
\]

for some positive constants \( k, l, m, k = \cosh(a) \). In order to determine where the maximum of this function occurs, we first note that the convexity of \( \cosh(x) \) implies that the denominator is minimized when \( b = c \). Thus it suffices to show that the numerator is maximized by \( b = c \). We will first take the first and second derivatives of the numerator, then show that the only possible maximum occurs at \( b = c \). Let \( t = b + c \).

Then the numerator is maximized when the function

\[
f(b) = 2k\cosh(b)\cosh(t - b) - \cosh^2(b) - \cosh^2(t - b) - l
\]

is maximized. The first derivative of \( f(b) \) is

\[
f'(b) = \frac{(e^{4b} - e^{2t})e^{-2b-2t}(e^{2t} - 2ke^t + 1)}{2},
\]

which has zeroes where \( e^{2t} - 2ke^t + 1 = 0 \) or \( b = t/2 \), the desired result. The second derivative of \( f(b) \) is

\[
f''(b) = -(e^{4b} + e^{2t})e^{-2b-2t}(e^{2t} - 2ke^t + 1).
\]

The triangle inequality implies that \( c < a + b \), so \( k = \cosh(a) < \cosh(b + c) = 1/2(e^{b+c} + e^{-b-c}) \). Thus \( e^{2t} - 2ke^t + 1 = e^{2b+2c} - 2\cosh(a)e^{b+c} + 1 > e^{2b+2c} - (e^{b+c} + e^{-b-c})e^{b+c} + 1 = 0 \). Thus \( e^{2t} - 2ke^t + 1 > 0 \) and it follows that \( f''(b) \) is always negative, thus the only remaining zero \( b = t/2 \) must be the maximum. Thus area is maximized when \( b = c \). It follows that in \( H^2 \), isosceles triangles maximize area among triangles with fixed base length and perimeter.

\[\square\]

The following proposition proves that regular geodesic \( m \)-gons minimize perimeter among geodesic \( m \)-gons with fixed \( m \) and area. The \( R^2 \) case was proven by Zenodorus (see [16]), the \( S^2 \) case in ([21], Chapter VII, Section 30), see also [6],
and the $\mathbb{H}^2$ case in [3], see also [2]. Fejes Tóth [21] provides a beautiful, geometric proof of the result in $\mathbb{S}^2$.

**Proposition 2.15. (The Polygonal Isoperimetric Inequality)** Among $m$-gons in $\mathbb{R}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$, the regular $m$-gon uniquely minimizes perimeter for a given area (less than $2\pi$ for $\mathbb{S}^2$).

*Proof.* For any area, there exists an embedded regular geodesic $m$-gon (Lemma 2.13), unique up to isometry. We first show that the vertices of a perimeter-minimizing $m$-gon lie on a circle. Suppose there exists a perimeter-minimizing $m$-gon which is not inscribed in a circle. By Lemma 2.13, we can construct a new circular $m$-gon with the same edge lengths as the original polygon.

*Figure 3.* To prove irregular polygons not minimizing, we construct a curvilinear irregular polygon from any irregular polygon in the following way (clockwise from top left): (1) construct a circular $m$-gon with the same side lengths as the original irregular polygon, (2-3) bulge out the sides of this new $m$-gon so that it becomes a circle, and finally (4) replace the edges of the original irregular polygon with the circular arcs from the bulged circular polygon. This curvilinear $m$-gon will have the same perimeter as the circle, but we assumed that it encloses more area, the desired contradiction. Thanks to Anthony Marcuccio for the picture.

Bulge out the sides of this new circular polygon (which has the same geodesic length as the original polygon), so that it becomes the circle, as in Figure 3. Back on the original polygon, bulge out each edge the same amount. The resulting curvilinear $m$-gon will have the same perimeter but enclose more area than a circle, a contradiction of Lemma 2.12.

Consider a non-equilateral circular $m$-gon. Connect some pair of nonconsecutive vertices so as to create a triangle which is not isosceles. Replacing the non-isosceles triangle with an isosceles one encloses area more efficiently, by Lemma 2.14. □

The following formula is the 2D case of the second variation formula given in [10], Lemma 3.3 and Remark 3.4.
Proposition 2.16. Consider an equilibrium soap bubble on a surface with Gauss curvature $G$ and smooth variation vectorfield $u$, where $u$ corresponds to a deformation for which $dA/dT = 0$ on each region at time zero. Any such $u$ is the initial velocity of many smooth area-preserving flows. Then the second variation of perimeter equals the sum over $0 < i < j$ of

\[
\int_{cl(R_i) \cap cl(R_j)} (u_{ij}^2 - \kappa_{ij}^2 u_{ij}^2) - \Sigma_p (u_{ij}^2(p) q_{ij}(p)) - \int_{cl(R_i) \cap cl(R_j)} G u_{ij}^2
\]

where $u_{ij}$ is the component of $u$ in the direction normal to the interface between $R_i$ and $R_j$, zero at a point not on the interface, and $\kappa_{ij}$ its curvature, nonnegative where $R_i$ is convex, and, at a point $p$ where $R_1, R_2,$ and $R_3$ meet,

\[
q_{12}(p) = \frac{\kappa_{13} + \kappa_{23}}{\sqrt{3}}
\]

Definition 2.17. An equilibrium soap bubble $B$ is stable if it is at least as efficient as its neighbors, by which we mean that there does not exist a smooth area-preserving ambient diffeomorphism which initially reduces perimeter. Note that positive second variation implies stability, which implies nonnegative second variation.

Lemma 2.18. A perimeter-minimizing soap bubble is stable.

Proof. Suppose not, then there exists some ambient diffeomorphism which generates a similar soap bubble with the same areas and less perimeter, a contradiction. □

Lemma 2.21 and Corollary 2.22 will show that certain stable $k$-bubbles have at most $k$ convex components. Propositions 2.19 and 2.20 verify this for geodesic bubbles. Lemma 2.21 treats nongeodesic bubbles. Conjecture 2.23 conjectures that geodesic $k$-bubbles have connected components, for all $k \leq 10$ or $k = 12$. These ten nets are shown in Figure 1.

Proposition 2.19. In $S^2$, a stable double bubble with geodesic components is standard, i.e., it consists of three constant-curvature curves meeting at 120 degrees.

Proof. We use a result of Heppes [9], which states that there exist precisely ten nets of geodesics meeting at 120 degrees on the sphere (see Figure 1). Any component of a double bubble has an even number of edges, since the other region and exterior must alternate around it, so we may immediately exclude all competitors except the cube arrangement of six squares (Figure 1(g)). The cube arrangement is unstable, as expanding and shrinking opposite faces (which must belong to the same region) at unit speed preserves area by symmetry but has negative second variation, since all the terms except the $\int G u_{ij}^2$ term vanish, and this term is positive for nontrivial deformations on surfaces with positive Gauss curvature. □

Proposition 2.20. Consider a least-perimeter partition of the sphere into four equal areas. If it is geodesic, then it is the tetrahedral partition, i.e., it consists of four congruent, geodesic, equilateral triangles.

Proof. We consider the ten cases of Figure 1, which Heppes [9] proved to be the only spherical nets of geodesics meeting at 120 degrees.
We may disregard nets (a), and (b), since they have fewer than four components. Net (c) is the tetrahedral partition.

Net (d), the triangular prism, can not occur in this type of partition as any division of these components into four regions leaves the two triangles identified, with the other three regions occupying 4-gons. But a geodesic 4-gon with interior angles 120 degrees has area $2\pi/3$, which contradicts the equal area hypothesis.

Net (e), twelve pentagons, cannot be separated into four regions without two components of one region sharing an edge, since there are only three sets of three 5-gons which are pairwise disjoint in any partition into twelve 5-gons.

Net (f), consisting of eight 5-gons and two 4-gons, cannot be split into four regions of area $\pi$, since any region consisting of only 5-gons must have three components, and there is no way to identify three 5-gons without them sharing an edge.

Net (g), six 4-gons, cannot be divided into four regions of area $\pi$.

Net (h), the pentagonal prism, also cannot be divided into four regions of area $\pi$.

Net (i), composed of four 4-gons and four 5-gons does not minimize perimeter. Indeed, each region must be composed of one 4-gon and one 5-gon. Gauss-Bonnet implies that all 4-gons have the same area and all 5-gons have the same area. Since each 4-gon is adjacent to three 5-gons, there is a unique division of the eight components into four equal-area regions. We will denote the components of $R_i$ as $R_{i5}$ and $R_{i4}$. The unique division results in $R_{14}$ adjacent to $R_{25}$ and $R_{15}$ adjacent to $R_{24}$. Switching $R_{14}$ and $R_{15}$ preserves the areas of each region and the length of the partition, but results in a partition which has adjacent components of the same region, thus it is not perimeter minimizing.

Net (j), three 4-gons and six 5-gons, would break down into four equal area regions as: $((4,5),(4,5),(4,5),(5,5,5))$, but such a division would result in two components of the same region sharing an edge, since any two 5-gons which don’t share an edge interface every other 5-gon. 

The following lemma, a generalization of the planar lemma given in ([18], Prop. 3.1), allows us to considerably limit the number of cases in the proof of Proposition 3.1 and in the partial proof of Conjecture 5.1.

**Lemma 2.21.** On a surface with positive Gauss curvature, a stable soap bubble of $k$ regions has at most $k$ convex components, unless the soap bubble is geodesic.

**Proof.** To obtain a contradiction, let $B$ be such a soap bubble cluster of $k$ regions with more than $k$ convex components. Let $u_l$ be the vectorfield which corresponds to a deformation which expands the $l$th convex component at unit rate while leaving all other components unchanged. If the vectorfields $u_l$ are linearly dependent, then some nontrivial linear combination $\Sigma a_l u_l$ vanishes. Thus the components for which $a_l$ is nonzero completely surround each other and every component is convex, thus every component is geodesic, a contradiction.

If the $u_l$ are linearly independent, there are more than $k$ free variables and just $k$ linear area constraints; thus some linear combination $u = \Sigma a_l u_l$ is area preserving.

We apply the second variation formula (equation 5) to $u$. The $u^2_{il}$ terms are all zero because the $u_{ij}$ are constant. The $\kappa^2_{ij} u^2_{ij}$ terms are always nonnegative. To show that the $q_{ij}(p)$ terms are always nonnegative, we consider three cases.
Case 1: only one component at $p$ is convex. The sum of the $q_{ij}(p)$ terms is nonnegative since at each point at which $u_{ij}$ nonzero, two incident edges are convex, and their difference is the curvature of the third component, so even if the third curvature is negative, it is smaller in magnitude than the value of the two others.

Case 2: Only two components at $p$ are convex: then the curvature of the interface between the two convex components is zero, and the curvature between these and the nonconvex component is positive, so the $u_{ij}q_{ij}(p)$ terms which do not vanish are nonnegative.

Case 3: All three components are convex: in this case every incident curvature is zero, so the $u_{ij}q_{ij}(p)$ terms vanish at these points. So at each point at which a convex component is incident the $u_{ij}q_{ij}(p)$ terms are nonnegative.

The $\Sigma \int Gu^2_{ij}$ term is positive for any nontrivial variation, since $G > 0$.

Therefore, the resulting initial second variation of perimeter is strictly negative, the desired contradiction. □

**Corollary 2.22.** On the sphere, a stable double bubble has at most two convex components. A perimeter-minimizing triple bubble has at most three convex components.

**Proof.** Lemmas 2.19 and 2.20 prove the result for geodesic bubbles, and Lemma 2.21 proves the result for nongeodesic bubbles □

Some notes on generalization: for $m$-bubbles with $m > 3$, the above proof requires either a hypothesis that there is at least one nongeodesic component or a result analogous to 2.19 and 2.20 as in Conjecture 2.23 below, which shows that a geodesic soap bubble has connected components. The original proof in [18] required the hypothesis that the convex components be nongeodesic and nonadjacent. Our approach requires positive Gauss curvature.

**Conjecture 2.23.** A stable geodesic soap bubble has connected regions and exterior.

3. **The Double Bubble Problem on the Sphere**

Joe Masters [12] proved in 1994 that the standard double bubble, consisting of three constant-curvature arcs meeting at 120 degrees, minimizes perimeter among partitions of the sphere into any three areas. In Proposition 3.1, we give a new, simplified proof of the equal-area case of Masters’ result. Remark 3.2 discusses an incomplete proof of Masters’ more general result. In Conjecture 3.3, we conjecture that the standard double bubble is the unique stable double bubble on the sphere. We conclude with a partial proof of Conjecture 3.3.

**Proposition 3.1.** The standard double bubble minimizes perimeter among enclosures of two equal areas in $S^2$.

**Proof.** By Theorem 2.1, a perimeter-minimizing soap bubble of given areas $A_1, A_2$ exists. We observe that any component of such a double bubble will have an even number of sides, since the other region and the exterior must alternate around it. We now proceed by cases on the region with greater or equal pressure. Since it has greater pressure than the other region and the exterior, its components must be convex. Since perimeter-minimizing double bubbles are stable, it has two or fewer components. By Lemma 2.6, any component of an equilibrium double bubble has 120 degree interior angles, by Gauss-Bonnet such a convex component will have
fewer than six edges. Therefore we must consider only the cases where the higher pressure region is composed of (1) one digon; (2) two digons; (3) one 4-gon; (4) two 4-gons; or (5) one digon and one 4-gon. Cases 2 and 5 are excluded immediately by Lemma 2.10.

Case 1: Lemma 2.10 implies that if the higher pressure region is one digon, then the double bubble must be standard.

Case 3: Since vertices in a perimeter-minimizing bubble cluster have valence three, in a double bubble, each vertex must lie on both regions and the exterior. If the higher pressure region is one 4-gon, then its four vertices are the only vertices in the double bubble. As a result, either the lower pressure region or the exterior is two digons, contradicting Lemma 2.10.

Case 4: The high pressure region has two 4-gon components. Note that the larger component has area at least $2\pi/3$. Applying Gauss-Bonnet to this component:

\[
2\pi - \sum (\pi - \alpha_i) - \int_{\partial R} \kappa_i = \int \int_R GdA
\]

\[
2\pi - 4\left(\frac{\pi}{3}\right) - \int_{\partial R} \kappa_i = A
\]

\[
\frac{2\pi}{3} - \int_{\partial R} \kappa_i \geq \frac{2\pi}{3}
\]

\[
\int_{\partial R} \kappa_i \leq 0.
\]

Since it has greater or equal pressure than its adjacent regions, it must be geodesic, so all regions have the same pressure, and all the components of the partition are geodesic. Heppes [9] proved that there exist precisely ten nets of spherical geodesics meeting at 120 degrees. Only one of these is composed of six 4-gons, namely the cubical partition of six congruent 4-gons (see Figure 1g).

We show the six regular 4-gon partition to be unstable by use of the second variational formula, equation 5. Let $u$ be the vectorfield which is the outward unit normal to one component of the first region and the inward unit normal to the other component of the first region. By symmetry, initially $dA/dt$ vanishes for all three regions. Applying the second variational formula to this deformation, the only nonvanishing term is the $-\int u^2_{ij}$ term, since the deformation is constant and all the curvatures are zero. So the second variation of perimeter is negative, the desired contradiction.

Thus the unique minimizer must be the standard double bubble consisting of three constant-curvature curves meeting at 120 degrees. □

Remark 3.2. The above proof of Proposition 3.1 generalizes almost entirely to the nonequal-area double bubble problem, except the section which eliminates Case 4, the four 4-gon double bubble.

Conjecture 3.3. The standard double bubble is the unique stable double bubble.

Partial Proof. As with the previous proof, we proceed on the cases for the higher pressure region, $R_1$. Components of a stable bubble have an even number of edges, and components of the higher pressure region will have fewer than six edges. Since $R_1$ has greater or equal pressure than $R_2$, its components are convex, so the number of components of $R_1$ is limited to two by Lemma 2.21.
Case 1: \( R_1 \) is composed of one digon. Since all three regions must alternate around each vertex, such a bubble cluster will have only two vertices. Thus it is standard.

Case 2: \( R_1 \) is composed of one 4-gon. Again, the only vertices are the four around the high pressure region, so either \( R_2 \) or the exterior is two digons. Expanding one of the digons along its interface with either \( R_2 \) or the exterior while shrinking the other along its interface with the exterior at unit rate preserves area while decreasing perimeter, thus the double bubble is not stable.

Case 3: \( R_1 \) is composed of two digons. Just as in Case 2, expanding and shrinking the two digons at unit rate along their interface with the exterior preserves area while initially decreasing perimeter, thus this case is unstable.

Case 4: \( R_1 \) is composed of one digon and one 4-gon. This case remains open.

Case 5: \( R_1 \) is composed of two 4-gons. There are a number of subcases to Case 5, each of which remain open.

4. **The Double Bubble Problem in \( \mathbb{RP}^2 \)**

We now consider \( \mathbb{RP}^2 \), the sphere with antipodal points identified, or equivalently, a hemisphere with antipodal points on the equator identified. A round disk or the complement of a disk minimizes perimeter among enclosures of one area [17]. The round disk minimizes when enclosed area \( A \leq \pi \), the disk complement minimizes when \( A \geq \pi \). For small areas, the standard double bubble minimizes perimeter ([18], Thm 3.3).

**Conjecture 4.1.** The standard double bubble enclosing areas \( A_1, A_2 < 2\pi - (A_1 + A_2) \) is a perimeter-minimizing double bubble in \( \mathbb{RP}^2 \).

5. **The Honeycomb Problem for \( n = 4 \)**

Conjecture 1.1 says that the tetrahedral arrangement of four equilateral triangles meeting at 120 degrees minimizes perimeter among partitions of the sphere into four equal areas. We give an partial proof. Our Theorem 5.2 proves the result given any of a set of hypotheses. The partial case analysis below attempts to prove that the hypothesis that the high pressure region be connected is unnecessary, thereby proving the result.

**Conjecture 5.1.** The perimeter-minimizing partition of the sphere into four equal areas is the tetrahedral arrangement of four geodesic triangles meeting at 120 degrees.

**Theorem 5.2.** A perimeter-minimizing partition of the sphere into four equal areas is tetrahedral if any of these five conditions are met: (1) the high pressure region \( R_1 \) is connected, (2) the low pressure region \( R_4 \) contains a triangle, (3) the partition contains a geodesic \( m \)-gon with \( m \) odd, (4) the high pressure region \( R_1 \) has the same pressure as some other region, or (5) the partition is geodesic.

We will need the following lemma before we can prove Theorem 5.2.

**Lemma 5.3.** In an equilibrium partition of \( S^2 \) into four equal areas, a connected convex region is a geodesic triangle. Furthermore, if such a partition contains a geodesic triangle, then it is geodesic.
Proof. Let $R$ be a connected convex region with $n$ sides. $R$ has area $\pi$, and its angle measures are $2\pi/3$ from Lemma 2.6. A simple calculation using the Gauss-Bonnet formula shows that this region is geodesic and has three edges. If an equilibrium partition into four regions contains a geodesic triangle, every adjacent region and hence every region has the same pressure; thus the partition is geodesic.

Proof of Theorem 5.2. First we will show that each of the first four conditions implies the fifth. Note that any triangle must interface all three other regions.

(1) Suppose $R_1$ is connected. By Lemma 5.3 it must be a triangle. By Gauss-Bonnet, a triangle of area $\pi$ and interior angles $2\pi/3$ has integral curvature zero. Since each edge of a highest pressure region must have nonnegative curvature, each edge must have zero curvature, so the component is geodesic. Since any triangle must interface all three other regions, all four regions must have the same pressure, and the whole partition must be geodesic.

(2) Suppose $R_4$ contains a triangle. By Proposition 5.11 it is connected. As above, Gauss-Bonnet implies that a triangle of area $\pi$ and interior angles $2\pi/3$ has integral curvature zero. Since each edge of a lowest pressure region must have nonpositive curvature, the region must be geodesic. Just as above, this implies that the partition is geodesic.

(3) Since all three other regions are necessary to completely surround an $m$-gon when $m$ is odd, a geodesic odd $m$-gon implies that every region has the same pressure. Thus the partition is geodesic.

(4) Suppose that $R_1$ has the same pressure as $R_2$. Then components of both regions are convex. Thus there are at most three total. Then at least one region is connected and convex, thus by Lemma 5.3 it is a geodesic triangle and the partition is geodesic by (3) above.

(5) If the partition is geodesic, then Proposition 2.20 implies that it is standard. □

The remainder of this section provides a partial proof that the high pressure region must be connected. There will be fifteen cases which must be eliminated in order to prove this. Lemma 5.14 will eliminate any of these in which the high pressure region contains two or three triangles. Lemmas 5.15 and 5.16 will also completely eliminate the cases where the high pressure region is composed of (1) one 4-gon and one 5-gon and (2) three 5-gons. We will first prove a series of lemmas on the general structure of 4-Honeycombs.

Lemma 5.4. In a perimeter-minimizing partition of $S^2$ into four equal areas, if two 3-gons share an edge, then the partition is the tetrahedral arrangement.

Proof. The union of two adjacent 3-gons is a digon, and if the partition is not tetrahedral, then the two edges incident to the union of the 3-gons are distinct. It follows from corollary 2.11 that such a partition is not perimeter minimizing. □

Lemma 5.5. In a perimeter-minimizing partition of the sphere into four equal areas, $R_2$ has at most six components.

Proof. We first note that $R_1, R_2,$ and $R_3$ have at most three convex components altogether, by Lemma 2.21. Since any component of $R_2$ not adjacent to a component of $R_1$ is convex, there are at most $3 - c$ components of $R_2$ which are not adjacent to $R_1$, where $c$ is the total number of components of $R_1$. On a 3-gon component of $R_1$, there can be at most one adjacent component of $R_2$; on a 4-gon and 5-gon,
at most 2. If $c = 1$, then by Lemma 5.13 $R_1$ is a 3-gon; then $R_2$ has at most 3 components, since $3 - c = 2$ and only one component of $R_2$ can be adjacent to $R_1$. If $c = 2$, $R_2$ has at most $3 - c = 1$ nonadjacent component and at most 4 adjacent components: at most five components total. If $c = 3$, $R_2$ has $3 - c = 0$ nonadjacent components and at most 6 adjacent components: at most 6 components total. □

**Lemma 5.6.** In a perimeter-minimizing partition of the sphere into four equal areas, a 5-gon cannot share two edges with 3-gons when the two 3-gons belong to the same region.

**Proof.** Suppose not. Consider a 5-gon with two 3-gons adjacent. We first note that the two 3-gons do not occupy consecutive edges, otherwise they would share an edge, and by Proposition 5.4, if two 3-gons share an edge, then the partition is either tetrahedral or not minimal. There are three incomplete vertices in the union of these three components (we will call this union the center), so each of the components surrounding the center bounds both of the other two surrounding components. Thus each of the surrounding components must belong to a different region. Additionally, each surrounding component intersects both the 5-gon and at least one of the 3-gons. One of the surrounding components intersects all three (the 5-gon and the two 3-gons). We will say that this component is part of $R_3$ and that the 5-gon is part of $R_2$. If the two 3-gons are components of the same region, say $R_1$, then each of the three surrounding components must be either $R_3$ and $R_1$, thus two components of the same region would share an edge, a contradiction. □

The following result of Bezdek and Naszódi [4] (which my advisor learned about from Bezdek at a workshop at the Banff International Research Station in April, 2007) allows us to eliminate a number of cases from the proof of Conjecture 5.1.

**Lemma 5.7.** [4, see proof of Theorem 0.2] Two spherical triangles are determined by the curvatures of the edges and the three interior angles.

**Lemma 5.8.** Suppose a perimeter-minimizing partition of the sphere into four equal areas is not tetrahedral. Then the region $R_2$ of second highest pressure has at least one component adjacent to $R_1$ and at most one component not adjacent to $R_1$. A component of $R_3$ not adjacent to $R_1$ is a 4-gon (with area at most $2\pi/3$).

**Proof.** Since the partition is not tetrahedral, $R_1$ has more than one component, and the partition is not geodesic. Hence $R_1$ has two or 3 components by Lemma 2.21. Suppose $R_2$ has two components not adjacent to $R_1$, then these two components are convex. By Lemma 2.21, there are at most three convex components in the partition. Thus $R_1$ is connected, which implies that the partition is tetrahedral by Theorem 5.2.

Suppose $R_2$ has no components adjacent to $R_1$. Then every component of $R_2$ is convex. Since $R_2$ has at most one component not adjacent to $R_1$, $R_2$ must be connected. Lemma 5.3 implies that $R_2$ is a geodesic triangle. Theorem 5.2 implies that any partition containing a geodesic component with an odd number of sides is tetrahedral.

Finally, any component of $R_2$ not interfacing $R_1$ is convex, and thus has fewer than six edges by Gauss-Bonnet. Since only two regions alternate around it, it must have an even number of edges. Thus $R_2$ has two or four edges. Since this partition is taken to be perimeter minimizing with $n > 3$, this component of $R_2$ must be a 4-gon by Lemma 2.10. □
We will now prove that there exist just sixteen possible decompositions of the high pressure region $R_1$. In order to prove Conjecture 5.1, we must verify that $R_1$ is connected. In order to establish this, we must show that all other possible decompositions of the high pressure region cannot occur in a perimeter-minimizing partition.

**Proposition 5.9.** In a perimeter-minimizing partition of the sphere into four equal areas, the high pressure region falls into one of the following sixteen cases: (1) one 3-gon; (2) one 3-gon and one 4-gon; (3) one 3-gon and one 5-gon; (4) two 4-gons; (5) one 4-gon and one 5-gon; (6) two 3-gons; (7) three 3-gons; (8) one 3-gon and two 4-gons; (9) one 3-gon and two 5-gons; (10) one 3-gon, one 4-gon, and one 5-gon; (11) three 4-gons; (12) one 4-gon and two 5-gons; (13) two 4-gons and one 5-gon; (14) three 5-gons; (15) two 3-gons and one 5-gon; and (16) two 3-gons and one 4-gon.

The proof of Proposition 5.9 requires four lemmas: Lemmas 5.10 and 5.12 will use the Gauss-Bonnet formula and the facts that $R_1$ has nonnegative integral curvature and $R_4$ nonpositive integral curvature to generate bounds on the areas of components of highest or lowest pressure regions which depend only on the number of edges of the component. Lemmas 5.11 and 5.13 then use these bounds to limit the possible decompositions of $R_1$ and $R_4$.

**Lemma 5.10.** Consider a perimeter-minimizing partition of $S^2$ into four equal areas. Let $a_n$ denote the area of an $n$-gon component of the region $R_4$ of lowest pressure. Then

\begin{align*}
(8) & \quad a_3 \geq \pi \\
(9) & \quad a_4 \geq 2\pi/4 \\
(10) & \quad a_5 \geq \pi/3 \\
(11) & \quad a_6, 7, \ldots \geq 0.
\end{align*}

*Proof.* Any component of $R_4$ must have nonpositive integral curvature and the total area of $R_4$ is $\pi$. We apply Gauss-Bonnet to any $n$-gon component of $R_4$, where each interior angle $\alpha_i = 2\pi/3$:

\[
\begin{align*}
A &= 2\pi - \sum (\pi - \alpha_i) - \int_{\delta R} \kappa_i \\
A - 2\pi + n(\pi - \frac{2\pi}{3}) &= -\int_{\delta R} \kappa_i \\
-A + 2\pi - n\left(\frac{\pi}{3}\right) &= \int_{\delta R} \kappa_i \leq 0 \\
-A + 2\pi - n\left(\frac{2\pi}{3}\right) &\leq 0 \\
A \geq 2\pi - \frac{n\pi}{3}.
\end{align*}
\]

The bounds (8)-(11) follow immediately. \qed

**Lemma 5.11.** In a perimeter-minimizing partition of $S^2$ into four equal areas, consider a region $R_4$ of lowest pressure. The following are true:
i) If \( R_4 \) contains a 3-gon, then \( R_4 \) is a geodesic 3-gon;
ii) \( R_4 \) cannot contain more than one 4-gon;
iii) \( R_4 \) cannot contain more than three 5-gons, and if it contains three 5-gons, then these are its only components and they are geodesic; and
iv) If \( R_4 \) contains a 4-gon and a 5-gon, then these are its only components and they are geodesic.

**Proof.** Using the bounds in Lemma 5.10: if \( R_4 \) contains a 3-gon component, then that component has area greater than or equal to \( \pi \); therefore it is the whole region, so \( R_4 \) is connected. If \( R_4 \) contains a 4-gon component, it has area greater than or equal to \( 2\pi/3 \), so if \( R_4 \) contains two 4-gon components, the area of \( R_4 \) would be greater than or equal to \( 4\pi/3 \), a contradiction. If \( R_4 \) contains a 5-gon component, it has area greater than or equal to \( \pi/3 \), so if \( R_4 \) contains more than three 5-gon components, it would have area greater than or equal to \( 4\pi/3 \), a contradiction. If \( R_4 \) contains three 5-gons, then Gauss-Bonnet implies that each has zero curvature, and each edge of a component of \( R_4 \) must have nonpositive curvature, each 5-gon must be geodesic. If \( R_4 \) contains a 4-gon and 5-gon component, then the sum of their areas is greater than or equal to \( \pi \), so they must be the only components of \( R_4 \). Note from Gauss-Bonnet above that both a 4-gon of area \( 2\pi/3 \) and a 5-gon of area \( \pi/4 \) have integral curvature zero, and since \( R_4 \) has lower or equal pressure than its adjacents, each edge must be zero curvature and therefore geodesic. □

**Lemma 5.12.** Consider a perimeter-minimizing partition of the sphere into four equal areas. Let \( a_n \) be the area of an \( n \)-sided component of the region \( R_1 \) of highest pressure. Then

\[
\begin{align*}
    a_3 &\leq \pi \\
    a_4 &\leq 2\pi/3 \\
    a_5 &\leq \pi/3 \\
    a_6, 7, ... &\leq 0.
\end{align*}
\]

**Proof.** Any component of \( R_1 \) must have nonnegative curvature and the total area of \( R_1 \) is \( \pi \). By Lemma 2.21, \( R_1 \) has at most three components. We apply Gauss-Bonnet to any \( n \)-gon component of \( R_1 \), where each interior angle \( \alpha_i = 2\pi/3 \):

\[
    A = 2\pi - \sum (\pi - \alpha_i) - \int_{\delta R} \kappa_i
\]

\[
    A - 2\pi + n(\pi - \frac{2\pi}{3}) = -\int_{\delta R} \kappa_i
\]

\[
    -A + 2\pi - n(\frac{\pi}{3}) = \int_{\delta R} \kappa_i \geq 0
\]

\[
    -A + 2\pi - n(\pi - \frac{2\pi}{3}) \geq 0
\]

\[
    A \leq 2\pi - \frac{n\pi}{3}.
\]

The bounds (12)-(15) follow immediately. □

**Lemma 5.13.** In a perimeter-minimizing partition of \( S^2 \) into four equal areas, consider a region \( R_1 \) of highest pressure. The following are true:
i) $R_1$ has at most three components; and

ii) if $R_1$ contains only one component, then that component is a geodesic 3-gon; and

iii) if $R_1$ contains two components, they are not both 5-gons,

iv) components of $R_1$ are either 3-gons, 4-gons, or 5-gons.

Proof. It follows from the bounds in Lemma 5.12 that (1) if $R_1$ has one component, then that component has area $\pi$, so it must be a 3-gon, (2) if $R_1$ has a 4- or 5-gon component, then it has area strictly less than $\pi$, so $R_1$ must have at least one additional component, and (3) if $R_1$ has two components, they are not both 5-gons. It also follows that an $n$-gon with $n \geq 6$ would be nonconvex, so it cannot be a component of a highest pressure region. It follows from these three facts, as well as the upper limit on the total number of components, that these cases are the only possibilities. 

We are now ready to proceed with the proof of Proposition 5.9.

Proof. Part (i) of Lemma 5.13 shows that the high pressure region has at most three components. Part (ii) shows immediately that there is only one case where the high pressure region has one component: the geodesic triangle. If $R_1$ has two components, then part (iii) shows that they are not both 5-gons, thus the possible decompositions are $((3,3),(3,4),(3,5),(4,4),(4,5))$. Finally, if $R_1$ has three components, there are no such constraints, so there are as many decompositions as combinations of 3, 4, and 5 into groups of three: ten. Thus there are exactly sixteen possible decompositions of the high pressure region $R_1$.

Finally, we eliminate some of the nonstandard decompositions of the high pressure region. Lemma 5.14 eliminates all decompositions which contain more than one triangle.

**Lemma 5.14.** Cases 6, 7, 15, and 16: $R_1$ contains at least two triangles. Any equilibrium partition of the sphere in which the region of highest pressure contains at least two triangles is not stable.

Proof. Any triangle component of the high pressure region $R_1$ interfaces $R_2$, $R_3$, and $R_4$. By Proposition 2.5, every interface between the same two regions has the same curvature. By Proposition 2.6, every component of an equilibrium soap bubble has 120 degree interior angles. Thus, by Lemma 5.7, every triangle component of the high pressure region in an equilibrium partition is congruent.

We deform this partition by shrinking one of these triangles and expanding another one, both at unit rate. Since the triangles are congruent, $dA/dt = 0$ for this deformation. We calculate the second variation by equation 5. Since the deformation is constant, the $\int u_{ij}^2$ term vanishes. The $u_{ij}^2 \kappa_{ij}$ term is always nonnegative. The sum of the $u_{ij}^2 q_{ij}(p)$ terms is nonnegative since at each point at which $u_{ij}$ is nonzero, two incident edges are convex, and their difference is the curvature of the third component, so even if the third curvature is negative, it is smaller in magnitude than the value of the two others. The $\int u_{ij}^2$ term is strictly positive for nontrivial deformations. Thus the initial second variation of perimeter of this deformation is negative. Thus this partition is not stable.
Lemma 5.15. Case 5: \( R_1 \) consists of one 4-gon and one 5-gons. In a perimeter-minimizing partition of the sphere into four equal areas, if a highest pressure region consists of one 4-gon and one 5-gons, then it is the tetrahedral partition.

Proof. Since the sum of their areas must be \( \pi \), each component must have the maximum possible area from equations (13,14). By Gauss-Bonnet, a 4-gon of area \( 2\pi/3 \) and a 5-gon of area \( \pi/3 \) with interior angles 120 degrees have zero integral curvature. Since any component of a high pressure must have nonnegative curvature on each edge, components with zero integral curvature must have geodesic edges. Thus the partition contains a geodesic pentagon, so by Theorem 5.2, the partition is tetrahedral. \( \square \)

Lemma 5.16. Case 14: \( R_1 \) consists of three 5-gons. In a perimeter-minimizing partition of the sphere into four equal areas, if a highest pressure region consists of three 5-gons, then it is the tetrahedral partition.

Proof. If a highest pressure region consists of three 5-gons, then by equation 14 each of these has area at most \( \pi/3 \), and since the sum of their areas is \( \pi \), they must each have area \( \pi/3 \). By Gauss-Bonnet, a 5-gon of area \( \pi/3 \) with 120 degree interior angles has integral curvature zero. Since each of these components is convex, it must be geodesic. Thus the partition contains a geodesic pentagon, so by Theorem 5.2, the partition is tetrahedral. \( \square \)

6. The Triangular Isoperimetric Inequality

A separate method, more similar to Hales’ complex computer methods, of proving Proposition 5.1 would utilize the verification of a certain spherical isoperimetric inequality, which states that the perimeter of any region is greater than the perimeter of a regular spherical triangle minus linear error terms which penalize a region having more than three sides and edges which bulge out to include greater area. Previous work shows that the verification of such an isoperimetric inequality requires either lengthy case analysis of various classes of curvilinear \( n \)-gons, as in the proofs of the planar Hexagonal Honeycomb Theorem [7] and Spherical Honeycomb Theorem for \( n = 12 \) [8] or the proof of the convexity of the geometric function of Lemma 6.1, as shown for the \( \mathbb{R}^2 \) case in [5].

The following lemma was originally proved in the plane by Hales [7], and then generalized by Carroll et al. [5] to \( \mathbb{H}^2 \) and \( \mathbb{S}^2 \).

Lemma 6.1. The Chordal Isoperimetric Inequality ([5], Prop. 2.6). Consider an immersed curvilinear polygon \( P \) in \( \mathbb{S}^2 \) of perimeter \( L \), net excess area \( X \) over the chordal polygon, where \( |X| \leq \pi/8 \), with each chord length at most 1. Let \( L_0 \) be the perimeter of the chordal polygon. Then

\[
L \geq L_0 \arccos \left( \frac{|X|}{L_0} \right)
\]

The function \( L_0 \arccos \left( \frac{|X|}{L_0} \right) \) is increasing with respect to \( L_0 \), fixing \( X \).

Carroll et al. [5] argue that the convexity of \( L_0 \arccos \left( \frac{|X|}{L_0} \right) \) implies a polygonal isoperimetric inequality which implies the honeycomb conjecture for almost every case.
Conjecture 6.2. See ([7] Thm. 4, and [8] Thm. 2). Let $L(P)$ be the perimeter of any region (not necessarily connected), $N(P)$ the number of sides, $T(P)$ the sum of the signed areas between each edge and its chord, and $A(P)$ its area, then for any region on the sphere,

\begin{equation}
L(P) \geq A(P)p_3 - aT(P) - b(3 - N(P))
\end{equation}

for some constants $a, b$ and $p_3$ the length of a regular spherical triangle.

The proof of the above conjecture, using the methods of [7] and [8], depends on a series of isoperimetric inequalities on the perimeter of a component enclosing certain area $A$.

**Appendix A. Partial Converse to Proposition 2.6**

Quinn Maurmann '09  
Department of Mathematics  
Brown University  
Providence, RI 02912  
Quinn_Maurmann@brown.edu

We prove that for a bubble $B$ in $S^2$, the consequences of equilibrium given in Proposition 2.6 imply that each connected component of each region can be assigned a pressure in the sense of Proposition 2.5. If we additionally assume that all components of any one region have the same pressure, these consequences in fact imply $B$ is in equilibrium. We begin with a more general result.

**Proposition A.1.** Let $G$ be a locally finite graph embedded in a simply connected 2-manifold $S$. For any two adjacent components of the complement of $G$, $R_{\alpha}$ and $R_{\beta}$, suppose that a number $\kappa_{\alpha\beta}$ has been assigned, called the curvature from $R_{\alpha}$ to $R_{\beta}$. Furthermore we assume these curvatures have been defined such that $\kappa_{\alpha\beta} = -\kappa_{\beta\alpha}$ for all such pairs $R_{\alpha}, R_{\beta}$ and such that the sum of curvatures in the cycle of components adjacent to any vertex $v$ of $G$ vanishes. Under these conditions, if $\gamma$ is any oriented closed curve in $S$, intersecting no vertex of $G$ and intersecting itself and the edges of $G$ only finitely many times, then the sum of all curvatures crossed by $\gamma$ vanishes.

The proposition will follow immediately from two lemmas.

**Lemma A.2.** With $S, G$, and $\gamma$ defined as above, the result of the proposition holds under the additional assumption that $\gamma$ is simple.

**Proof.** Observe that $\gamma$ bounds a region $R$ which is topologically a disk. Assume without loss of generality that the orientation of $\gamma$ defines $R$ as the interior region bounded by $\gamma$ (otherwise we will consider $\gamma$ under the opposite orientation, and the proof will go forward).

The proof is by induction on the number of vertices of $G$ contained in $R$. If $R$ contains no vertices, then we take a disk $D$ contained in $R$ whose boundary crosses no edge of $G$. Since $R$ and $D$ are equivalent, $D$ can be deformed to $R$ while any intermediate stage remains inside $R$. In this deformation process, wherever the boundary of $D$ crosses an edge of $G$, it does so twice; in particular, it crosses the same edge twice since no vertices are contained in $R$, and the crossings are in
Figure 4. A curve $\gamma$, enclosing the vertex $v$ (among others), can be decomposed into curves $\alpha$, $\beta$, and $\delta$, with $\alpha$ and $\beta$ each enclosing fewer vertices.

opposite directions so that they do not contribute to the sum. Thus the sum of curvatures crossed by $\gamma$, the boundary of $R$, is the same as the sum of curvatures crossed by the boundary of $D$, which is 0.

Suppose for some $n > 0$, we know the result holds through $n - 1$, and suppose $R$ contains $n$ vertices. We then take a small circle $\delta$ around one of these vertices $v$ so that $\delta$ has an obvious interior containing $v$ and no other vertices, and that $\delta$ intersects only those edges incident with $v$. We define an orientation on $\delta$ that agrees the orientation on $\gamma$, in the sense that $\gamma$ can be represented by $\delta$ and two other simple, closed, oriented curves $\alpha$ and $\beta$ as in Figure 4.

The sum of crossings by $\gamma$ is exactly the same as the sum of crossings by $\alpha$, $\beta$, and $\delta$, since each additional crossing by one of these curves is cancelled by a crossing in the other direction by another curve. But $\alpha$ and $\beta$ bound strictly fewer vertices than $\gamma$, so by the induction hypothesis, the sums of crossings by $\alpha$ and $\beta$ are identically 0. By construction, $\delta$ crosses only those edges incident with $v$, so the sum of crossings by $\delta$ also vanishes. By induction, the sum of curvatures crossed by $\gamma$ vanishes for any number of vertices in $R$. □

Lemma A.3. With $S$ and $\gamma$ defined as in the proposition, $\gamma$ can be written as the finite union of simple closed curves, each preserving the orientation of $\gamma$. This union is disjoint except for those points where $\gamma$ intersects itself.

Proof. Starting at any intersection point, construct a curve $\alpha$ by following $\gamma$ until $\alpha$ intersects itself (at some intersection point $p$ of $\gamma$). Then $\alpha$ contains a cycle, which can be taken as one of the simple closed curves with its original orientation. This cycle can be removed from $\gamma$ (except the intersection points of $\gamma$) to create a new closed, oriented curve $\gamma'$ which begins and ends at $v$. Since $\gamma'$ has strictly fewer self-intersections than $\gamma$, this process can be continued inductively until $\gamma$ has been completely written as a union of simple closed curves. This proves Proposition A.1. □

In the special case that $G$ is a graph on $S^2$ whose edges are arcs of constant geodesic curvature (and this geodesic curvature is taken as the "curvature between components" in the proposition), we see that the condition that the sum of curvatures at any vertex vanishes implies that each component of the complement of
Proposition 2.5. In particular, if the edges of $G$ meet only in threes so that $G$ may be considered a soap bubble on $S^2$, then each component of each region of the bubble has a pressure. With these observations, the proposition just proved suggests a partial converse to Proposition 2.6:

**Corollary A.4.** Let $B$ be a soap bubble in $S^2$. If the edges of $B$ are all constant-curvature arcs meeting at 120 degrees, the sum of the curvatures at any vertex vanishes, and all components of any one region have the same pressure, then $B$ is in equilibrium.

**Proof.** With the regions of $B$ labeled $R_1, \ldots, R_n$ and the pressure of any $R_i$ denoted $P_i$, we recall the first variation formula for the perimeter of $B$ and note that it reduces to $-\sum_{0<i<j} \int \kappa_{ij} u_{ij}$ since the sum of unit tangent vectors at each vertex vanishes when edges are assumed to meet at 120 degrees. By our definition of pressure, $\kappa_{ij} = P_j - P_i$ so that the first variation is

$$\sum_{0<i<j} \int (P_i - P_j) u_{ij} = \sum_{0<i<j} \left( P_i \int u_{ij} + P_j \int u_{ji} \right)$$

since $u_{ij} = -u_{ji}$. Collecting like terms, we write the first variation as

$$\sum_{j=1}^n \sum_{k \neq j} P_j \int u_{jk}$$

which vanishes since each $\int \sum u_{jk}$ is the flux of the vector field $u$ over the boundary of $R_j$, and $u$ preserves area, hence $B$ is indeed in equilibrium.

**References**


