Rose-Hulman Undergraduate Mathematics Journal

Volume 8
Issue 2

Article 7

Improving the Mathematical Model of the Tacoma Narrows Bridge

Brian Fillenwarth
University of Evansville, bf36@evansville.edu

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Brian A. Fillenwarth
Department of Mathematics
University of Evansville
Advisor: Dr. Talitha M. Washington

1 Introduction

Since the opening day on July 1, 1940, the Tacoma Narrows Bridge has been a topic of interest for physicists, engineers, and mathematicians. Immediately following opening day, the bridge could be seen oscillating vertically up and down [**6**]. On November 7, 1940, the vertical movement of the Tacoma Narrows Bridge changed to a violent torsional rotation [**14**]. This twisting motion continued for about 45 minutes until the bridge ultimately collapsed [**5**].

After the collapse of the Tacoma Narrows Bridge, it became important to determine what factors caused this catastrophic failure so that these factors would be taken into consideration for the design of future suspension bridges. Although debate persists about the exact reason for the Tacoma Narrows Bridge failure, mathematical models have been developed to illustrate how the bridge behaved during its final moments. There are models that illustrate both the vertical motion, as well as the torsional motion exhibited by the bridge. Since the torsional rotation of the bridge is what ultimately caused its failure [**7**], the model of the torsional rotation will be analyzed in depth.

The base mathematical model that will be used for all modifications presented in this paper was derived in [**9**] by P.J. McKenna, and is

$$\ddot{\theta} = -\delta\dot{\theta} - \frac{lk}{m}\cos\theta\sin\theta + f(t) \tag{1.1}$$

where

$$\begin{aligned}
\delta &= \text{damping coefficient} \\
l &= \text{width of half the bridge} \\
k &= \text{spring constant} \\
m &= \text{mass per foot of bridge} \\
\theta &= \text{angle of rotation} \\
f(t) &= \text{external forcing term} \\
t &= \text{time}
\end{aligned}$$

For all mathematical models presented in this paper, Newton's notation for differentiation is used in which a single dot represents the first derivative taken with respect to time, and a double dot represents the second derivative taken with respect to time. In Equation 1.1, the first term, $\delta\dot{\theta}$, represents the wind resistance of the bridge and the second term, $\frac{lk}{m}\cos\theta\sin\theta$, represents the cable resistance. A detailed derivation of Equation 1.1 can be found in [**3**].

McKenna's model was chosen because this model was one of the first that corrected small angle linear approximations that had been made in previous models of the Tacoma Narrows Bridge, such as the model presented in [**5**]. The small angle linear approximations are simply

$$\sin\theta = \theta \tag{1.2}$$

and

$$\cos\theta = 1 \tag{1.3}$$

Unfortunately these approximations are only valid for small angles of displacement. Since the angular displacements of the Tacoma Narrows Bridge were very large, the use of the small angle approximations is not valid.

In McKenna's model (Equation 1.1), the external forcing term was assumed to be periodic. This may be represented by

$$f(t) = \lambda\sin(\mu t) \tag{1.4}$$

where λ is the amplitude of the forcing term, and μ is the period of this term. The problem with this forcing term is that it may not accurately depict exactly how the Tacoma Narrows Bridge behaved in torsion. The external forcing term represented in Equation 1.4 will be modified in all but the last case presented to see how the bridge would have responded due to various types of physical assumptions which will be explained in subsequent sections.

All the models presented in this paper are numerical, and the ode15s solver was used in Matlab to solve and graph these solutions. This solver is a variable-step solver, which simply means the step size is varied during the simulation of the model. This solver reduces the step size to increase the accuracy when the model's state changes rapidly, and increases the step size to reduce the simulation time when the model's state changes slowly [**11**].

Initially a fixed-step solver was used, but after preliminary tests, this was found to be impractical due to the large time interval that was used and the small step size that was required for the model to be accurate. For all cases presented, initial conditions representing a large torsional displacement were used and are $\theta(0) = 1.2$ and $\dot{\theta}(0) = 0$. The initial conditions used can change the behavior of the model, and a detailed look at the initial conditions can be found in [**8**].

2 Deriving the Constants

We approximate values found in [**1**] of the characteristics of the Tacoma Narrows Bridge to determine the constants for Equation 1.1. It has been estimated that the bridge weighed 5,000 lbs per foot, so we set m in Equation 1.1 to be 2,500 kgs. The bridge deflected about 0.5 meters per foot of the bridge with a 100 kg load applied. From this information we can determine the spring constant knowing that the cables act as springs in tension and there are two resisting cables [**14**], [**8**], one on each side of the bridge. This can be shown as

$$2k\Delta = m_L g \tag{2.1}$$

where k represents the spring constant, Δ is the change in length of the cables (0.5 meters), m_L is mass of the applied load (100 kgs), and g is acceleration due to gravity (9.8 m/s^2). If we solve for k we get approximately 1,000 N/m. It is also known that the width of half the bridge (l) was 6 meters, and it has been determined that the damping coefficient (δ) should be about 0.01 [**1**]. With all of these constants in place, Equation 1.1 becomes

$$\ddot{\theta} = -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + f(t) \tag{2.2}$$

For the forcing term shown in Equation 1.4, McKenna determined that the μ value should be somewhere between 1.2 and 1.6 [**9**]. In this paper we will assume that the value is 1.3. The λ value has been set to cause a small forcing term, and this value will be 0.05 for all the cases explored in this paper. With the constants in place, the external forcing term becomes

$$f(t) = 0.05\sin(1.3t) \tag{2.3}$$

This forcing term assumes that the forcing parameters on the bridge do not change with respect to time. That is, the amplitude and period of the forcing term remain constant the entire time the bridge is in torsional rotation. However, the amplitude or period of the external force may have varied with time. The forces on the bridge for a specified time, say $0 \le t < 8$, may be completely different from forces on the bridge at another time such as $8 \le t < 10$. In addition, the forces applied to the bridge may have not been periodic.

Also, it is unlikely that the current mathematical model accurately depicts the torsional behavior of the Tacoma Narrows Bridge since the rotation of the bridge would have caused fatigue on the hanger cables and bridge deck. This fatigue would allow the bridge to rotate further the longer that the bridge stayed in motion. Fatigue in the bridge cables would not be accounted for in the external forcing term but rather in the $\frac{lk}{m}\cos\theta\sin\theta$ term that represents the force due to the cables.

The first modification that we present investigates an external forcing term on the Tacoma Narrows Bridge with varying amplitudes on both sides of the bridge. We then discuss the effects of an external forcing term on the bridge having different periods on each side. The next modification involves how the bridge reacts to a change in the amplitude of the periodic external forcing term for a large block of time. We will also look at the effects of a constant force for short time intervals as well as for a large block of time. The final modification to McKenna's model that will be presented investigates the bridge response as a result of weakening hanger cables. Reasons for these modifications will be given and from the results, a more accurate mathematical model will be presented.

3 Modifying the Mathematical Model

3.1 Standard Bridge Response

The standard response of the bridge will be defined as the mathematical model created by McKenna, using the constants presented in Section 2. With the constants placed into Equation 1.1, the standard bridge response may be shown as

$$\ddot{\theta} = -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) \tag{3.1}$$

Figure 3.1-1 shows the rotation of the bridge as a result of the mathematical model developed by McKenna.

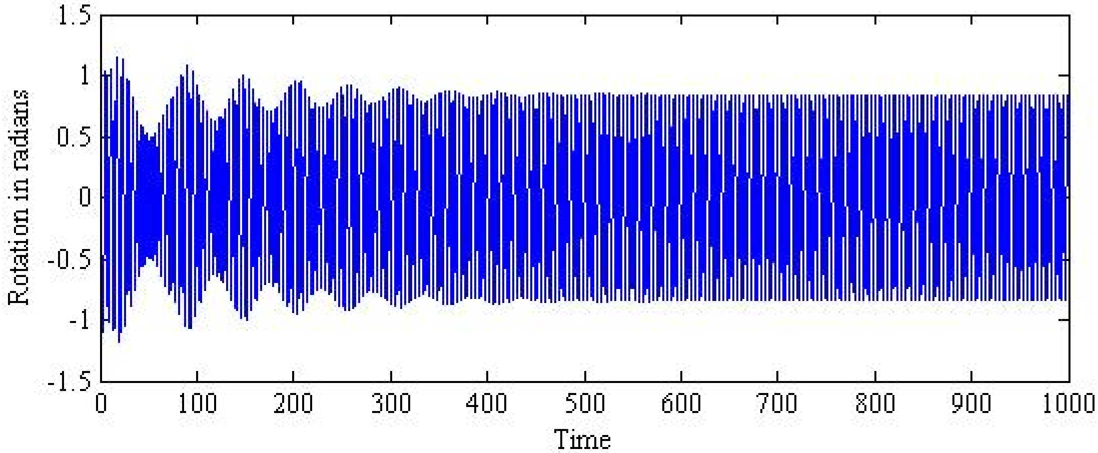

Figure 3.1-1—The standard response of the bridge

Our simulation shows that the rotation of the bridge does not stop, which is consistent with observations made the day that the bridge failed.

3.2 Modifications with a Periodic Forcing Term

The first modification to the mathematical model is varying the period or amplitude of the forcing function for specified time intervals. We will keep the modified periodic forcing terms in the same form as the original forcing term (Equation 1.1), but the amplitude or period of these terms will be varied. The effects of changing these terms in short but frequent time intervals as well as a single long continuous block of time will be investigated in depth.

Short and Frequent Periodic Forces

First we changed the external force by adding another periodic force for many short but frequent time intervals. To illustrate the effect of increasing the amplitude of the original force for short and frequent times, this additional force will have the same period as the original force. This model can be illustrated with

$$\ddot{\theta} = \begin{cases} -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) + \lambda_2\sin(1.3t) & 10n+9 < t \le 10(n+1)+1 \\ -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) & 10(n+1)+1 < t \le 10(n+1)+9 \end{cases} \tag{3.2}$$

where λ_2 represents the amplitude of the additional forcing term and n denotes a positive integer.

This model will illustrate the response of the bridge if different periodic forces were acting on each side. If the external forces on one side were larger than the side with the original external force on it, the additional forcing term would be positive. Similarly, if the force on one side were smaller than the original, the additional forcing term would be negative. Unfortunately due to the complexity of finding the rate at which the period of this function varies, we will have to settle for looking at the effects of an additional external forcing function over the arbitrary time interval specified in Equation 3.2. If we modeled the function with the different amplitude over the actual time interval that the bridge were on one side, the responses would likely be very similar to the ones found in this example.

We will first look at how a positive additional force changes the bridge response. By setting λ_2 equal to 0.01, we find that initially the rotation of the bridge varies significantly, but then the rotation levels out and becomes fairly constant (Figure 3.2-1). In this Figure, the peaks of the maximum amplitude occur at a later time than they did in the standard response, and the amplitude in this response fluctuates a little more after a time of about 500.

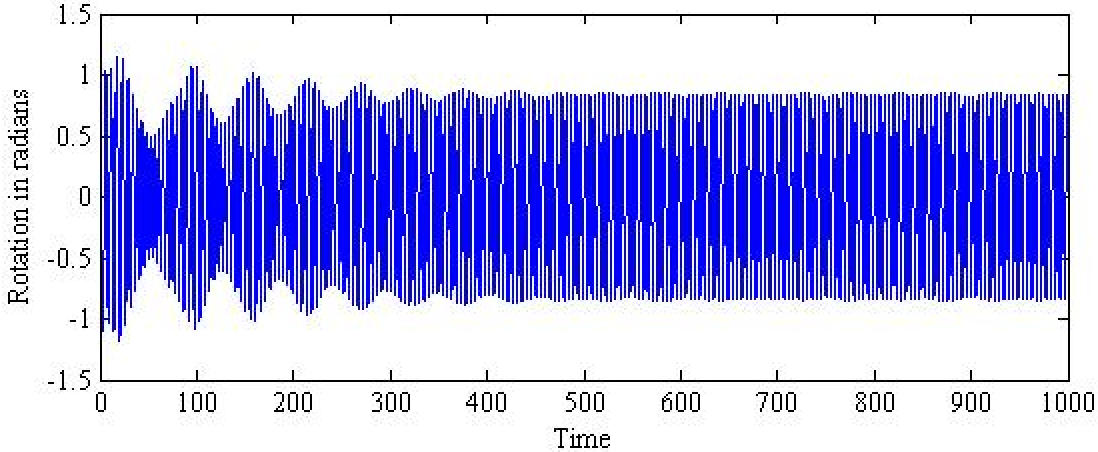

Figure 3.2-1—The bridge response of a small additional periodic forcing term

Thus with a slightly larger amplitude of forcing on one side, there does not appear to be a significant change in the response of the bridge when compared to the standard response.

By adding a larger force, such as by doubling the original forcing term by setting λ_2 equal to 0.05, a slightly different response can be produced. As shown in Figure 3.2-2, there is a more gradual fluctuation in the amplitude, and it never becomes constant.

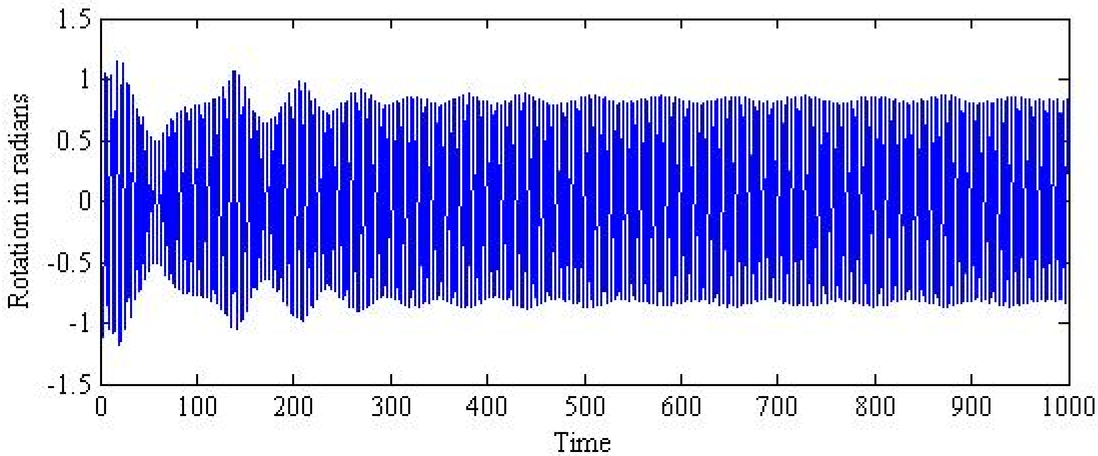

Figure 3.2-2—The bridge response of doubling the original periodic forcing term

Although there is a noticeable change in the response as a result of a large additional periodic forcing term, it is very unlikely that the external forces acting on each side of the Tacoma Narrows Bridge varied this significantly.

We obtained similar results when we used a negative λ_2 of the same magnitude. By using a -0.05 amplitude for the additional forcing term, we are modeling the bridge as if it would lose all external forcing for a small period of time. Even with this the bridge still remains in torsion, and the response resembles that of Figure 3.2-2.

External Forces with Differing Periods

We now consider what would happen if the external forcing function on the Tacoma Narrows Bridge had a varying period. The model that will be used in this case is

$$\ddot{\theta} = \begin{cases} -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(\mu_2 t) & 10n + 9 < t \le 10(n+1) + 1 \\ -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) & 10(n+1) + 1 < t \le 10(n+1) + 9 \end{cases} \tag{3.3}$$

where μ_2 represents the new period of the external forcing function and n is a positive integer.

The case illustrated by this model may correspond to one of the cables holding the bridge in place being stronger than the cable on the other side. This may cause the bridge to resist twisting on one side better than the other side, which could result in a shorter period force for one side of the bridge. Once again, due to the difficulty of determining the rate that the period of this function varies, we will model the different period for an arbitrary length of time. If we modeled the function over the actual time intervals that the bridge was on one side, the responses would not change a significant amount from this model.

The first case we will look at is if the period of the external forcing function were to change a small amount for a short length of time. We will set μ_2 equal to 1.32 representing a very small change in the period of the external forcing function. Figure 3.2-3 shows a large deviation from the standard response in Figure 3.1-1.

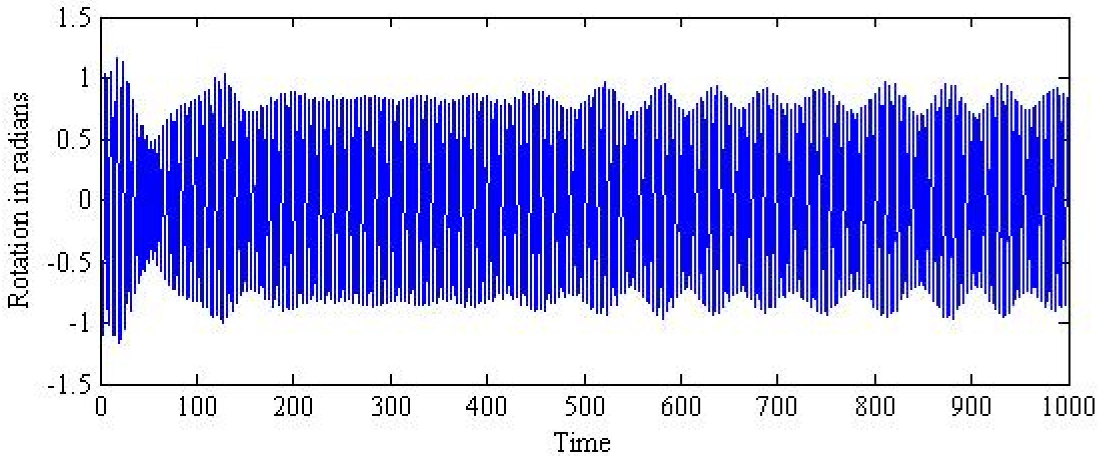

Figure 3.2-3—The bridge response for a small change to the period for a short time

In this case, the amplitude of the bridge deck's rotation levels out fairly quickly and remains level from a time of about 200 to a time of about 450. After this time, the amplitude of the rotation begins to vary and continues undulating over the shown time. Thus, a slight change in the period of the external forcing function will result in a substantial change in the response of the bridge.

If the μ_2 term is further increased to 1.35, an unexpected bridge response occurs (Figure 3.2-4).

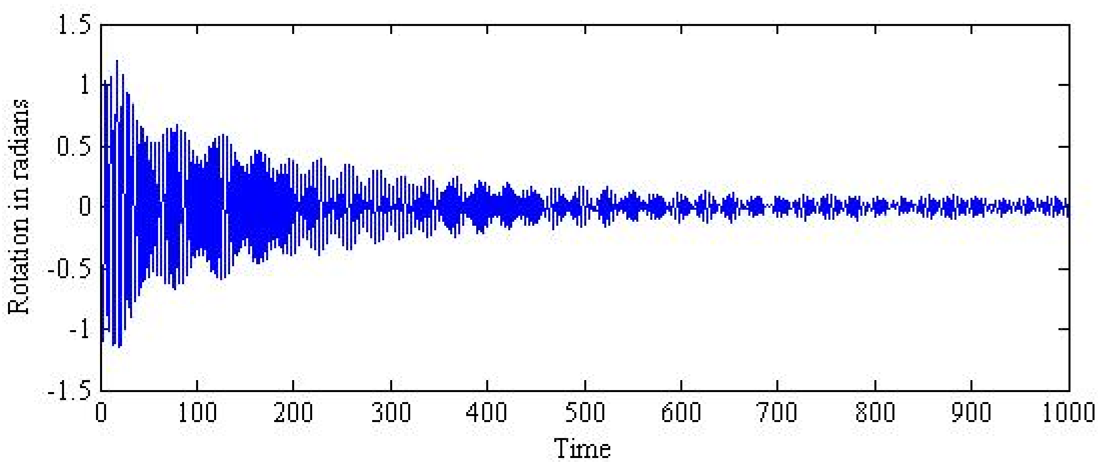

Figure 3.2-4—The bridge response for $\mu_2 = 1.35$

Figure 3.2-4 shows the amplitude of rotation in the Tacoma Narrows Bridge decreasing towards zero. For this reason, it is unlikely that the external forces on the bridge had a varying period for repeated short time intervals. Further investigation of the μ_2 term resulted in finding that the responses of the bridge continued to change between remaining in torsion (responses similar to Figure 3.2-3), and having the torsion diminish (responses similar to Figure 3.2-4).

Additional Periodic Force for a Single Block of Time

The next model describes how the Tacoma Narrows Bridge response is affected by a change of the amplitude in the external forcing term for a single large block of time. Similar to the first models, the period of the additional forcing term will be the same as the period in the original forcing term (Equation 2.3). Thus, we have the following equation

$$\ddot{\theta} = \begin{cases} -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) + \lambda_2\sin(1.3t) & S < t \le E \\ -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) & t \le S, t > E \end{cases} \quad (3.4)$$

where λ_2 still represents the amplitude of the additional force, S is the starting time for the additional force, and E is the end time of the additional force.

One reason this may have occurred in the bridge would be due to a minor structural failure with the bridge in torsion. A minor structural failure may cause the external forcing term on the Tacoma Narrows Bridge to induce a larger rotation in the bridge. The larger rotation in the bridge will be modeled by simply modifying the external forcing term to have a different amplitude than the original forcing term.

From all the cases tested with this model, we found that where the time interval starts can significantly change the results. For this reason, the critical values for modeling this case were found, and the model was tested at these locations. We found these critical values to be around times $t = 50$ and $t = 90$. These represent the time at which the low and high points of the original function occur, and are identified in Figure 3.2-5.

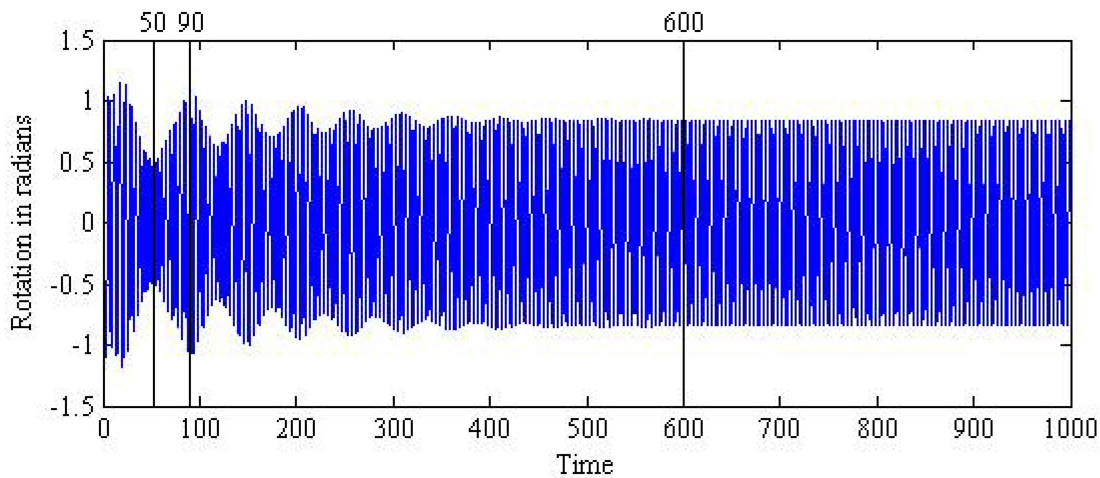

Figure 3.2-5—Three critical times in the original function

All cases presented with this model will also be looked at for a third critical start time of $t = 600$, since at this time the torsional motion of the standard bridge response is fairly stable. For many of these cases we found that if the time interval of the different forcing term were sufficiently large, the twisting bridge response would end. We show the responses of the bridge for the time interval directly prior to the time interval at which the twisting bridge response stops. In other words, we will show the bridge response where the amplitude of rotation does not approach zero since we know that the rotation of the Tacoma Narrows Bridge did not stop.

The first case of this model predicts how cutting the general forcing function's amplitude in half changes the response. That is, we will set the λ_2 equal to -0.025. The critical cases for this forcing term are shown in Figures 3.2-6 – 3.2-8.

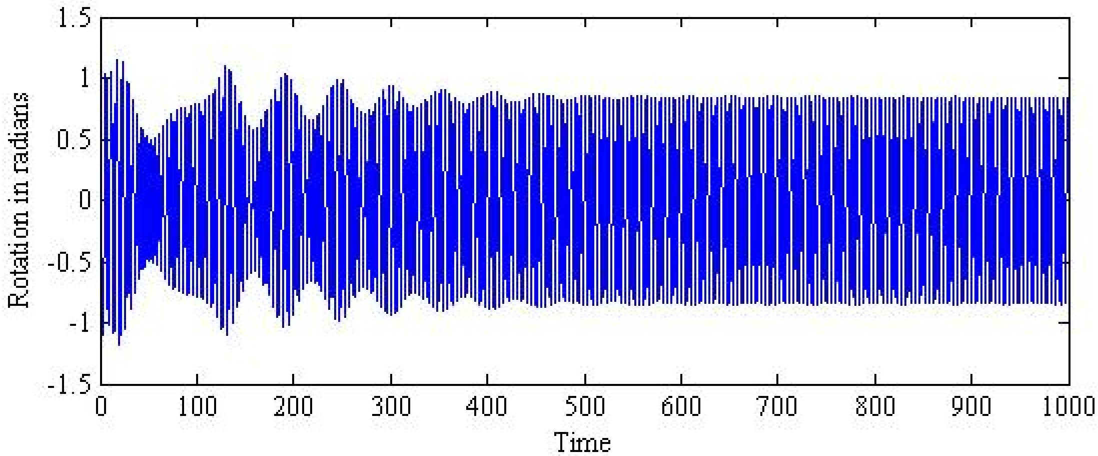

Figure 3.2-6—The bridge response with $\lambda_2 = -0.025$ for $S = 50$ and $E = 61.4$

After many numerical experiments with the start time at 50, we determined that for any block of time that ended at a time greater than about 61.4, the amplitude of the bridge rotation would eventually approach zero. Since the rotation of the Tacoma Narrows Bridge did not stop, Figure 3.2-6 shows the case with an ending time for the additional force of 61.4, which is the time directly prior to the amplitude going to zero. From this Figure it can be seen that a large but gradual variance in the rotation occurs initially from a time of about 50 to a time of about 125. From the time of 125 to about 300, the amplitude of rotation varies more significantly, but levels out at a time of around 500.

Figure 3.2-7 shows the case with a start time of 90, and unlike the start time of 50, the rotation did not stop even as the time interval for the force approached infinity. Initially the amplitude of rotation varies a significant amount, and at a time of about 125, this variance becomes more gradual until it levels out at a time of about 500.

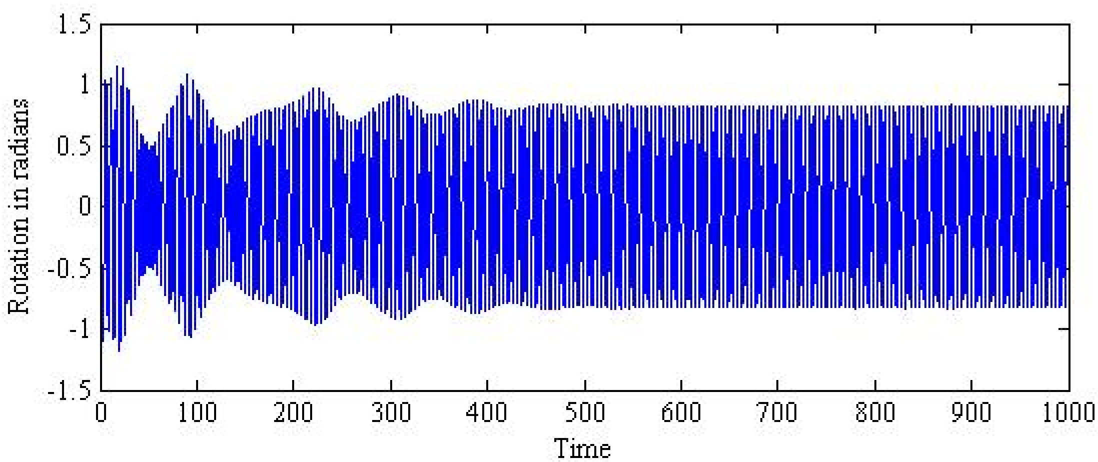

Figure 3.2-7—The bridge response with $\lambda_2 = -0.025$ for $S = 90$ and $E \to \infty$

The final critical case where the λ_2 equals -0.025 is shown in Figure 3.2-8 and has a start time of 600. In Figure 3.2-8, there is a slight dip in the limits of the torsional rotation after a time of 600 as a result of the lower amplitude. Similar to the case with the start time of 90, the amplitude of rotation in this case does not approach zero as the time interval for the force approaches infinity.

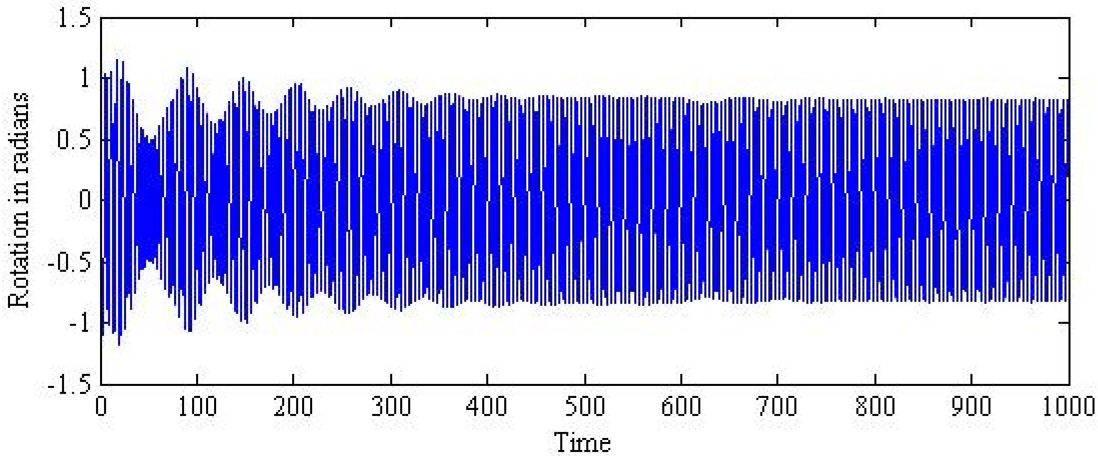

Figure 3.2-8—The bridge response with $\lambda_2 = -0.025$ for $S = 600$ and $E \to \infty$

The response that began at a time of 50 (Figure 3.2-6) was the only case where the torsional rotation of the bridge approached zero when the amplitude of the forcing function was cut in half for a set block of time. The end time value at which the response changed from large oscillations to small oscillations was found to be around 61.4. Thus, if there was a change in the periodic forcing term by having the amplitude reduced, the bridge's torsional rotation would not have stopped if the rotation was already well established.

The consequence of the periodic forcing term being completely removed from the Tacoma Narrows Bridge for a block of time will now be considered. This can be done by simply setting λ_2 equal to the opposite value of the standard bridge response's amplitude, making $\lambda_2 = -0.05$. The critical cases of this are shown in Figures 3.2-9 – 3.2-11, and in all of these cases the torsional rotation stopped if the block of time was large enough.

For the first critical case with a start time of 50 (Figure 3.2-9), we found that if the end time was greater than 57.3, the amplitude of rotation of the bridge deck went to zero. Figure 3.2-9 shows that initially there is a large variance in the amplitude of rotation until it levels out at a time of about 500.

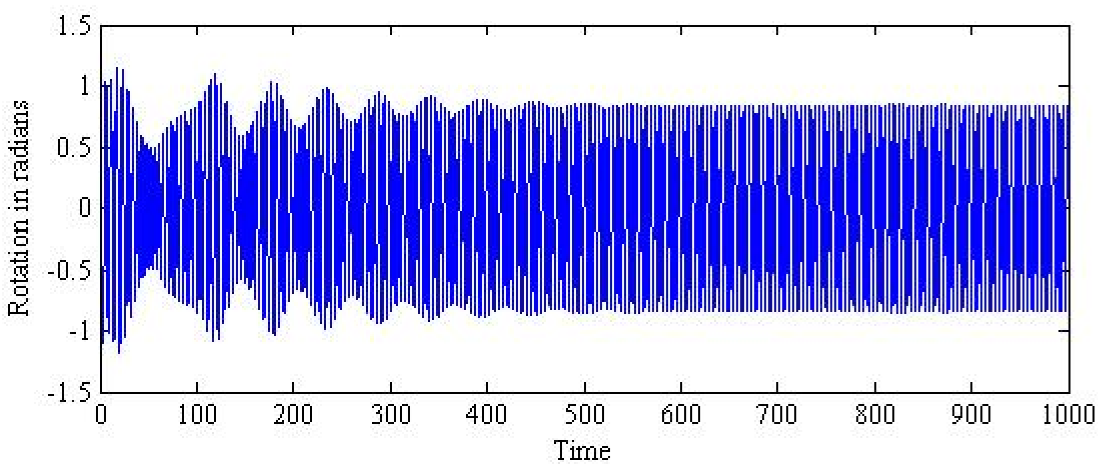

Figure 3.2-9—The bridge response with $\lambda_2 = -0.05$ for $S = 50$ and $E = 57.3$

The second critical case with a start time of 90 (Figure 3.2-10) has the time at which the bridge deck rotation would go to zero if the external forcing term were removed for any longer at about 208.7. Figure 3.2-10 clearly illustrates the effect of removing the external forcing function in the mathematical model. From a time of 90 to about 208.7, the amplitude of the rotation in the bridge deck can be seen to gradually decrease. At the time of 208.7, the external forcing function is added back into the model, and the amplitude of rotation of the bridge can be seen to jump back up and level out as it has in many previous cases.

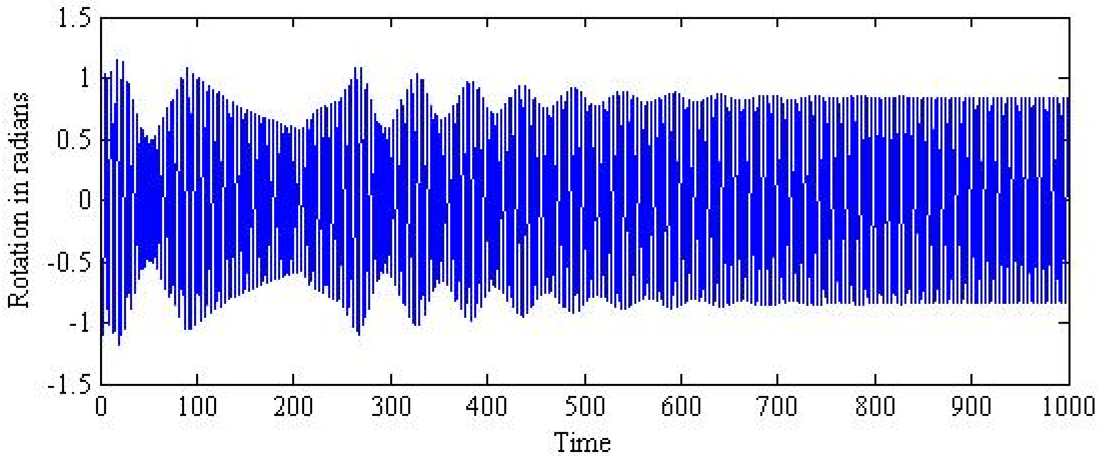

Figure 3.2-10—The bridge response with $\lambda_2 = -0.05$ for $S = 90$ and $E = 208.7$

The final case in which the external forcing function was removed from the mathematical model has a start time of 600 (Figure 3.2-11). After many numerical experiments, we found that with the external force removed for any longer than a time of 100 (end time 700), the amplitude of rotation of the bridge deck would go to zero. Similar to Figure 3.2-10, from the start time of 600 to the end time of 700, the amplitude of rotation in the bridge deck gradually decrease.

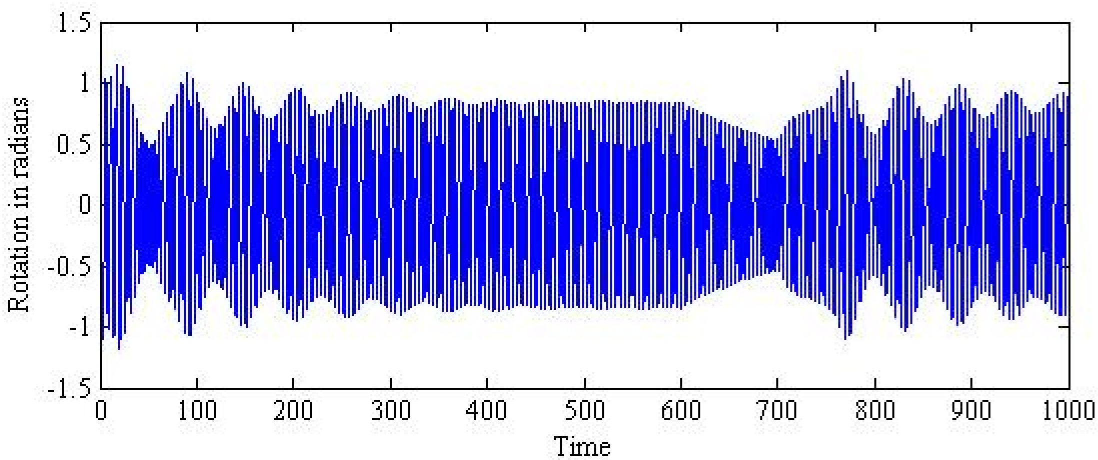

Figure 3.2-11—The bridge response with $\lambda_2 = -0.05$ for $S = 600$ and $E = 700$

In summary, we found that the amplitude of rotation in all three of these cases approached zero if the block of time that the external force was removed from the system was large enough. Once again, since we know the rotation of the bridge did not stop, we have shown the cases directly prior to those that the amplitude of rotation went to zero. The critical values for these trials were found to be at $t \approx 57.3$ for the case that began at $t = 50$ (Figure 3.2-9), $t \approx 208.7$ for the case that began at $t = 90$ (Figure 3.2-10), and $t \approx 700.0$ for the case that began at $t = 600$ (Figure 3.2-11). This suggests that not only is the external forcing function needed to start the bridge in torsional rotation, but without an external forcing function the bridge will not remain in torsional rotation.

All cases tested with an increase in the amplitude of the external periodic forcing function resulted in responses that maintained their large oscillations.

3.3 Modifications with a Constant Forcing Term

The second modification to the mathematical model that we will consider is the effects of the addition of a constant forcing term to the standard response in short but frequent time intervals and a long and continuous interval.

Short and Frequent Constant Forces

We now consider how the Tacoma Narrows Bridge torsional response changes due to short but frequent additional external forces on the bridge. This will be modeled as

$$\ddot{\theta} = \begin{cases} -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) + C & 10n + 9 < t \le 10(n+1) + 1 \\ -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) & 10(n+1) + 1 < t \le 10(n+1) + 9 \end{cases} \tag{3.5}$$

where C represents the additional constant force on the bridge and n represents any positive integer.

The most obvious reason for this change to occur would be many spikes in the wind speed on the bridge possibly resulting from gusty winds. This would induce many additional constant forces on it acting at various times. If the wind gusts were greater than the current wind speed then the C value would be a positive value, and similarly if the wind speed decreased suddenly the C value would be negative. If the C value were chosen to be a large enough negative number, this can represent a change in the wind direction at the Tacoma Narrows Bridge.

The first case that will be investigated is how an additional force of half the amplitude of the original forcing term will change the response. This means that C will be set to 0.025. In Figure 3.3-1, which can be seen below, the amplitude of the rotation in the bridge deck never quite level out as a result of the additional constant force added periodically.

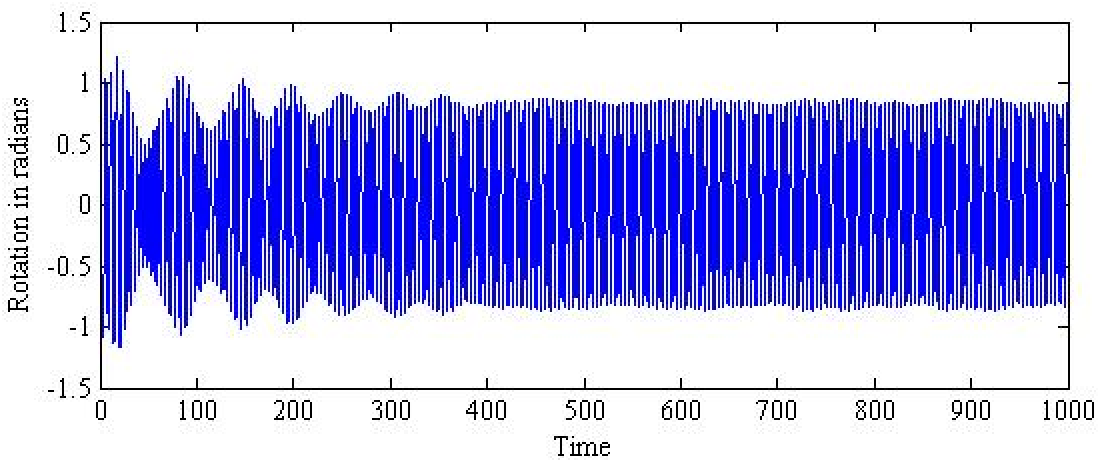

Figure 3.3-1—The bridge response as a result of an additional constant force of $C = 0.025$

From comparing Figure 3.3-1 to Figure 3.1-1, not much change is observed in the bridge response as a result of an additional constant force that is half the value of the amplitude the original forcing function has. When we increased the magnitude of the C value, we found that the bridge response became more chaotic, but followed a similar pattern to that illustrated in Figure 3.3-1. The result of adding a negative constant forcing term of -0.025 also results in a similar response to the positive force illustrated above.

If this value increased to -0.075, a very different response occurs (Figure 3.3-2). That is, the rotation of the bridge deck is very chaotic, and the amplitude of rotation eventually goes to zero at a time of about 600.

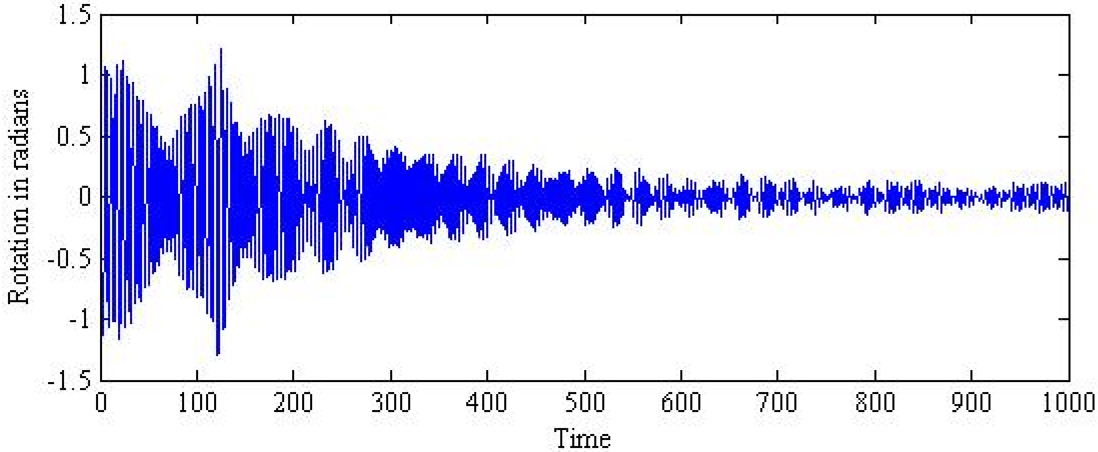

Figure 3.3-2—The bridge response as a result of an additional constant force of $C = -0.075$

Therefore, if a sufficiently large force opposite that of the original force is applied to the bridge at short and frequent intervals, the torsional motion of the bridge will stop. However, it is unlikely that the actual wind forces on the Tacoma Narrows Bridge changed directions this violently.

Additional Constant Force for a Single Block of Time

The next model demonstrates how the Tacoma Narrows Bridge response changes due to an additional external force for a single large block of time. This will be modeled as

$$\ddot{\theta} = \begin{cases} -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) + C & S < t \le E \\ -0.01\dot{\theta} - 2.4\cos\theta\sin\theta + 0.05\sin(1.3t) & t \le S, t > E \end{cases} \tag{3.6}$$

where C still represents the additional constant force on the bridge, S is the starting time for the additional force, and E is the end time of the additional force. Similarly to the short and frequent case of a constant force, we will use this case to demonstrate a change in wind speed. We now focus on a constant change in the speed which is more likely to have occurred at Tacoma Narrows.

No significant changes could be noted from the standard response with any of the cases that were examined with this model. The responses observed simply shifted the amplitude of rotation of the bridge, and made the amplitude slightly more random. The case where $C = 0.075$, $S = 600$, and $E \to \infty$ can be seen in Figure 3.3-3. This Figure shows that at a time of 600, which corresponds to the start time of the additional constant force, a slight jump in the amplitude of rotation of the bridge deck happens. As would be expected, if a negative C value would be introduced, a downward shift in the amplitude would occur instead of a shift up in the oscillations.

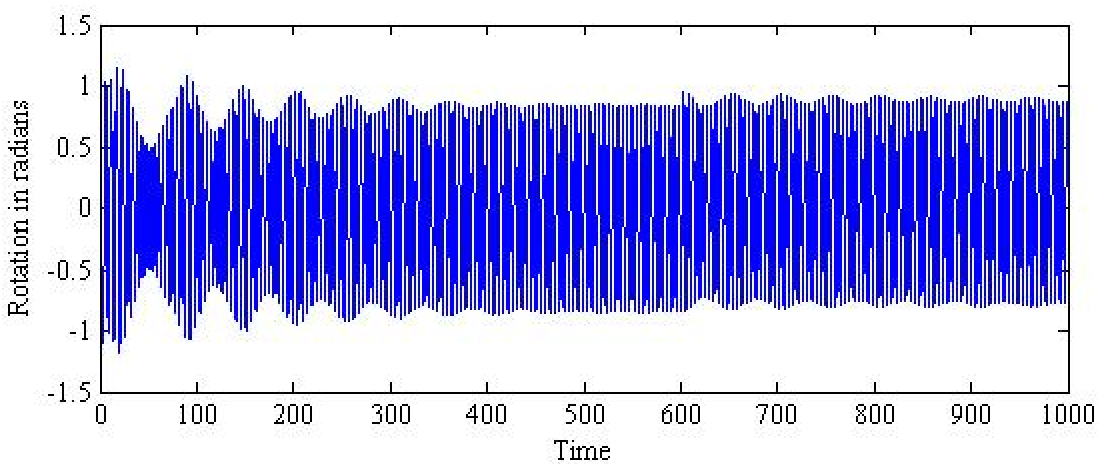

Figure 3.3-3—The bridge response due to a large additional constant force of $C = 0.075$ for $S = 600$ and $E \to \infty$

Figure 3.3-3 indicates that it was easier for the bridge to rotate to one side than the other since the amount of rotation in the bridge shifted up. This makes sense since if a force is acting on only one side of a body, the body will want to move in the direction of the applied force.

3.4 Weakening Hanger Cables on the Tacoma Narrows Bridge

We now consider a model to illustrate how the Tacoma Narrows Bridge responds if the hanger cables holding the bridge weakened as a result of the large forces applied on them from the torsional rotation. This will be modeled as

$$\ddot{\theta} = -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) \tag{3.7}$$

with w representing a reducing factor for the amount the cables weaken as time passes, and b representing a rate factor.

As previously stated, this will illustrate how the Tacoma Narrows Bridge behaves as a result of the cables weakening as time passes. It is very likely that this is what actually happened to the bridge, but the magnitude of this weakening of the cables is uncertain. For this reason, the model will be tested using different reducing and rate factors.

The first reducing factor that will be tested illustrates a minor loss in strength of the cables as time passes. In order to accomplish this, the w value will be set to 1,000 (Figure 3.4-1). Also for this case, the rate factor will be set to 1.00, which represents the cables losing strength at a constant rate as time passes. For this case, initially there is a large variance in the amplitude of the rotation and the lower limit of the variance can be seen to gradually increase over the entire time.

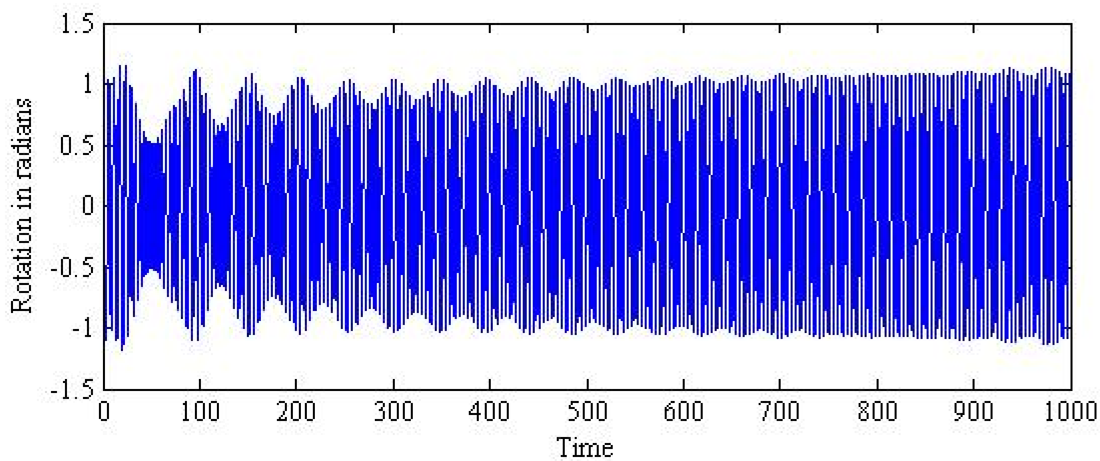

Figure 3.4-1—The bridge response due to weakening cables with a reduction factor of w = 1,000 and rate factor of b = 1.00

Figure 3.4-1 shows that a small reduction in strength of the cables due to fatigue will cause the bridge to rotate further as time passes. This is what would be expected, however the process is very gradual with a reduction factor of 1,000.

We will now look at what happens if the hanger cables lost strength at an increasing rate. This may have happened since the more that the bridge deck was allowed to rotate, the larger the forces on the cables would be. The reason the forces would be larger is simply because the bridge deck would be falling from a higher point as it rotated more, so the deck would gain more velocity before the cables began picking up the weight.

We can model this by setting the rate factor at a value greater than 1.00. For this case we will set b to 1.20 representing a slow increase in the rate that the cables lost strength, and w will remain at 1,000. The results of this are shown in Figure 3.4-2.

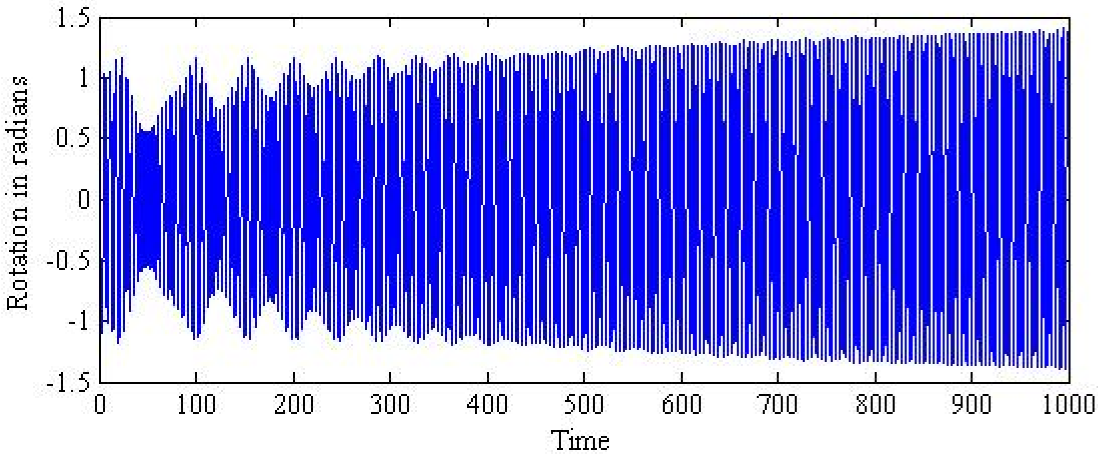

Figure 3.4-2—The bridge response due to weakening cables with a reduction factor of $w = 1{,}000$ and rate factor of $b = 1.20$

From Figure 3.4-2, we can see that the response is very similar to Figure 3.4-1, but with the amplitude of rotation increasing at a quicker rate which is what was expected. Increasing the rate factor further simply increases the amplitude of rotation more quickly.

The reduction factor will now be set at 250 and rate factor set at 1.00 in order to show what happens to the bridge if the cables lose a fairly substantial amount of strength due to the oscillations. This case can be seen in Figure 3.4-3 and we find that the increase of the lower limit of rotation occurs faster than in the case with the reduction factor at 1000 and rate factor at 1.00. Thus, the bridge is allowed to rotate further the weaker the cables get.

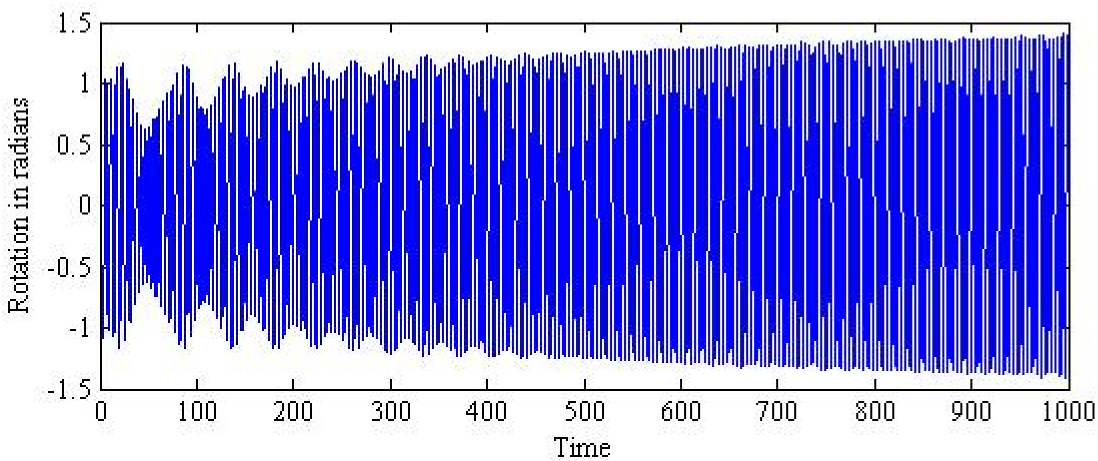

Figure 3.4-3—The bridge response due to weakening cables with a reduction factor of $w = 250$ and rate factor of $b = 1.00$

Once this rotation reaches about $\frac{\pi}{2}$ radians, the bridge deck completely flipped over. The bridge deck flipping over for the case with a reduction factor of 250 is shown in Figure 3.4-4. In this Figure, the deck flips over once at a time of about 1,180, and then flipped over again at a time of about 1,220. After the deck flips over a second time, the amplitude of rotation decreases significantly.

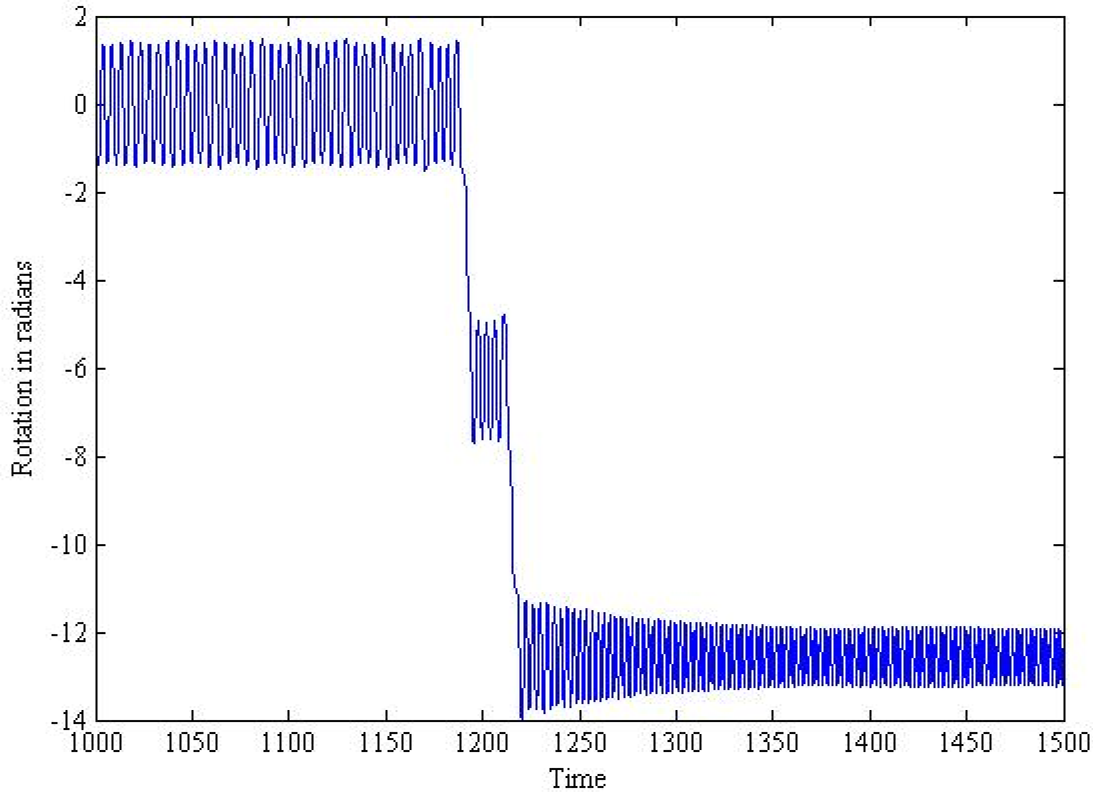

Figure 3.4-4—The Tacoma Narrows deck flipping over completely

When the case with a reduction factor of 1,000 and rate factor of 1.00 was modeled over a very large time, we found that the deck flipped over at a time of about 4,340, and this case rotated 2π radians as opposed to the -4π radian rotation that we see in Figure 3.4-4.

Obviously due to physical restraints, the bridge deck could not have flipped over completely. This model does help illustrate the large impact fatigue in the cables may have had on the bridge though. From these results it can be seen that a possible reason for the final collapse of the bridge was due to fatigue in the cables which resulted in a cable or cable connection failing.

4 Summary of Results

By comparing Figure 3.2-1 – Figure 3.2-4, Figure 3.2-6 – 3.2-11, Figure 3.3-1 – Figure 3.3-3, and Figure 3.4-1 – Figure 3.4-3, with Figure 3.1-1, we have shown various modifications to the mathematical model that have a significant impact on the overall response of the Tacoma Narrows Bridge. Since the rotation in most of these cases does not stop as a substantial amount of time passes, we can conclude that any one of these may very well be a more accurate representation of what actually happened to the Tacoma Narrows Bridge during its final moments. It would be more likely that the actual response of the Tacoma Narrows Bridge was a combination of some of the cases presented. It is also possible that as time passed the cables actually become weaker which enabled the bridge to twist further, until the cables could no longer support the momentum of the bridge and ultimately failed. For this reason the general direction that the bridge response probably most closely followed is the direction presented in Figure 3.4-1 as opposed to Figure 3.1-1.

If a few of these cases are combined, many very unique and interesting results occur. One such example of the infinite possibilities occurs if the cables weakened with a reducing factor of 1,000 and rate factor of 1.00 the entire time the bridge was in torsional rotation, the bridge had a slightly higher amplitude for the periodic force acting on it at a relatively frequent interval for the entire duration of rotation ($\lambda_2 = 0.021$ in this case), the bridge experienced a large increase in wind speed ($C = 0.05$) for the time interval $50 < t \leq 200$, and the bridge experienced a decrease in the amplitude of the periodic forcing function ($\lambda_3 = -0.03$ in this case) over the interval $400 < t \le 600$. This can be mathematically represented with the following function

$$\ddot{\theta} = \begin{cases} -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) + \lambda_2\sin(1.3t) & 20n-6 < t \le 20n-1,\ t \le 50 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) & 20n-1 < t \le 20(n+1)-6,\ t \le 50 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) + \lambda_2\sin(1.3t) + C & 20n-6 < t \le 20n-1,\ 50 < t \le 200 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) + C & 20n-1 < t \le 20(n+1)-6,\ 50 < t \le 200 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) + \lambda_2\sin(1.3t) & 20n-6 < t \le 20n-1,\ 200 < t \le 400 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) & 20n-1 < t \le 20(n+1)-6,\ 200 < t \le 400 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) + \lambda_2\sin(1.3t) + \lambda_3\sin(1.3t) & 20n-6 < t \le 20n-1,\ 400 < t \le 600 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) + \lambda_3\sin(1.3t) & 20n-1 < t \le 20(n+1)-6,\ 400 < t \le 600 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) + \lambda_2\sin(1.3t) & 20n-6 < t \le 20n-1,\ t > 600 \\ -0.01\dot{\theta} - (2.4 + \frac{t^b}{w})\cos\theta\sin\theta + 0.05\sin(1.3t) & 20n-1 < t \le 20(n+1)-6,\ t > 600 \end{cases} \tag{4.1}$$

where

$b = 1.00 =$ rate factor

$w = 1{,}000 =$ reducing factor

$\lambda_2 = 0.021 =$ amplitude of the periodic forcing for the entire torsional rotation

$\lambda_3 = -0.03 =$ decreased amplitude for the block of time specified

$C = 0.05 =$ large additional wind force for the block of time specified

$n =$ any positive integer

This case is presented in Figure 4-1 below.

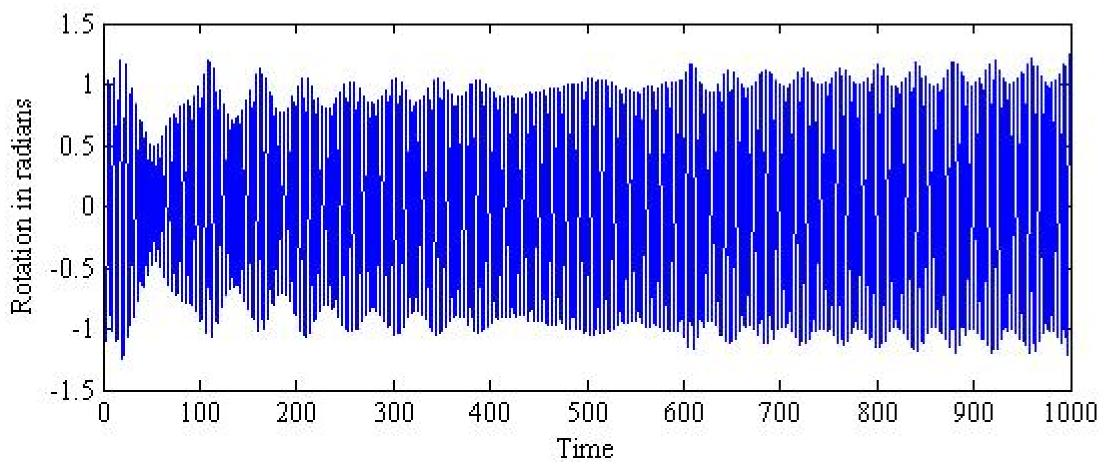

Figure 4-1— The bridge response due to a combined loading situation

It is quite clear that Figure 4-1 differs significantly from Figure 3.1-1. Unfortunately, it is impossible to say exactly what forces were acting on the Tacoma Narrows Bridge and when these forces were acting on it. However this does emphasize that there is still some work needed to be done on the mathematical model of the Tacoma Narrows Bridge in order to paint a more accurate picture of the torsional motion of the bridge.

5 Development of a Theory

From the results of the modifications made to McKenna's mathematical model, we will now develop a theory as to why the Tacoma Narrows Bridge failed. Since the case demonstrating the hanger cables weakening produced the most interesting results, we will take a closer look at the hanger cables. First we should determine what the actual forces on the hanger cables were. A good place to start is by assuming that the only force on the hangers was a result of the bridge deck free falling from a height equal to the amplitude of the bridge decks rotation. From Newton's Second Law of Motion we know that

$$F = ma \tag{5.1}$$

where F is force, m is mass, and a represents acceleration [**12**]. Unfortunately, the only variable known without further investigation is the mass of the bridge.

To determine the acceleration of the bridge deck, we will analyze the basic distance-velocity-acceleration relationships which are

$$a = \dot{v} \tag{5.2}$$

and

$$v = \dot{x} \tag{5.3}$$

with v denoting velocity and x representing distance [**4**]. The acceleration that we seek represents the acceleration the bridge deck is subjected to while the cable resists the force of the bridge deck. We will approximate the acceleration to be constant since we are only concerned with obtaining a general idea of what the forces were in the cables. In order to find this, we first need to integrate Equation 5.2

$$a = \frac{dv}{dt} \Rightarrow a \cdot dt = dv \qquad \text{Equation 5.2}$$

$$a \cdot \int_0^t dt = \int_{v_o}^{v_f} dv \Rightarrow a \cdot t = v_f - v_o \qquad \text{Integrating 5.2}$$

$$a_1 = \frac{v_{f_1} - v_{o_1}}{t} \tag{5.4}$$

In Equation 5.4, v_{f_1} is known to be zero since this would correspond to the velocity of the deck at the point when the deck is changing direction from falling to being pulled up by the cable. However, the time (t) it takes for the cable to stop the bridge deck, and the velocity of the deck at the point when the cable first begins resisting it (v_{o_1}) are still unknown.

When the cable begins resisting, the velocity of the bridge deck ($-v_{o_1}$) would be the same as the final velocity of the bridge deck as it falls from the top of the rotation to the point where the cable begins resisting (v_{f_2}). From further manipulations to Equation 5.2 and Equation 5.3, we can arrive at Equation 5.5.

$$v = \frac{dx}{dt} \Rightarrow dt = \frac{dx}{v} \qquad \text{Equation 5.3}$$

$$a = \frac{dv}{dt} \qquad \text{Equation 5.2}$$

$$a = v \cdot \frac{dv}{dx} \qquad \text{Combining (5.2) and (5.3)}$$

$$a \cdot \int_{x_o}^{x_f} dx = \int_{v_o}^{v_f} v \cdot dv \Rightarrow a \cdot (x_f - x_o) = \frac{v_f^{\,2}}{2} - \frac{v_o^{\,2}}{2} \qquad \text{Integrating combined Equation}$$

$$v_{f_2}^{\,2} = v_{o_2}^{\,2} + 2a_2(x_f - x_o) \tag{5.5}$$

In Equations 5.5, v_{o_2} is zero since this is the velocity of the deck at the point where the deck is changing directions from rising to falling. Also in this equation, a_2 represents the acceleration due to gravity since we are assuming the deck is free falling from the apex of its rotation to the point that the cables begin resisting forces. Other variables in Equation 5.5 are x_f and x_o which represent the final and initial positions of the bridge deck respectively.

When we substitute v_{f_2} in for v_{o_1}, Equation 5.4 simplifies to Equation 5.6.

$$a_1 = \frac{\sqrt{2g(x_f - x_o)}}{t} \tag{5.6}$$

If we now combine Equation 5.6 and Equation 5.1, we get

$$F = m\frac{\sqrt{2g(x_f - x_o)}}{t} \tag{5.7}$$

which we can use to find the force in each hanger cable with some minor assumptions.

As F. B. Farquharson was observing the bridge while it was in torsional rotation, he noted that "The motion had a frequency of 14 cycles per minute" [**1**]. Also, in [**9**], it is mentioned that the double amplitude of the bridge was about 28 feet. From this information, we can conclude that one cycle took approximately 4.3 seconds and had an amplitude of approximately 4.2 meters.

As mentioned earlier, the bridge deck weighed approximately 2,500 kgs. per foot of bridge [**1**]. The spacing between groups of hanger cables on the bridge was approximately 50 feet, and there were four cables grouped together per side of the bridge [**13**]. If we assume that the bridge deck weight is evenly distributed between all eight cables for each section of bridge, we find that $m \approx 15{,}625$ kgs per cable.

To proceed any further, we will have to make the assumption as to when the cables began picking up load. To maintain simplicity, we will assume the load was picked up halfway through the fall of the bridge deck, and the duration of the cables resisting forces was one-forth the time the deck took to make once cycle. With these assumptions, we obtain $x_f - x_o \approx 4.2$ meters and $t \approx 1.1$ seconds. We can now substitute these values into Equation 5.7 to get $F \approx 120{,}000$ N.

Now that we have an approximate force in one hanger cable, we can find the stress in this cable by using

$$\sigma = \frac{F}{A} \tag{5.8}$$

where σ is stress and A is cross-sectional area [**10**]. Due to the diameter of the hanger cables on the Tacoma Narrows Bridge not being readily available, we will have to assume a diameter for these. From inspection of many photos of the bridge, a 1" diameter cable seems reasonable and will be used. With this assumption, we will find that the stress in each suspender cable is approximately 230 MPa. Typical structural steel that is used today has a yield strength in the range of 250 MPa to 350 MPa [**2**], which shows that the forces in the hanger cables during the torsional rotation of the bridge were probably significant.

Although this example did not take into account all of the forces acting on the bridge, it does give an idea of how large the forces may have been. From these results, we can see that the steel used for the hanger cables may have been repeatedly stressed up to its yield point. If this occurred, the hanger cables may have undergone strain hardening which would have increased the chances of a brittle failure in the hanger cables [**10**]. A more in depth investigation of the forces on the hanger cables needs to be conducted in order to give a better idea of this possibility.

6 Future Study

This paper concentrated on numerically analyzing various loading situations of the Tacoma Narrows Bridge and developing a theory as to why the bridge ultimately failed. The next step would be to verify these cases by creating a scale model of a section of the bridge and simulating these loading conditions. Creating a scale model and subjecting it to these tests would also help to find the magnitude of these various forces. From this information, an even more accurate response could be modeled. Also, with a better idea of the forces acting on the bridge, the theory presented could be developed further.

By analyzing the forces that have the greatest impact on the torsional rotation of the Tacoma Narrows Bridge, engineers can focus on designing bridges to resist these forces in future suspension bridges which would help to improve their safety. This would allow engineers to push the limits of their designs with more certainty. In doing this, significant amounts of money could be saved by not over-designing these large structures, and engineers would be given more freedom to innovate new sound, structural designs.

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