Zero-Divisor Graphs of $\mathbb{Z}_n$ and Polynomial Quotient Rings over $\mathbb{Z}_n$

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**Recommended Citation**

Endean, Daniel; Manlove, Erin; and Henry, Kristin (2007) "Zero-Divisor Graphs of $\mathbb{Z}_n$ and Polynomial Quotient Rings over $\mathbb{Z}_n,"$

*Rose-Hulman Undergraduate Mathematics Journal*; Vol. 8 : Iss. 2 , Article 5.  
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Zero-Divisor Graphs of $\mathbb{Z}_n$ and Polynomial Quotient Rings over $\mathbb{Z}_n$

Daniel Endean, Kristin Henry, Erin Manlove

November 12, 2007

Abstract

Critical to the understanding of a graph are its chromatic number and whether or not it is perfect. Here we prove when $\Gamma(\mathbb{Z}_n)$, the zero-divisor graph of $\mathbb{Z}_n$, is perfect and show an alternative method to [D] for determining the chromatic number in those cases. We go on to determine the chromatic number for $\Gamma(\mathbb{Z}_p[x]/(x^n))$ where $p$ is prime and show that an isomorphism exists between this graph and $\Gamma(\mathbb{Z}_{p^n})$.

Introduction

We set out to study the zero-divisors of an algebraic ring. Graphically representing these elements leads to insight about their behavior and provides methods of categorizing the ring. In general, a graph $G$ is a set of vertices $V(G)$ combined with a corresponding edge set $E(G)$ such that every element in $E(G)$ represents an unordered pairing of distinct elements in $V(G)$. The order of $G$, denoted $|G|$, is equal to the cardinality of the vertex set. Given a commutative ring $R$ with unity, let $Z(R)$ be the non-zero zero-divisors of $R$. We define $\Gamma(R)$ as the zero-divisor graph of $R$ whose vertex set is $Z(R)$. The edge set of $\Gamma(R)$ is the set of all pairs of distinct vertices $(x,y)$ that are adjacent, that is $xy = 0$.

A graph $G$ has a chromatic number $\chi(G)$ corresponding to the minimum number of colors required to color each vertex such that no two adjacent vertices are colored the same color. If every pair of vertices in $G$ is adjacent then $G$ is called complete and is denoted $K_n$ where $n = |G|$. $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In particular, given $U \subseteq V(G)$ the subgraph induced by $U$ is the graph with vertex set $U$ and all edges in $E(G)$ whose end vertices are in $U$. Any subgraph of $G$ is called a clique if it is complete, and the clique number $\omega(G)$ denotes the order of the largest clique in $G$. Clearly, the clique number provides a lower bound to the chromatic number of $G$ ([CL], p. 279). Anna Duane determined the chromatic number of $\Gamma(\mathbb{Z}_n)$ for any $n$ [D].

In this paper, we will first offer a second method of determining the chromatic number of $\Gamma(\mathbb{Z}_n)$ for certain $n$. Following this we will explore the chromatic number of the zero-divisor graph of certain polynomial quotient rings over $\mathbb{Z}_n$. Using these results we discover a connection between these two sets of zero-divisors.

1 Zero-Divisor Graphs of $\mathbb{Z}_n$

We begin by exploring the family of rings $\mathbb{Z}_n$. 

1
In [D], the chromatic number of $\Gamma(\mathbb{Z}_n)$ was determined by defining a *proper coloring* of the vertices, where no two adjacent vertices share the same color. To establish the chromatic number, it was then shown that the graph could not be colored using fewer colors.

A second method of establishing the chromatic number takes advantage of the clique number of the graph. In order to use a clique to determine the exact chromatic number of a graph, the graph must first be proven to be perfect. A graph is defined to be *perfect* if for every subgraph $H \subseteq G$, $\omega(H) = \chi(H)$. The following theorem provides one tool for proving that a graph is perfect. Note that $P_n$ is the graph of $n$ vertices such that the vertices $v_i$ and edges $e_j$ form the alternating sequence $v_1, e_1, v_2, e_2, \ldots, v_{n-1}, e_{n-1}, v_n$, where $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, n$ and $v_i \neq v_j$ for all $i \neq j$ as shown in Figure 1.

![Figure 1: $P_4$](image)

**Theorem 1.1** ([CL]) *If a graph $G$ does not contain $P_4$ as an induced subgraph, then $G$ is perfect.*

In most cases, $\mathbb{Z}_n$ does not yield a perfect zero-divisor graph. However, we now identify two cases which are perfect.

**Theorem 1.2** The graph $\Gamma(\mathbb{Z}_{p^n})$, where $p$ is prime, is perfect.

**Proof:** Suppose $p = 2$ and $n = 2$ or $n = 3$. Then $\Gamma(\mathbb{Z}_{p^n})$ has $\{2\}$ and $\{2, 4, 6\}$ as vertex sets respectively. Also, if $p = 3$ and $n = 2$ then $\Gamma(\mathbb{Z}_{p^n})$ has vertices labeled 3 and 6. Thus in these cases, the order of $\Gamma(\mathbb{Z}_{p^n})$ is less than 4, and therefore it cannot contain $P_4$.

For all other cases of $p$ and $n$, we must show that given four distinct elements $v_1, v_2, v_3, v_4 \in Z(\mathbb{Z}_{p^n})$ such that $v_1v_2 = 0, v_2v_3 = 0, v_3v_4 = 0$ there exists at least one more pair $v_qv_r = 0$ where $q, r \in \{1, 2, 3, 4\}$ and $q \neq r$. Each vertex $v_k$ can be written as $m_kp^k$ where $\gcd(p, m_k) = 1$. When two vertices $v_q$ and $v_r$ are multiplied together we have $v_qv_r = m_qp^{i_q}m_rp^{i_r} = m_qm_rp^{i_q+i_r}$. This yields the condition $i_q + i_r \geq n$ for an edge to exist between $v_q$ and $v_r$. Based on our definition of the vertices we know:

\[
\begin{align*}
i_1 + i_2 &\geq n \\
i_2 + i_3 &\geq n \\
i_3 + i_4 &\geq n
\end{align*}
\]

We note that either $i_2 \geq i_3$ or $i_3 > i_2$. In the first case we can say that $i_2 + i_4 \geq i_3 + i_4 \geq n$ implies an edge between $v_2$ and $v_4$. In the second case we can say that $i_1 + i_3 \geq i_1 + i_2 \geq n$ implies an edge between $v_1$ and $v_3$. Thus $\Gamma(\mathbb{Z}_{p^n})$ cannot contain $P_4$ as an induced subgraph and is therefore perfect.

**Theorem 1.3** The graph $\Gamma(\mathbb{Z}_{p_1p_2})$, where $p_1$ and $p_2$ are distinct primes, is perfect.
\section*{Proof}

Suppose that $p_1$ or $p_2$ equals 2. Without loss of generality, assume $p_1 = 2$. Then $\Gamma(Z_{p_1p_2})$ consists of one central vertex corresponding to $p_2$ which is connected to the remaining vertices of the form $2k$ such that $k < p_2$. The vertices that are divisible by 2 do not contain with each other since their product will not be divisible by $p_2$. Therefore, the graph cannot contain $P_4$ since the longest path in $\Gamma(Z_{p_1p_2})$ contains two vertices divisible by 2 connected to the central vertex.

If $p_1 \neq 2$ and $p_2 \neq 2$ then let $v_1, v_2, v_3$ and $v_4$ be four distinct vertices in $\Gamma(Z_{p_1p_2})$ such that $v_1v_2 = 0$, $v_2v_3 = 0$, and $v_3v_4 = 0$. Since $v_1v_2 = 0$, without loss of generality $p_1 | v_1$ and $p_2 | v_2$. It follows that $p_1$ must divide $v_3$ since $v_2v_3 = 0$. From this we then know $p_2$ must divide $v_4$ since $v_3v_4 = 0$. Therefore $v_1v_4 = 0$ and $P_4$ is not an induced subgraph of $\Gamma(Z_{p_1p_2})$.

Not only can we prove that $\Gamma(Z_n)$ is perfect for these cases, but we also know these are the only cases when it is perfect.

\begin{thm}
The zero-divisor graph of $Z_n$ is perfect if and only if $n = p^k$ for some prime $p$ or $n = p_1p_2$ for some distinct primes $p_1, p_2$.
\end{thm}

\begin{proof}
If $n = p^k$ or $n = p_1p_2$ we know $\Gamma(Z_n)$ is perfect by Theorems 1.2 and 1.3. For all other cases of $n$, we know that $n = p_1p_2m$ for some distinct primes $p_1$ and $p_2$ and $m > 1$. Also, there exist four vertices in $\Gamma(Z_n)$ defined as $v_1 = p_1$, $v_2 = p_2m$, $v_3 = p_1m$, and $v_4 = p_2$. Since $v_1v_2 = 0$, $v_2v_3 = 0$, and $v_3v_4 = 0$ but $v_1v_3 \neq 0$, $v_2v_4 \neq 0$, and $v_1v_4 \neq 0$ we know $\Gamma(Z_n)$ contains $P_4$. Therefore, $\Gamma(Z_n)$ is perfect if and only if $n = p^k$ or $n = p_1p_2$.

Since we have proven that $\Gamma(Z_{p^m})$ and $\Gamma(Z_{p_1p_2})$ are perfect, we can use this property to determine the chromatic number. The proofs simply require finding the clique number of the graph.

\begin{thm}
The graph $\Gamma(Z_{p^n})$, where $p$ is prime, has chromatic number $p^{\frac{n}{2}} - 1$ for $n$ even and $p^{\frac{n+1}{2}}$ for $n$ odd.
\end{thm}

\begin{proof}
Let $N_i$ be the subset of $Z(Z_{p^n})$ defined by $N_i = \{ kp^i | gcd(k, p) = 1, 0 < k < p^{n-i} \}$ for all $0 < i < n$. Given two elements $a, b \in N_i$, their product is $ab = k_ap^ikbp^i = k_ap^{2i}$. It follows that elements in $N_i$ are connected in $\Gamma(Z_{p^n})$ if and only if $2i \geq n$. Similarly given an element $c \in N_i$ and $d \in N_j$, their product is $cd = k_cp^ikdp^j = k_cp^{i+j}$. This shows elements of $N_i$ connect with elements of $N_j$ if and only if $i + j \geq n$. If $q = \lceil \frac{n}{2} \rceil$ then the set of vertices $S = \bigcup_{i=0}^{\lceil \frac{n}{2} \rceil} N_i$ induces a clique in $\Gamma(Z_{p^n})$. Since $S$ consists of all elements in $Z_{p^n}$ divisible by $p^q$, elements in $S$ may be written as $tp^q$ where $0 < t < p^{n-q}$. Since there are $p^{n-q} - 1$ integer values for $a$ between 0 and $p^{n-q}$, it follows that $|S| = p^{n-q} - 1$. For all $N_i$ such that $i < q$ it follows that no two elements in $N_i$ are connected since $2i < n$. Similarly, if $i < j < q$, for all $c \in N_i$ and $d \in N_j$, $c$ and $d$ are not connected in $\Gamma(Z_{p^n})$. Thus, any complete subgraph in $\Gamma(Z_{p^n})$ can contain at most one element not divisible by $p^q$. We now consider two cases for the value of $n$:

\textbf{Case 1}: If $n$ is even, then $q = \frac{n}{2}$. In this case any element $a \in N_{q-1}$ does not connect to the elements in $N_q$ since $q + q - 1 = n - 1$. Therefore the subgraph induced by $S \cup \{ a \}$ is not a clique. This implies $S$ is the largest clique in $\Gamma(Z_{p^n})$. Thus $|S|$ is the clique number of $\Gamma(Z_{p^n})$, that is $|S| = p^{\frac{n}{2}} - 1 = \omega(\Gamma(Z_{p^n}))$.

\textbf{Case 2}: If $n$ is odd, then $q = \frac{n+1}{2}$. In this case each vertex in $N_{q-1}$ connects to all vertices in $S$. It follows that the subgraph induced by $S \cup \{ a \}$ is a clique. Since we have proved that any clique in $Z_{p^n}$ must contain two elements not divisible by $p^q$, we know $S \cup \{ a \}$ is the largest clique. Therefore $|S \cup \{ a \}| = p^{n-q} - 1 + 1 = p^n - p^{\frac{n+1}{2}} = p^{\frac{n-1}{2}} = \omega(\Gamma(Z_{p^n}))$.

Since we know definitively the clique number of $\Gamma(Z_{p^n})$, and Theorem 1.2 states that $\Gamma(Z_{p^n})$ is perfect, we know $\chi(\Gamma(Z_{p^n})) = \omega(\Gamma(Z_{p^n}))$.
\end{proof}
The following proof uses this same method of determining the chromatic number by way of the graph’s clique number. It should be noted that a much more straightforward proof uses the fact that all elements in $\Gamma(Z_{p_{1}p_{2}})$ fall into two separate sets which are unconnected with themselves leaving a 2-color graph. However we include this proof for completeness in showing the significance of the graph’s perfect property.

**Theorem 1.6** The graph $\Gamma(Z_{p_{1}p_{2}})$, where $p_{1}$ and $p_{2}$ are distinct primes, has chromatic number two.

**Proof:** Clearly $\Gamma(Z_{p_{1}p_{2}})$ has a clique of order two since any element divisible by $p_{1}$ is connected to an element divisible by $p_{2}$. Suppose there is a clique of order three. Then there must exist $q, r, s \in Z(Z_{p_{1}p_{2}})$ such that $qr = 0$, $rs = 0$ and $qs = 0$. Since there are only two primes each element is divisible by only one. If all three elements are divisible by $p_{1}$ then the conditions will not hold. The conditions also fail if all are divisible by $p_{2}$. Thus we may assume without loss of generality that $p_{1}|q$ and $p_{2}|r$. This implies that $p_{1}|s$ by assumption that $rs = 0$. However, if both $s$ and $q$ are divisible by $p_{1}$ their product cannot be zero and we have a contradiction. Therefore $\Gamma(Z_{p_{1}p_{2}})$ cannot have a clique of order three, and cannot have a clique of higher order since it would necessarily contain a clique of order three. This implies $\omega(\Gamma(Z_{p_{1}p_{2}})) = 2$ and since $\Gamma(Z_{p_{1}p_{2}})$ is perfect it has chromatic number two. $
$

## 2 Zero-Divisor Graphs of Polynomial Quotient Rings

To extend our study of zero-divisor graphs we examine the quotient ring $Q = Z_{p}[x]/\langle x^{n} \rangle$ where $p$ is prime and $n \geq 2$. The elements of this ring are congruence classes of polynomials modulo the principal ideal $\langle x^{n} \rangle$. For sake of simplifying notation, $q(x)$ will denote the congruence class of polynomials congruent to $q(x) \mod \langle x^{n} \rangle$. Thus, an element in $Q$ is of the form $q(x) = q_{0} + q_{1}x + q_{2}x^{2} + \ldots + q_{n-1}x^{n-1}$ where $q_{i} \in Z_{p}$ for each $i$. Two non-zero polynomials $q(x)$ and $r(x)$ are zero-divisors in this quotient ring if $q(x)r(x) \equiv 0 \mod \langle x^{n} \rangle$. To begin with, we demonstrate that using $Z_{p}$ as the base of the quotient ring greatly simplifies the zero-divisor set of this ring.

**Theorem 2.1** If an element $a(x) = a_{0} + a_{1}x + \cdots + a_{n-1}x^{n-1}$ is in $Z(Q)$, then $a_{0} \equiv 0 \mod p$.

**Proof:** Let $a(x) \in Z(Q)$. Then there exists $b(x) \in Q$ such that $a(x)b(x) \equiv 0 \mod \langle x^{n} \rangle$. Let $b(x) = b_{i}x^{i} + b_{i+1}x^{i+1} + \cdots + b_{n-1}x^{n-1}$ where $b_{i}$ is the first non-zero coefficient in the polynomial $b(x)$. The coefficient of $x^{i}$ in the product $a(x)b(x)$ is $a_{0}b_{i}$. Since $a(x)b(x) \equiv 0 \mod \langle x^{n} \rangle$ and $i < n$, we must have $a_{0}b_{i} \equiv 0 \mod p$. Since $b_{i} \neq 0$, we know $a_{0} \equiv 0 \mod p$.

**Corollary 2.2** There are $p^{n-1} - 1$ zero-divisors in $Q$.

**Proof:** Let $a(x)$ be in $Z(Q)$. Then by Theorem 2.1, $a_{0} \equiv 0 \mod p$, so $a(x) = a_{1}x + a_{2}x^{2} + \cdots + a_{n-1}x^{n-1}$ where $a_{i} \in Z_{p}$. Every element of $Q$ of this form is a zero-divisor because $a(x)x^{n-1} \equiv 0 \mod \langle x^{n} \rangle$. Since there are $n - 1$ coefficients each with $p$ choices, there are a total of $p^{n-1}$ zero-divisors. However, the case where all coefficients are zero must be removed, so there are $p^{n-1} - 1$ nonzero zero-divisors in the ring $Q$ if $p$ is prime.

**Theorem 2.3** The clique number of $\Gamma(Q)$ is $p^{n}/2 - 1$ for $n$ even and $p^{n-1}/2$ for $n$ odd.
Case 1: If \( n \) is even, then \( n = 2m \) for some \( m \in \mathbb{Z} \). Then \( x^m x^m = x^n \equiv 0 \), and \( x^{m+j} x^{m+k} \equiv 0 \) mod \( (x^n) \) for any \( j, k \in \mathbb{Z}^+ \). Thus, when \( i \geq m \), any two elements \( a(x), b(x) \in A_i \) are connected. Moreover, all elements in \( A_i \) are connected to all elements in \( A_j \) when \( i, j \geq m \). This gives a complete subgraph with
\[
\sum_{i=m}^{n-1} |A_i| = \sum_{i=m}^{n-1} (p-1)p^{n-i-1} = (p-1) \sum_{k=0}^{n-m-1} p^k = (p-1) \frac{p^{n-m-1} - 1}{p-1} = p^m - 1
\]

elements. The remaining elements in \( A_i \) where \( i < m \) will not contribute to the clique since \( i + m < n \). Furthermore there is no disjoint clique since none of the elements in \( A_i \) are connected to elements in \( A_j \) for \( i \leq j < m \). Therefore, \( \bigcup_{i=m}^{n-1} N_i \) is the largest clique and \( \omega(\Gamma(Q)) = p^m - 1 = p^{\frac{n}{2}} - 1 \) when \( n \) is even.

Case 2: If \( n \) is odd, then \( n = 2m + 1 \) for some \( m \in \mathbb{Z} \). Then \( x^m x^m \not\equiv 0 \), but \( x^{m+1} x^m \equiv 0 \), and \( x^{m+1} x^{m+1} \equiv 0 \). Thus, when \( i > m \), any two elements \( a(x), b(x) \in A_i \) are connected. Moreover, all elements in \( A_i \) are connected to all elements in \( A_j \) when \( i, j > m \). This gives a complete subgraph with
\[
\sum_{i=m+1}^{n-1} |A_i| = \sum_{i=m+1}^{n-1} (p-1)p^{n-i-1} = (p-1) \sum_{k=0}^{n-m-2} p^k = (p-1) \frac{p^{n-m-1} - 1}{p-1} = p^m - 1
\]

elements. The elements in the set \( A_m \) do not connect to each other but they do connect to all the elements in \( A_i \) where \( i > m \), so only one element from \( A_m \) can be included in the clique. Therefore, there is a complete subgraph connecting the \( p^m \) elements. The remaining elements in \( A_i \) where \( i < m \) and all elements except one from \( A_m \) will not contribute to the clique since \( i + m < n \). Therefore, \( \bigcup_{i=m}^{n-1} N_i \cup a \), for a single \( a \in A_m \), is the largest clique and \( \omega(\Gamma(Q)) = p^m = p^{\frac{n-1}{2}} \) when \( n \) is odd.

Corollary 2.4 The chromatic number of \( \Gamma(Q) \) is equal to the clique number of \( \Gamma(Q) \). 

Proof: Let \( A_i \) be as in the proof of Theorem 2.3.

Case 1: Let \( n \) be even. A clique is formed from the elements in \( A_i \) where \( i \geq \frac{n}{2} \). The remaining elements in \( A_i \) where \( i < \frac{n}{2} \) can be assigned the same color as the elements in \( A_{\frac{n}{2}} \) since \( i + \frac{n}{2} < n \). Therefore, \( \omega(\Gamma(Q)) = \chi(\Gamma(Q)) \) when \( n \) is even.

Case 2: Let \( n \) be odd. The clique is formed from the elements in \( A_i \) where \( i > \frac{n-1}{2} \) in addition to one element from \( A_{\frac{n-1}{2}} \). The remaining elements in \( A_i \) where \( i \leq \frac{n-1}{2} \) can be assigned the same color since \( i + \frac{n-1}{2} < n \). Therefore, \( \omega(\Gamma(Q)) = \chi(\Gamma(Q)) \) when \( n \) is odd.

Returning to our work in Section 1, we notice that \( \Gamma(Q) \) is similar to \( \Gamma(\mathbb{Z}_{p^n}) \), having the same clique number and chromatic number. In fact, we can prove the graphs are isomorphic. Note that two graphs are isomorphic if there exists a one-to-one function from one vertex set onto the other such that adjacency is preserved by the function.

Lemma 2.5 [L] Let \( b \) be greater than 1. Then every \( a > 0 \) can be uniquely represented in the form
\[
a = c_n b^n + c_{n-1} b^{n-1} + \cdots + c_1 b + c_0
\]
with \( c_n \neq 0 \), \( n \geq 0 \), and \( 0 \geq c_i < b \) for \( i = 0, 1, 2, \ldots, n \).
Theorem 2.6 If $p$ is prime then $\Gamma(Z_{p^n}) \cong \Gamma(Q)$.

Proof: Let $C = Z_{p^n}$ and $Q$ be defined as above. By Lemma 2.5 for each $q \in Z(C)$ we may uniquely write $q = q_1 p^k + q_{k+1} p^{k+1} + \ldots + q_{n-1} p^{n-1}$ where $k < n$ and $0 \leq q_i < p$ for each $i = k, \ldots, n - 1$. Then we define $\phi_p : Z(C) \rightarrow Z(Q)$ by $\phi_p(q) = q(x) = q_k x^k + q_{k+1} x^{k+1} + \ldots + q_{n-1} x^{n-1}$.

Suppose $q$ and $r$ are equivalent in $Z(C)$, then $q = r + s p^n$ for some $s \in Z$. Now $q(x) = r(x) + s x^n$ and we see that $\phi_p(q)$ is equivalent to $\phi_p(r)$ in $Z(Q)$. Hence $\phi_p$ is a well-defined function.

Let $q(x) \in Z(Q)$. By Theorem 2.1 we know $q(x) = q_1 x + \ldots + q_{n-1} x^{n-1}$ where $0 \leq q_i < p$ for all $i = 1, \ldots, n$ and $q_i \neq 0$ for at least one $i$. Clearly, $q = q_1 p + \ldots + q_{n-1} p^{n-1} \in Z(C)$ since $p \mid q$. Thus, $\phi_p(q) = q(x)$ and $\phi_p$ is onto.

To establish the order of $Z(C)$ we note that there are $p^{n-1} - 1$ non-zero integers less than $p^n$ which are divisible by $p$. Since these integers comprise $Z(C)$ we have $|Z(C)| = p^{n-1} - 1$. In the case of $Z(Q)$ Theorem 2.1 guarantees us that any $q(x) \in Z(Q)$ can be written as $q(x) = q_1 x + \ldots + q_{n-1} x^{n-1}$ where $0 \leq q_i < p$ for all $i = 1, \ldots, n$ and $q_i \neq 0$ for at least one $i$. Since there are $p^{n-1} - 1$ choices for the coefficients modulo $p$, the order of $Z(Q)$ must also be $p^{n-1} - 1$. Thus by the pigeon hole principle, since $\phi_p$ is onto we know that it must also be one-to-one.

Finally, we will show that $\phi_p$ preserves adjacency. Let $q \in Z(C)$ and write $q = a p^k$ for some $0 < k < n$ and $0 < a < p^n - k$ with $\gcd(a, p) = 1$. If $r \in Z(C)$ is adjacent to $q$, then $qr \equiv 0 \mod p^n$. Hence, we can write $r = b p^{n-k}$ for some $0 < b < p^k$ with $\gcd(b, p) = 1$. Now Lemma 2.5 assures us that $a = a_0 + a_1 p + \ldots + a_{n-k-1} p^{n-k-1}$ and $b = b_0 + b_1 p + \ldots + b_{k-1} p^{k-1}$ where $0 \leq a_i, b_j < p$ whenever $i = 1, \ldots, n - k - 1$ and $j = 1, \ldots, k - 1$. By definition, $\phi_p(q) = q(x) = a_0 x^k + a_1 x^{k+1} + \ldots + a_{n-k-1} x^{n-1}$ and $\phi_p(r) = r(x) = b_0 x^{n-k} + b_1 x^{n-k+1} + \ldots + b_{k-1} x^{n-1}$. Multiplying these elements we see $q(x) = f(x) x^n$ for some polynomial $f(x) \in Z_p[x]$. But clearly then $q(x) r(x) \equiv 0 \mod (x^n)$. Thus, $\phi_p$ preserves adjacency.

Having provided this final isomorphism we can conclude from earlier work that $\Gamma(Q)$ is a perfect graph.

We propose further study to explore isomorphisms of $\Gamma(Z_{(p_1 p_2)^n})$ with $\Gamma(Z_{p_1 p_2}[x]/(x^n))$. This task may be substantially more complex given the fact that $\Gamma(Z_{p_1 p_2}[x]/(x^n))$ is not perfect for $n > 1$ as seen by inducing $P_4$ with the vertex set $\{p_1 + p_2 x^{n-1}, p_2 x^{n-1}, p_1 x^n, p_2 + p_1 x^{n-1}\}$.

Finally, we would like to thank our advisor, Professor Jill Dietz, for her helpful comments and support on this project and the reviewer.

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