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A Multi-Objective Approach to Portfolio Optimization

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Abstract: Optimization models play a critical role in determining portfolio strategies for investors. The traditional mean variance optimization approach has only one objective, which fails to meet the demand of investors who have multiple investment objectives. This paper presents a multi-objective approach to portfolio optimization problems. The proposed optimization model simultaneously optimizes portfolio risk and returns for investors and integrates various portfolio optimization models. Optimal portfolio strategy is produced for investors of various risk tolerance. Detailed analysis based on convex optimization and application of the model are provided and compared to the mean variance approach.

1. Introduction to Portfolio Optimization

Portfolio optimization plays a critical role in determining portfolio strategies for investors. What investors hope to achieve from portfolio optimization is to maximize portfolio returns and minimize portfolio risk. Since return is compensated based on risk, investors have to balance the risk-return tradeoff for their investments. Therefore, there is no a single optimized portfolio that can satisfy all investors. An optimal portfolio is determined by an investor's risk-return preference.

There are a few key concepts in portfolio optimization. First, reward and risk are measured by expected return and variance of a portfolio. Expected return is calculated based on historical performance of an asset, and variance is a measure of the dispersion of returns. Second, investors are exposed to two types of risk: unsystematic risk and systematic risk. Unsystematic risk is an asset's intrinsic risk which can be diversified away by owning a large number of assets. These risks do not present enough information about the overall risk of the entire portfolio. Systematic risk, or the portfolio risk, is the risk generally associated with the market which cannot be eliminated. Third, the covariance between different asset returns gives the variability or risk of a portfolio. Therefore, a well-diversified portfolio contains assets that have little or negative correlations [1].

The key to achieving investors' objectives is to provide an optimal portfolio strategy which shows investors how much to invest in each asset in a given portfolio. Therefore, the decision variable of portfolio optimization problems is the asset weight vector $\vec{x} = \left[x_1 \, x_2 \cdots x_n \right]^T$ with x_i as the weight of asset i in the portfolio. The expected return for each asset in the portfolio is expressed in the vector form $\vec{p} = [p_1 p_2 \cdots p_n]^T$ with p_i as the mean return of asset i. The portfolio expected return is the weighted average of individual asset return $x_p = \vec{p}^T \vec{x} = \sum_{i=1}^{n} x_i p_i$. Variance and covariance of individual asset are characterized by a variance-covariance matrix $V = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}$, where $\sigma_{i,i}$ is the variance of asset i and $\sigma_{i,j}$ is the covariance between asset i and asset j. The portfolio variance is $\sigma_p{}^2 = \vec{x}^T V \vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{i,j}$ [2].

1.1 Problem Formulations

Modern portfolio theory assumes that for a given level of risk, a rational investor wants the maximal return, and for a given level of expected return, the investor wants the minimal risk. There are also extreme investors who only care about maximizing return (disregard risk) or minimizing risk (disregard expected return). There are generally five different formulations that serve investors of different investment objectives:

Model 1: Maximize expected return (disregard risk)

Maximize: $x_p = \vec{p}^T \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$

where $\vec{1}^T = [1 \cdots 1]$. The constraint $\vec{1}^T \vec{x} = 1$ requires the sum of all asset weights to be equal to 1.

Model 2: Minimize risk (disregard expected return)

Minimize: $\sigma_p{}^2 = \vec{x}^T V \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$

Model 3: Minimize risk for a given level of expected return $p*$

Minimize: $\sigma_p{}^2 = \vec{x}^T V \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$ and $\vec{p}^T \vec{x} = p*$

Model 4: Maximize return for a given level of risk σ^{2*}

Maximize: $x_p = \vec{p}^T \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$ and $\vec{x}^T V \vec{x} = \sigma^{2*}$

Model 5: Maximize return and Minimize risk

Maximize: $x_p = \vec{p}^T \vec{x}$ and Minimize: $\sigma_p{}^2 = \vec{x}^T V \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$

The five models above include both rational and extreme investors with different investment objectives. Model 3 and 4 are extensions of Model 1 and 2 with fixed constraints. The classic solution to portfolio optimization is the mean variance optimization proposed by Nobel Prize winner Harry Markowitz in 1990 [2]. The mean variance method aims at minimizing variance of a portfolio for any given level of expected return, which shares the same formulation of model 3. Since the mean variance method assumes all investors' objectives are to minimize risk, it may not be the best model for those who are extremely risk seeking. Also, the formulation does not allow investors to simultaneously minimize risk and maximize expected return.

1.2 Introduction to Multi-objective optimization

Multi-objective optimization, developed by French-Italian economist V. Pareto, is an alternative approach to the portfolio optimization problem [3]. The multi-objective approach combines multiple objectives $f_1(\vec{x}), f_2(\vec{x}), \ldots, f_n(\vec{x})$ into one objective function by assigning a weighting coefficient to each objective. The standard solution technique is to minimize a positively weighted convex sum of the objectives using single-objective method, that is,

$$Minimize\ F(\vec{x}) = \sum_{i=1}^{n} a_i f_i(\vec{x}), \qquad a_i > 0, \quad i = 1, 2, \ldots, n$$

The concept of optimality in multi-objective optimization is characterized by Pareto optimality. Essentially, a vector $\vec{x}^*$ is said to be Pareto optimal if and only if there is no $\vec{x}$ such that $f_i(\vec{x}) \le f_i(\vec{x}^*)$ for all $i = 1, 2, \ldots, n$. In other words, $\vec{x}^*$ is the Pareto point if $F(\vec{x}^*)$ achieves its minimal value [4].

Since investors are interested in minimizing risk and maximizing expected return at the same time, the portfolio optimization problem can be treated as a multi-objective optimization problem (Model 5). One can attain Pareto optimality in this case because the formulation of Model 5 belongs to the category of convex vector optimization, which guarantees that any local optimum is a global optimum [5]. This paper focuses on the analysis and application of the multi-objective approach to portfolio optimization based on convex vector optimization.

2. Methodology

Before going into details about multi-objective optimization, it is essential to introduce the concept of convex vector optimization.

2.1 Convex Vector Optimization

As shown in Figure 1, a set $S \subseteq \mathrm{R}^n$ is a convex set if it contains all line segments joining any pair of points in S , that is,

$$x, y \in S, \theta > 0 \Rightarrow \theta x + (1-\theta) y \in S$$

Convex

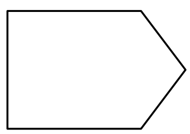

Figure 1 (a)

Convex

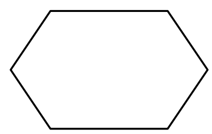

Figure 1 (b)

Non-Convex

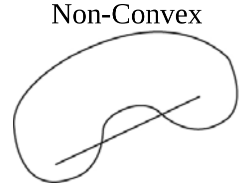

Figure 1 (c)

A function $f : \mathrm{R}^{\mathrm{n}} \to \mathrm{R}$ is convex if its domain $dom\, f$ is convex and for all $x, y \in dom\ f$, $\theta \in [0,1]$

$$f\left(\theta x + (1-\theta) y\right) \leq \theta f\left(x\right) + (1-\theta)\, f\left(y\right)$$

A function f is concave if $-f$ is convex. Geometrically, one can think of the curve of a convex function as always lying below the line segment of any two points. Here is an example of a convex function:

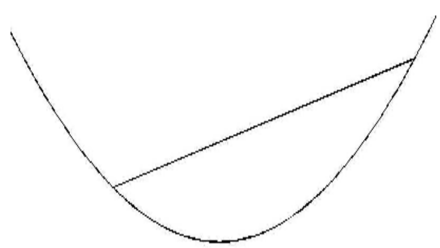

Figure 2 Convex Function

A vector convex optimization has standard form:

Minimize (w.r.t. $\vec{x}$) $\quad f_0\left(\vec{x}\right)$

Subject to $\quad f_i\left(\vec{x}\right) \leq 0, \;\; i = 1, \ldots, m$

$\quad h_j\left(\vec{x}\right) = 0, \;\; j = 1, \ldots, p$

Here $f_0(\vec{x})$ is the convex objective function, $f_i(\vec{x})$ are the convex inequality constraint functions, and $h_j(\vec{x})$ are the equality constraint functions which can be expressed in linear form $A\vec{x}+B$.

2.2 Multi-objective Formulation

The specific formulation for the portfolio optimization problem in Model 5 can be determined by recognizing that the two objectives minimizing portfolio risk $\sigma_p{}^2 = \vec{x}^T V\vec{x}$ and maximizing portfolio expected return $x_p = \vec{p}^T\vec{x}$ are equivalent to minimizing negative portfolio expected return $x_p = \vec{p}^T\vec{x}$ and portfolio risk $\sigma_p{}^2 = \vec{x}^T V\vec{x}$. This gives the new formulation of Model 5:

$$\text{Minimize w.r.t. } \vec{x} \quad \left(f_1(\vec{x}), f_2(\vec{x})\right) = \left(-\vec{p}^T\vec{x}, \vec{x}^T V\vec{x}\right) \qquad \text{(Model 5)}$$

$$\text{Subject to:} \quad \vec{1}^T\vec{x} = 1$$

This multi-objective optimization can be solved using scalarization, a standard technique for finding Pareto optimal points for any vector optimization problem by solving the ordinary scalar optimization [4]. Assign two weighting coefficients $\lambda_1, \lambda_2 > 0$ for objective functions $f_1(\vec{x})$ and $f_2(\vec{x})$ respectively. By varying λ_1 and λ_2, one can obtain different Pareto optimal solutions of the vector optimization problem. Without loss of generality, one can take $\lambda_1 = 1$ and $\lambda_2 = \mu > 0$:

$$\text{Minimize: } -\vec{p}^T\vec{x} + \mu\vec{x}^T V\vec{x} \qquad \text{(Modified Model 5)}$$

[1]

$$\text{Subject to: } \vec{1}^T\vec{x} = 1$$

The weighting coefficient μ represents how much an investor weights risk over expected return. One can consider μ as a risk aversion index that measures the risk tolerance of an investor. A smaller value of μ indicates that the investor is more risk-seeking, and a larger value of μ indicates that the investor is more risk-averse. All Pareto optimal portfolios can be obtained by

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[1] The objective function in Modified Model 5 is convex because V is positive semi-definite. A twice differentiable function f is convex if and only if the second derivative of f is positive semi-definite for all $x \in dom\ f$ [5].

varying μ except for two extreme cases where $\mu \to 0$ and $\mu \to \infty$. As $\mu \to 0$, the variance term $\mu \vec{x}^T V \vec{x} \to 0$ and the objective function is dominated by the expected return term $-\vec{p}^T \vec{x}$. This replicates Model 1 where investors only want to maximize return and disregard risk. In this case, the investor is being extremely risk seeking. The optimal strategy for this extreme case is to concentrate the portfolio entirely on the asset that gives the highest expected return. As $\mu \to \infty$, $\mu \vec{x}^T V \vec{x} \to \infty$. The objective function is dominated by the variance term $\mu \vec{x}^T V \vec{x}$. This replicates Model 2 where the investor only wants to minimize risk without regard to expected return. In this case, the investor is being extremely risk averse. The optimal strategy for such type of investor is to invest all resources on the asset that has the minimal variance. By varying μ, one can generate various optimization models that serve investors of any risk tolerance.

2.3 Solving Multi-objective optimization

The multi-objective optimization can be solved using Lagrangian multiplier:

$$L(\vec{x}) = -\vec{p}^T \vec{x} + \mu \vec{x}^T V \vec{x} + \lambda(\vec{1}^T \vec{x} - 1)$$

Set $\frac{\delta L}{\delta \vec{x}} = 0$, it follows that

$$\vec{x} = \frac{1}{2\mu}(V^{-1})(\vec{p} - \lambda \vec{1}) \tag{1.2}$$

To solve the Lagrangian multiplier λ, substitute equation 1.2 to the constraint $1^T x = 1$:

$$\lambda = \frac{\vec{1}^T V^{-1} \vec{p}}{\vec{1}^T V^{-1} \vec{1}} - \frac{2\mu}{\vec{1}^T V^{-1} \vec{1}} \tag{1.3}$$

Let $a_1 = \vec{1}^T V^{-1} \vec{1}$ and $a_2 = \vec{1}^T V^{-1} \vec{p}$, both of which are scalars, equation 1.3 can be written as:

$$\lambda = \frac{a_2}{a_1} - \frac{2\mu}{a_1}$$

The optimized solution for the portfolio weight vector x is

$$\vec{x}^* = \frac{1}{2\mu} V^{-1} \vec{p} - \frac{V^{-1}}{2\mu}\left(\frac{a_2}{a_1} - \frac{2\mu}{a_1}\right)\vec{1} \tag{1.4}$$

Detailed derivation of the optimal solution is provided in Appendix A.

3 Applications of Multi-objective Portfolio Optimization

The mathematical results from the multi-objective portfolio optimization (1.4) can be applied to portfolios consisting of any number of assets. As a specific example, assume that an investor is interested in owning a portfolio that contains five of his favorite stocks: IBM (IBM), Microsoft (MSFT), Apple (AAPL), Quest Diagnostics (DGX), and Bank of America (BAC). Assuming that the investor is not sophisticated in finance, cases involving of short selling are excluded in this example. The expected return and variance of each stock in the portfolio is calculated based on historical stock price and dividend payment from February 1, 2002 to February 1, 2007. (Appendix C)

Stock	Exp. Return	Variance
IBM	0.400%	0.006461
MSFT	0.513%	0.0039
AAPL	4.085%	0.012678
DGX	1.006%	0.005598361
BAC	1.236%	0.001622897

Table 1 Expected Return and Variances of Selected Stocks

Using Matlab to implement the multi-objective optimization on this portfolio, the investor can see the optimal asset allocation strategy for any value of risk aversion index μ. Figure 1 shows how much the investor should invest in each stock given different values of μ.

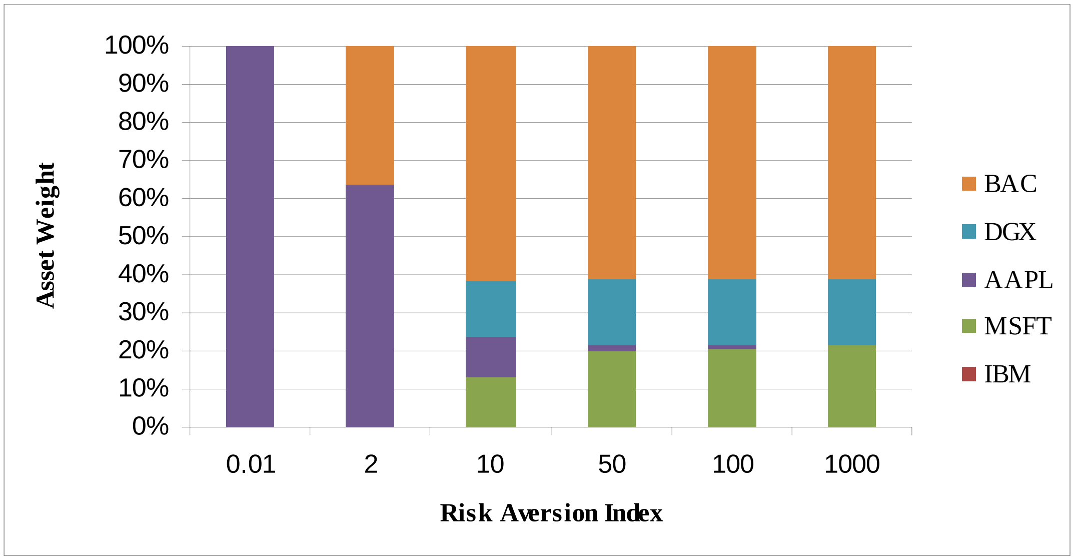

Figure 3 Risk Aversion Index vs. Optimal Asset Allocations

The optimized results of this simple example agree with intuition. An investor with $\mu = 0.01$ is highly risk seeking, and the optimal portfolio for such an investor is to concentrate 100% on the highest expected return stock AAPL. As μ increases, the investor is becoming more sensitive to risk, and the composition of portfolio starts to show a mix of other lower return (lower variance) stocks. When μ equals to 50, the optimal portfolio strategy shows that the investor should invest in a mix of assets; for this example the investor should invest 2.05% of total resources in AAPL stock, 61.22% in BAC stock, 19.77% in MSFT stock, and 16.96% in DGX stock. The allocation on AAPL stock has significantly decreased from 100% to 2.05% as μ increases from 0.01 to 50 because AAPL has the highest return variance. Note that none of the optimal portfolio strategies indicate any asset allocation in IBM stock. That is because IBM gives the lowest return but somewhat high variance compared to other four stocks in the portfolio.

Another important observation from Figure 3 is that there is no significant difference in asset allocation strategy as μ increases from 100 to 1000. The actual data suggests that $1 \le \mu \le 100$ is the meaningful range of risk index μ. Figure 4 illustrates that 1 and 100 are two thresholds of μ that are determinate to the investor's portfolio strategy. When $\mu < 1$, the optimal solutions indicate that the investor should invest all his resources on the highest return stock AAPL. For $1 \le \mu \le 100$, the optimal solutions indicate a variety of asset allocation strategies. As $\mu > 100$, the asset allocation strategy has little change.

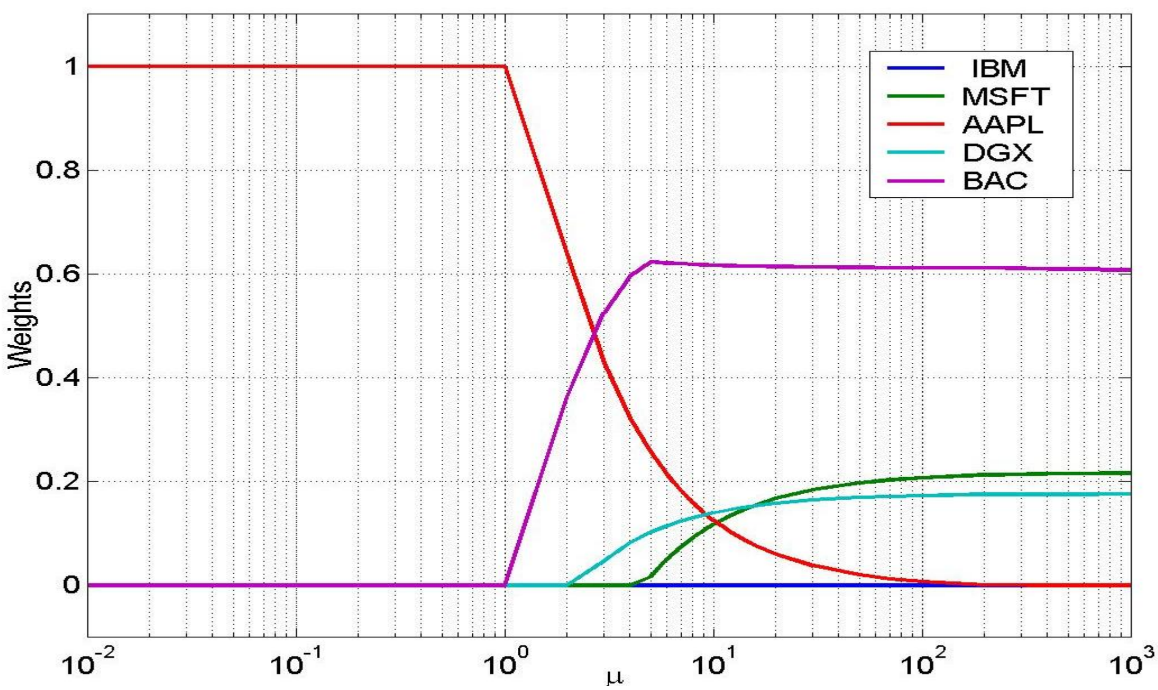

Figure 4 Risk Aversion Index vs. Asset Weight

4 Multi-objective optimization vs. Mean variance optimization

The previous sections have demonstrated the application of multi-objective approach on the portfolio optimization. This section compares the multi-objective approach with the traditional mean variance method. Applying both multi-objective optimization and mean variance optimization to the same portfolio, the numerical experiments generate efficient frontiers that show the set of all possible optimal portfolio points on a risk-return tradeoff curve. Figure 5 shows that efficient frontiers generated by both methods coincide, which indicates that both methods produce exactly the same set of optimal solutions.

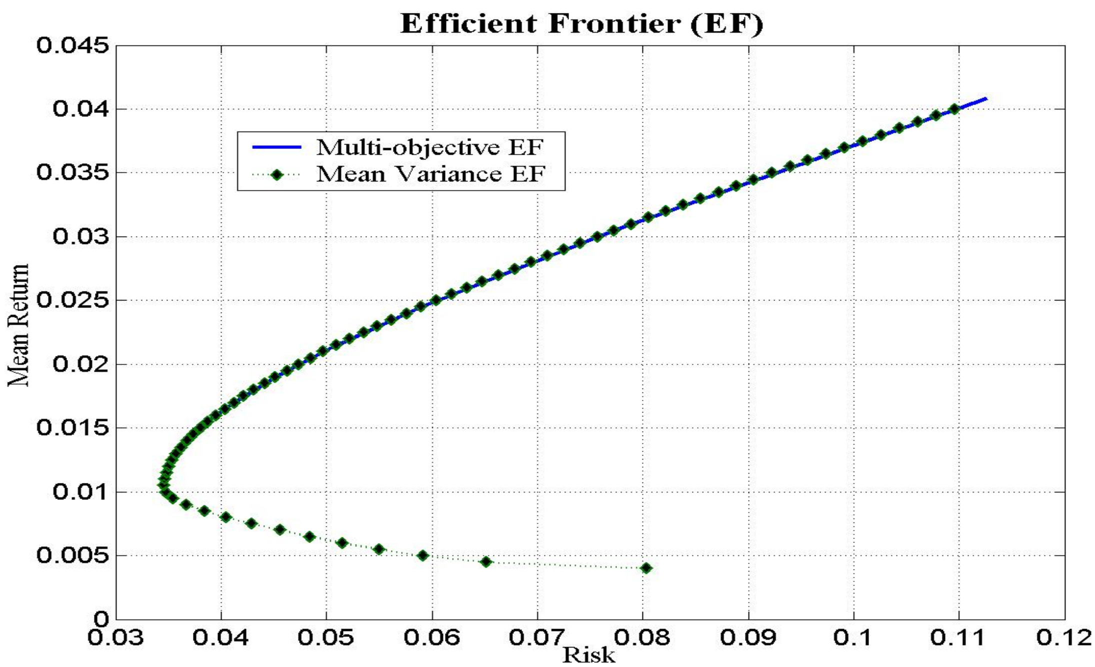

Figure 5 Efficient Frontiers of Multi-objective and Mean variance optimization

From the analytical point of view, one can prove that both methods produce the same optimal solution by rearranging the terms in their corresponding Lagrange multipliers. Examining the Lagrangian multipliers from both formulations, it is not hard to notice that the two Lagrangian multipliers share equivalent form. This is because when taking the first derivative $\frac{\delta L}{\delta \bar{x}}$, the constant terms vanish and the remaining parts can all be rearranged to have equivalent form. Details of proof are provided in Appendix B.

The main difference between the mean variance and multi-objective approach is their problem formulations. The multi-objective approach puts two optimization objectives (minimizing risk and maximizing expected return) into one objective function where as the mean variance approach has only one objective of minimizing risk. The mean variance method places the expected value as a constraint in the formulation, which forces the optimization model to provide the minimal risk for each specified level of expected return.

There are two comparative advantages for the multi-objective formulation over the mean variance formulation. First, since the mean variance approach assumes that the investor's sole objective is to minimize risk, it may not be a good fit for investors who are extremely risk seeking. The multi-objective formulation is applicable for investors of any risk tolerance. Second, the mean variance method requires investors to place an expected value constraint, but there are times when investors do not want to place any constraints on their investment or do not know what kind of return to expect from his investment. The multi-objective optimization provides the entire picture of optimal risk-return trade off.

Another key difference between these two methods lies in their approach to producing efficient frontiers. The efficient frontier of the multi-objective optimization is determined by the risk aversion index μ because different values of μ determine different values of risk and expected return. The efficient frontier of the mean variance method is generated by varying the proportion of two optimal portfolios because the Two Fund Separation Theorem guarantees that any optimized portfolio can be duplicated by a combination of two optimal portfolios [6]. Therefore, in the process of generating the efficient frontier for the mean variance optimization, one needs to use the minimum variance portfolio to replicate a secondary portfolio with the given expected return vector $\bar{p}$. As a result, using the mean variance method to generate the efficient frontier can be numerically more cumbersome than the multi-objective approach.

5 Concluding Remarks

The traditional single-objective approach, such as the mean variance method, solves the problem by having one of the optimization objectives in the objective function and fixes the other objective as a constraint. Consequently, investors have to choose the optimal solution based on given expected return or risk. The multi-objective optimization provides an alternative solution to the portfolio optimization problem, generating the same optimal solution as the mean variance method. It can be applied to investors of any risk tolerance, including those who are extremely risk-seeking and risk-averse. The risk-aversion index measures how much an investor weights risk over expected return. Given any specified value of risk-aversion index, the multi-objective optimization provides investors with optimal asset allocation strategy that can simultaneously maximize expected return and minimize risk.

Appendix A: Derivation of Analytic Solution to Multi-objective optimization

Objective function: Minimize (w.r.t. $\vec{x}$) $-p^T\vec{x}+\mu\vec{x}^T V\vec{x}$

Subject to: $\vec{1}^T\vec{x}=1$

Solution:

Using Lagrangian Multiplier to solve the multi-objective optimization problem:

$$L(\vec{x})=-\vec{p}^T\vec{x}+\mu\vec{x}^T V\vec{x}+\lambda(\vec{1}^T\vec{x}-1)$$

Set $\frac{\delta L}{\delta\vec{x}}=0$. Then we have

$$2\mu V\vec{x}=\vec{p}-\lambda\vec{1}$$

thus $\vec{x}=\frac{1}{2\mu}(V^{-1})(\vec{p}-\lambda\vec{1})$ (1)

To solve the Lagrangian multiplier λ, substitute equation (1) to the constraint $1^T\vec{x}=1$:

$$\vec{1}^T\left[\frac{1}{2\mu}(V^{-1})(\vec{p}-\lambda\vec{1})\right]=1$$

thus $\frac{1}{2\mu}\vec{1}^T V^{-1}\vec{p}-\frac{\lambda}{2\mu}\vec{1}^T V^{-1}\vec{1}=1$

thus $\lambda=\frac{\vec{1}^T V^{-1}\vec{p}}{\vec{1}^T V^{-1}\vec{1}}-\frac{2\mu}{\vec{1}^T V^{-1}\vec{1}}$ (2)

Set $a_1=\vec{1}^T V^{-1}\vec{1}$ and $a_2=\vec{1}^T V^{-1}\vec{p}$. Both a_1 and a_2 are scalars. Substitute a_1 and a_2 into equation (2):

$$\lambda=\frac{a_2}{a_1}-\frac{2\mu}{a_1}$$

The optimized solution for the asset weight vector:

$$\vec{x}^*=\frac{1}{2\mu}V^{-1}\vec{p}-\frac{V^{-1}}{2\mu}(\frac{a_2}{a_1}-\frac{2\mu}{a_1})\vec{1}$$

Appendix B: Proof of Equivalent Analytic Solutions for Multi-objective and Mean variance optimization

Lagriangian Equation for Multi-objective optimization:

$$L(\vec{x})^{Multi-Objective} = -\vec{p}^T \vec{x} + \mu \vec{x}^T V \vec{x} + \lambda(\vec{1}^T \vec{x} - 1)$$

Since μ can be assigned to either $-\vec{p}^T \vec{x}$ or $\vec{x}^T V \vec{x}$,

The Lagrangian Equation for Multi-objective optimization can be rewritten as:

$$L(\vec{x})^{Multi-Objective} = -\mu \vec{p}^T \vec{x} + \vec{x}^T V \vec{x} + \lambda(\vec{1}^T \vec{x} - 1) \qquad (1)$$

The Lagriangian Equation for Mean variance optimization:

$$L(\vec{x})^{Mean\ Variance} = \vec{x}^T V \vec{x} + \lambda_1 (p^* - \vec{p}^T \vec{x}) + \lambda_2 (1 - \vec{1}^T \vec{x}) \qquad (2)$$

Now compare equation (1) and (2), let $\mu = \lambda_1$ and $\lambda = -\lambda_2$,

Solving for $\frac{\delta L(\vec{x})}{\delta \vec{x}}$ =0 for both the Mean variance and Multi-objective Lagrangian Equations:

(Multi-Objective) $2V\vec{x} - \lambda_1 \vec{p} + \lambda_2 \vec{1}^T = 0 \Rightarrow \vec{x}^{*Multi-Objective} = \frac{1}{2} V^{-1} (\lambda_1 \vec{p} - \lambda_2 \vec{1}^T)$

(Mean variance) $2V\vec{x} - \lambda_1 \vec{p} + \lambda_2 \vec{1}^T = 0 \Rightarrow \vec{x}^{*Mean\ Variance} = \frac{1}{2} V^{-1} (\lambda_1 \vec{p} - \lambda_2 \vec{1}^T)$

Therefore, $\vec{x}^{*Multi-Objective} = \vec{x}^{*Mean\ Variance}$

Appendix C: Expected Return of Five Selected Assets

Date	IBM	MSFT	AAPL	DGX	BAC
2/1/2007	-0.152%	-0.972%	-1.155%	1.105%	0.494%
1/3/2007	2.055%	3.349%	1.049%	-0.794%	-1.517%
12/1/2006	5.696%	1.703%	-7.441%	-0.320%	-0.854%
11/1/2006	-0.120%	2.621%	13.049%	6.910%	1.013%
10/2/2006	12.673%	4.952%	5.326%	-18.543%	0.547%
9/1/2006	1.193%	6.443%	13.456%	-4.856%	4.083%
8/1/2006	5.025%	7.200%	-0.162%	6.928%	0.971%
7/3/2006	0.763%	3.241%	18.666%	0.486%	7.135%
6/1/2006	-3.856%	2.890%	-4.183%	7.502%	-0.633%
5/1/2006	-2.611%	-5.860%	-15.087%	0.018%	-2.026%
4/3/2006	-0.160%	-11.223%	12.229%	8.855%	9.608%
3/1/2006	2.780%	1.242%	-8.425%	-2.972%	0.410%
2/1/2006	-1.050%	-4.216%	-9.297%	6.948%	3.680%
1/3/2006	-1.101%	7.642%	5.035%	-3.802%	-4.160%
12/1/2005	-7.535%	-5.533%	6.001%	2.780%	0.568%
11/1/2005	8.835%	8.036%	17.764%	7.237%	6.027%
10/3/2005	2.070%	-0.119%	7.424%	-7.420%	3.908%
9/1/2005	-0.493%	-6.020%	14.331%	1.112%	-2.157%
8/1/2005	-3.170%	7.211%	9.941%	-2.638%	-0.147%
7/1/2005	12.471%	3.122%	15.865%	-3.459%	-4.421%
6/1/2005	-1.785%	-3.718%	-7.420%	1.466%	-0.558%
5/2/2005	-0.818%	2.307%	10.261%	-0.766%	2.847%
4/1/2005	-16.418%	4.659%	-13.463%	0.810%	2.126%
3/1/2005	-1.294%	-3.946%	-7.111%	5.776%	-4.548%
2/1/2005	-0.714%	-3.947%	16.671%	4.300%	0.586%
1/3/2005	-5.235%	-1.653%	19.410%	-0.106%	-1.319%
12/1/2004	4.605%	-0.345%	-3.967%	1.929%	2.564%
11/1/2004	5.212%	6.833%	27.977%	7.078%	3.311%
10/1/2004	4.676%	1.159%	35.191%	-0.600%	3.372%
9/1/2004	1.238%	1.300%	12.348%	3.067%	-2.713%
8/2/2004	-2.532%	-3.908%	6.679%	4.290%	5.821%
7/1/2004	-1.227%	-0.241%	-0.615%	-3.216%	0.472%
6/1/2004	-0.488%	8.884%	15.966%	-1.396%	2.776%
5/3/2004	0.679%	0.395%	8.844%	2.151%	3.285%
4/1/2004	-4.001%	4.788%	-4.660%	2.022%	-0.609%
3/1/2004	-4.825%	-6.015%	13.043%	-0.049%	-0.166%

Date	IBM	MSFT	AAPL	DGX	BAC
1/2/2004	7.062%	1.049%	5.519%	16.517%	1.294%
12/1/2003	2.364%	6.429%	2.297%	0.196%	7.730%
11/3/2003	1.378%	-1.625%	-8.654%	7.867%	-0.393%
10/1/2003	1.302%	-5.440%	10.425%	11.542%	-2.959%
9/2/2003	7.705%	4.787%	-8.400%	1.057%	-0.525%
8/1/2003	1.137%	0.437%	7.306%	0.411%	-4.028%
7/1/2003	-1.522%	3.017%	10.598%	-6.320%	4.502%
6/2/2003	-6.284%	4.174%	6.125%	0.678%	7.410%
5/1/2003	3.882%	-3.747%	26.301%	6.064%	0.189%
4/1/2003	8.246%	5.627%	0.566%	0.103%	10.805%
3/3/2003	0.614%	2.143%	-5.859%	13.111%	-2.548%
2/3/2003	-0.120%	0.195%	4.596%	-1.903%	-1.175%
1/2/2003	0.901%	-8.199%	0.279%	-5.468%	0.710%
12/2/2002	-10.835%	-10.397%	-7.613%	1.981%	0.203%
11/1/2002	10.313%	7.882%	-3.487%	-12.600%	0.374%
10/1/2002	35.368%	22.234%	10.759%	3.759%	9.412%
9/3/2002	-22.639%	-10.854%	-1.762%	9.748%	-8.134%
8/1/2002	7.311%	2.268%	-3.277%	-7.184%	5.366%
7/1/2002	-2.223%	-12.278%	-13.883%	-29.805%	-5.480%
6/3/2002	-10.506%	7.461%	-23.948%	-1.569%	-6.433%
5/1/2002	-3.754%	-2.571%	-4.036%	-4.878%	4.597%
4/1/2002	-19.470%	-13.364%	2.534%	10.919%	6.567%
3/1/2002	5.992%	3.374%	9.124%	16.825%	6.342%
Exp. Returns	**0.400%**	**0.513%**	**4.085%**	**1.006%**	**1.236%**

Appendix D: Variance-Covariance Matrix of Five Selected Assets

	IBM	MSFT	AAPL	DGX	BAC
IBM	0.006461	0.002983	0.00235487	0.00235487	0.00096889
MSFT	0.002983	0.0039	0.00095937	-0.0001987	0.00063459
AAPL	0.002355	0.000959	0.01267778	0.00135712	0.00134481
DGX	0.002355	-0.0002	0.00135712	0.00559836	0.00041942
BAC	0.000969	0.000635	0.00134481	0.00041942	0.0016229

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