Rose-Hulman Undergraduate Mathematics Journal

Volume 8
Issue 1

Article 5

Euclid's Partition Problem and Ceva's Theorem

Max Nosiglia
Willamette University, mnosigli@willamette.edu

Euclid's Partition Problem and Ceva's Theorem

Max Nosiglia, Willamette University

Abstract

Euclid's Partition Problem is the problem of constructing one-nth of a given segment using only a compass and straightedge. There are many well-known constructions that solve this problem, including the standard construction involving parallel lines. This new construction uses Ceva's Theorem and is simpler than many of the other constructions. Furthermore, it easily generalizes to construct $m - n$ths of any given segment using only compass and straightedge.

The classical construction for n-secting a segment involves the construction of parallel lines, which is a tedious construction. We present here a new construction based on Ceva's Theorem (see statement of theorem below). In addition, we construct a segment of length m/n; for example, 3/7 of an arbitrary segment $\overline{AB}$. We will follow the conventions of [Kay]. In particular, the length of segment $\overline{AB}$ is AB, and $A - B - C$ means that A, B, and C are collinear with B between A and C.

Definition 1. A segment from a vertex of a triangle to the opposite side (extended) is called a **cevian**. The **foot** of a cevian is the point of intersection of the cevian with the side of the triangle (extended if necessary).

Example 1. In Figure 1, $\overline{AD}$ and $\overline{BE}$ are cevians of $\triangle ABC$. Note that cevian $\overline{BE}$ is outside of the triangle.

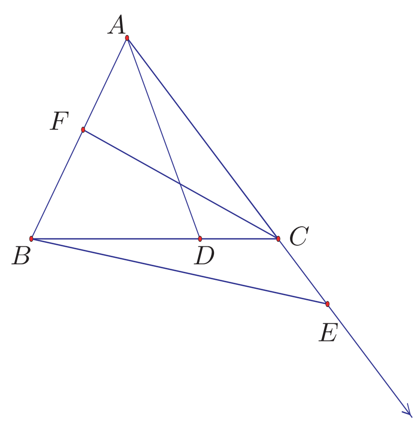

Figure 1: Example of Cevians

Definition 2. Given a $\triangle ABC$, let D, E, and F be feet of the cevians, then the **linearity number** $\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$ is defined as $\dfrac{AF}{FB} \cdot \dfrac{BD}{DC} \cdot \dfrac{CE}{EA}$.

Definition 3. Segments are **concurrent** if they intersect at a common point.

Theorem 1 (Ceva). *The cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$, of $\triangle ABC$ are concurrent iff the linearity number* $\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} = 1$.

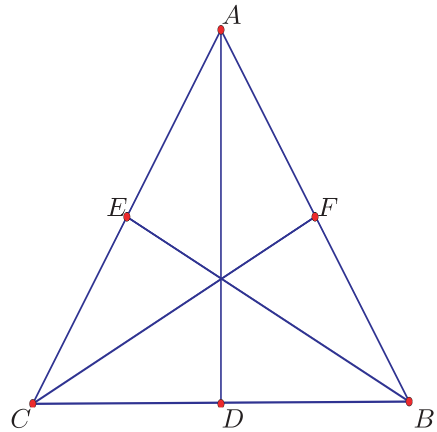

Figure 2: Medians of a Triangle

Example 2. Ceva's Theorem provides a simple proof of the well known fact that the medians of a triangle are concurrent. Let D, E, F be the midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively. Then $\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} = \frac{(1/2)AB}{(1/2)AB} \cdot \frac{(1/2)BC}{(1/2)BC} \cdot \frac{(1/2)CA}{(1/2)CA} = 1$, so the medians are concurrent by Ceva's Theorem.

Consider the following construction: Let $m, n \in \mathbb{Z}^+$ with $m < n$. Given an arbitrary segment $\overline{AB}$ construct ray $\overrightarrow{AX}$ such that $\overrightarrow{AX} \not\parallel \overrightarrow{AB}$ and $\overrightarrow{AX} \neq \overrightarrow{AB}$. On $\overrightarrow{AX}$ construct an arbitrary segment $\overline{AA_1}$. Copy $\overline{AA_1}$ to $\overrightarrow{A_1X}$, creating $\overline{A_1A_2}$ where $A - A_1 - A_2$. Repeat this from the new endpoint A_2, constructing a sequence of points $A - A_1 - A_2 - \cdots - A_n$ such that $A_{i-1}A_i = AA_1$ for each i. Let $E = A_m$ and $C = A_n$.

Construct segment $\overline{BC}$ to form $\triangle ABC$. Bisect segment $\overline{BC}$; label this point D. (See Figure 3)

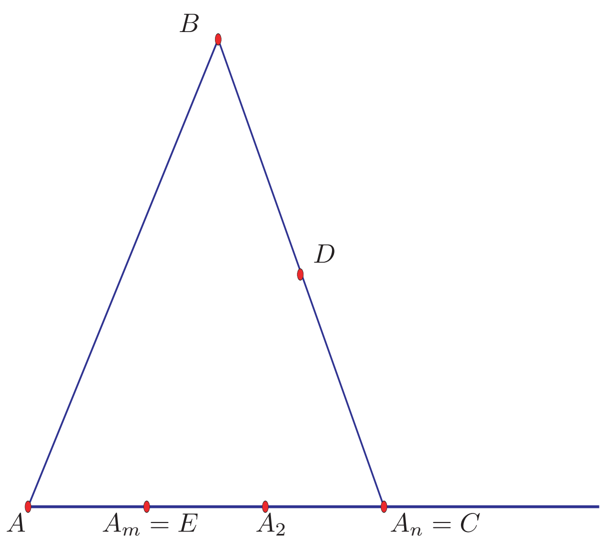

Figure 3: Step 1 of the Construction $m < n$

Then construct cevians $\overline{AD}$ and $\overline{BE}$. Let Y be the point of intersection, and construct the cevian $\overline{CF}$ through Y as in Figure 4.

Theorem 2. *The preceding construction produces segment* $\overline{AF}$ *such that* $AF = \frac{m}{n}AB$.

Proof. By Ceva's Theorem, $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$ because the cevians are concurrent by construction. Thus $\frac{AF}{FB} \cdot \frac{1/2BC}{1/2BC} \cdot \frac{\frac{(n-m)}{n}CA}{\frac{m}{n}CA} = 1$, so $\frac{AF}{FB} = \frac{\frac{m}{n}}{\frac{(n-m)}{n}} = \frac{m}{n-m}$. We have $AF + FB = AB$, so $\frac{AF}{FB} + 1 = \frac{AB}{FB}$, and $\frac{m}{n-m} + \frac{n-m}{n-m} = \frac{n}{n-m} = \frac{AB}{FB}$. Thus $\frac{FB}{AB} = \frac{n-m}{n}$, so $\frac{AF}{AB} = \frac{m}{n}$. Thus $AF = \frac{m}{n}AB$, as desired. □

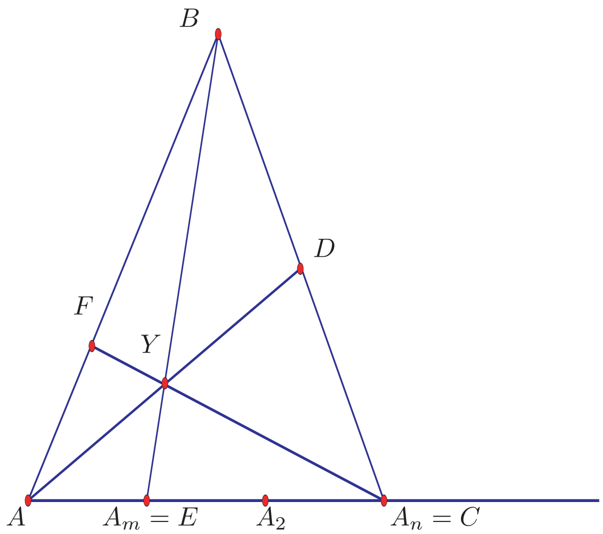

Figure 4: Step 2 of the Construction $m < n$

A similar construction applies when $m > n$: let $m, n \in \mathbb{Z}^+$ with $m > n$. Given an arbitrary segment $\overline{AB}$, construct ray $\overrightarrow{AX}$ such that $\overrightarrow{AX} \nparallel \overrightarrow{AB}$ and $\overrightarrow{AX} \neq \overrightarrow{AB}$. On $\overrightarrow{AX}$ construct an arbitrary segment $\overline{AA_1}$, and construct a sequence of points $A - A_1 - A_2 - \cdots - A_n$ as in the case $m < n$ such that $A_{i-1}A_i = AA_1$ for each i. Let $E = A_m$ and $C = A_n$.

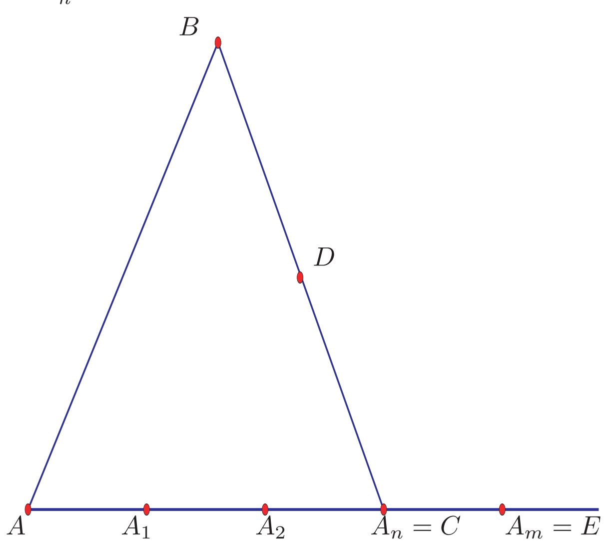

Figure 5: Step 1 of the Construction for $m > n$

Construct segment $\overline{BC}$ to form $\triangle ABC$. Bisect segment $\overline{BC}$; label this point D (See Figure 5). Construct cevians $\overline{AD}$ and $\overline{BE}$, and let Y be the point of intersection. Construct the cevian $\overline{CF}$ through Y. Note that $\overline{AB}$ must be extended (See Figure 6).

Theorem 3. *The preceding construction produces segment* $\overline{AF}$ *such that* $AF = \frac{m}{n}AB$.

Proof. By Ceva's Theorem, $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$ because the cevians are concurrent.

Thus $\frac{AF}{FB} \cdot \frac{1/2BC}{1/2BC} \cdot \frac{\frac{(m-n)}{n}CA}{\frac{m}{n}CA} = 1$, so $\frac{AF}{FB} = \frac{\frac{m}{n}}{\frac{(m-n)}{n}} = \frac{m}{m-n}$. We have $AF - FB = AB$, so $\frac{AF}{FB} - 1 = \frac{AB}{FB}$, and $\frac{m}{m-n} - \frac{m-n}{m-n} = \frac{n}{m-n} = \frac{AB}{FB}$. Thus $\frac{FB}{AB} = \frac{m-n}{n}$, so $\frac{AF}{AB} = \frac{m}{n}$. Thus $AF = \frac{m}{n}AB$, as desired.

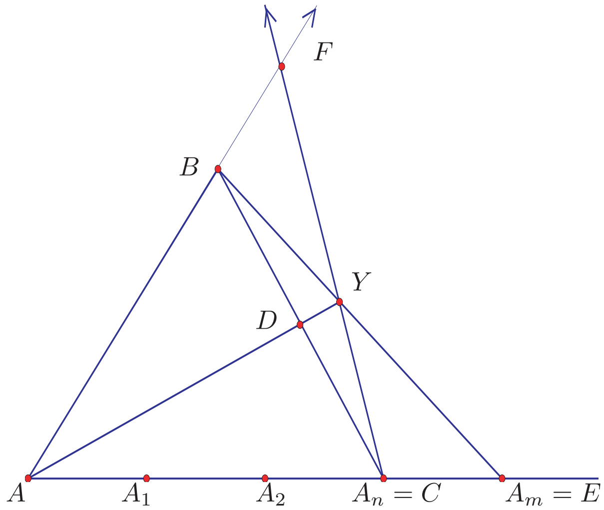

Figure 6: Step 2 of the Construction $m > n$

□

Theorem 4. *Given an arbitrary segment of length a, it is possible to construct a segment of length* $\frac{1}{n}(a)$ *using compass and straightedge.*

Proof. Let $\overline{AB}$ be an arbitrary segment. Using the construction above in Theorem 2 with $m = 1$, we find $AF = \frac{1}{n}AB$. □

References

[Kay] Kay, David C., *College Geometry: A Discovery Approach.* Addison Wesley Longman, Inc., USA 2001.