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A Study of Approximated Solutions of Heat Conduction Problems Using Approximated Eigenfunctions

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Abstract

Let L be the length of a thin rod and $u(x,t)$ be its temperature for $(x,t) \in [0,L] \times [0,\infty)$. We assume that the initial and boundary temperatures of the rod are $f(x)$ and zero respectively. This heat conduction problem is formulated as the following first initial-boundary value problem:

$$u_t = \frac{1}{\sigma(x)} u_{xx} \text{ for } 0 < x < L,\ t > 0,$$

$$u(x,0) = f(x) \text{ for } 0 \le x \le L,\ u(0,t) = 0 = u(L,t) \text{ for } t > 0,$$

where $\sigma(x)$ is a positive function and $f(x)$ is a nonnegative function. The solution $u(x,t)$ of this problem can be expressed as a series using eigenfunctions. This is a standard method and known. However, for any given $\sigma(x)$, the analytic solution of eigenfunctions may not be obtained. In this paper, we explain and demonstrate a method for estimating the eigenfunctions with a non-constant weight function $\sigma(x)$.

1. Introduction

Suppose that a thin rod is placed along the x-axis with $x = 0$ at the left end of the rod and $x = L$ at the right end. The heat conduction problem of the rod is described by (cf. Nagle et al. [4, p. 578]):

$$c\rho \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u(x,t)}{\partial x}\right) + Q(x,t),$$

where c, ρ, K_0, Q, and u are corresponding to specific heat, mass density, thermal conductivity, heat source, and temperature respectively. In the following discussion, we assume that there is no heat source, K_0 is a positive constant, and c and ρ are positive functions of x. Then, the above expression becomes

$$u_t = \frac{1}{\sigma(x)} u_{xx},$$

where $\sigma(x) = c\rho/K_0$ and $1/\sigma(x)$ is called the thermal diffusivity.

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[1] This paper is the research work of the independent study in Spring 2006 semester under the supervision of Prof. W. Y. Chan in Department of Mathematics of Southeast Missouri State University.

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We assume that σ is a positive twice-differentiable function of x. Furthermore, the rod satisfies the homogeneous boundary condition. That is, the temperature of the rod at each end is zero. Also, the rod has an initial temperature $f(x)$ which is a differentiable function. Then, the following first initial-boundary value problem is formulated:

$$u_t = \frac{1}{\sigma} u_{xx} \text{ for } 0 < x < L,\ t > 0, \tag{1}$$

$$u(x,0) = f(x) \text{ for } 0 \leq x \leq L,\ u(0,t) = 0 = u(L,t) \text{ for } t > 0. \tag{2}$$

To solve the problem (1)-(2), the method of separation of variables is used. We assume that $u(x,t) = \phi(x)h(t)$. Substitute this expression into (1); it leads to the following two ordinary differential equations:

$$\frac{dh}{dt} + \lambda h = 0, \tag{3}$$

$$\frac{d^2\phi}{dx^2} + \lambda\sigma\phi = 0,\ \phi(0) = 0 = \phi(L), \tag{4}$$

where λ is a constant. Equation (4) is called Sturm-Liouville eigenvalue problem. Let λ_n and ϕ_n be the corresponding eigenvalues and eigenfunctions of (4) respectively. $\{\phi_n\}$ forms an orthogonal set with the weight function $\sigma(x)$. That is,

$$\int_0^L \sigma(x)\,\phi_n(x)\,\phi_m(x)\,dx \begin{cases} = 0 \text{ if } n \neq m, \\ \neq 0 \text{ if } n = m. \end{cases}$$

To each λ_n, the solution of (3) is $h_n(t) = e^{-\lambda_n t}$. From the result of Haberman [2, p. 142], the solution of the problem (1)-(2) is given by the following infinite series:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t}, \tag{5}$$

where

$$a_n = \frac{\int_0^L \sigma(x)\, f(x)\, \phi_n(x)\, dx}{\int_0^L \sigma(x)\, \phi_n^2(x)\, dx}. \tag{6}$$

However, the analytic solution to $\phi_n(x)$ in (4) may not be obtained for any given $\sigma(x)$. In Section 2, we present a method for approximating $\phi_n(x)$ for a positive twice-differentiable weight function $\sigma(x)$. We use $\hat{\phi}_n(x)$ to denote the approximation of $\phi_n(x)$. In Section 3, we give examples where the solution $u(x,t)$ is approximated with the approximated eigenfunctions $\hat{\phi}_n(x)$.

2. Sturm-Liouville Eigenvalue Problem

The general form of Sturm-Liouville eigenvalue problem (cf. Gustafson [1, p.175]) is given by

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda r\phi = 0$$

where p, q, and r are functions of x, and λ is the eigenvalue. If we let $p = 1$, $q = 0$, and $r = \sigma$, then it gives

$$\frac{d^2\phi}{dx^2} + \lambda\sigma\phi = 0. \tag{7}$$

The Liouville-Green approximation is used to determine the approximated value of $\phi(x)$ when λ is large. The following theorem comes from Olver [5, pp. 190-191]. The book only provides the outline of the proof. We give the detail of the proof.

Theorem 1. *If $\lambda \gg 1$, the approximation for $\phi(x)$ is given by*

$$\phi(x) \approx \tilde{A}\sigma^{-\frac{1}{4}} e^{\int_0^x (-\lambda\sigma)^{1/2} ds} + \tilde{B}\sigma^{-\frac{1}{4}} e^{-\int_0^x (-\lambda\sigma)^{1/2} ds},$$

where $\tilde{A}$ and $\tilde{B}$ are arbitrary constants.

Proof. Let $g(x) = -\lambda\sigma(x)$, then (7) becomes

$$\frac{d^2\phi}{dx^2} = g(x)\,\phi. \tag{8}$$

Let $\xi(x) = \int_0^x g^{1/2}(s)\,ds$ and $\phi(x) = (\xi'(x))^{-1/2} W(\xi)$. Then,

$$\frac{d\xi}{dx} = g^{1/2}$$

and

$$\frac{d\phi}{dx} = \frac{dW}{dx}(\xi')^{\frac{-1}{2}} - \frac{W}{2}(\xi')^{\frac{-3}{2}}\xi''.$$

Also,

$$\begin{aligned}
\frac{d^2\phi}{dx^2} &= \frac{d^2W}{dx^2}(\xi')^{\frac{-1}{2}} - \frac{dW}{dx}(\xi')^{\frac{-3}{2}}\xi'' - \frac{W}{2}\frac{d}{dx}\left[(\xi')^{\frac{-3}{2}}\xi''\right] \\
&= \frac{d^2W}{dx^2}(\xi')^{\frac{-1}{2}} - \frac{dW}{dx}(\xi')^{\frac{-3}{2}}\xi'' - \frac{W}{2}\left[\frac{-3}{2}(\xi')^{\frac{-5}{2}}(\xi'')^2 + (\xi')^{\frac{-3}{2}}\xi'''\right] \\
&= \frac{d}{d\xi}\left(\frac{dW}{d\xi}\frac{d\xi}{dx}\right)\left(\frac{d\xi}{dx}\right)(\xi')^{\frac{-1}{2}} - \frac{dW}{d\xi}\frac{d\xi}{dx}\left[(\xi')^{\frac{-3}{2}}\xi''\right] \\
&+ \frac{3W}{4}(\xi')^{\frac{-5}{2}}(\xi'')^2 - \frac{W}{2}(\xi')^{\frac{-3}{2}}\xi'''.
\end{aligned}$$

By $\frac{d\xi}{dx} = g^{1/2}$, $\frac{d^2\xi}{dx^2} = \frac{1}{2}g^{-1/2}g'$, and $\frac{d^3\xi}{dx^3} = \frac{-1}{4}g^{-3/2}\left(g'\right)^2 + \frac{1}{2}g^{-1/2}g''$, we have

$$\begin{aligned}
\frac{d^2\phi}{dx^2} &= \frac{d}{d\xi}\left(\frac{dW}{d\xi}g^{\frac{1}{2}}\right)g^{\frac{1}{4}} - \frac{dW}{d\xi}g^{-\frac{1}{4}}\left(\frac{1}{2}g^{\frac{-1}{2}}g'\right) + \frac{3W}{4}g^{\frac{-5}{4}}\left[\frac{1}{4}g^{-1}\left(g'\right)^2\right] \\
&- \frac{W}{2}g^{\frac{-3}{4}}\left[\frac{-1}{4}g^{\frac{-3}{2}}\left(g'\right)^2 + \frac{1}{2}g^{\frac{-1}{2}}g''\right] \\
&= \frac{d^2W}{d\xi^2}g^{\frac{3}{4}} + \frac{1}{2}\frac{dW}{d\xi}g^{-\frac{1}{4}}\frac{dg}{d\xi} - \frac{1}{2}g^{\frac{-3}{4}}\frac{dg}{dx}\frac{dW}{d\xi} \\
&- \frac{W}{2}\left[\frac{-3}{8}g^{\frac{-9}{4}}\left(g'\right)^2 - \frac{1}{4}g^{\frac{-9}{4}}\left(g'\right)^2 + \frac{1}{2}g^{\frac{-5}{4}}g''\right] \\
&= \frac{d^2W}{d\xi^2}g^{\frac{3}{4}} + \frac{1}{2}g^{-\frac{1}{4}}\frac{dW}{d\xi}\frac{dg}{d\xi} - \frac{1}{2}g^{\frac{-3}{4}}\frac{dg}{d\xi}\frac{dW}{d\xi}\frac{d\xi}{dx} \\
&- \frac{W}{2}\left[\frac{-5}{8}g^{\frac{-9}{4}}\left(g'\right)^2 + \frac{1}{2}g^{\frac{-5}{4}}g''\right] \\
&= \frac{d^2W}{d\xi^2}g^{\frac{3}{4}} - \frac{W}{2}\left[\frac{-5}{8}g^{\frac{-9}{4}}\left(g'\right)^2 + \frac{1}{2}g^{\frac{-5}{4}}g''\right].
\end{aligned}$$

Then (8) becomes,

$$\frac{d^2W}{d\xi^2}g^{\frac{3}{4}} - \frac{W}{2}\left[\frac{-5}{8}g^{\frac{-9}{4}}\left(g'\right)^2 + \frac{1}{2}g^{\frac{-5}{4}}g''\right] = g^{\frac{3}{4}}W.$$

It is equivalent to

$$\frac{d^2W}{d\xi^2} = W\left[\frac{-5}{16}g^{-3}\left(g'\right)^2 + \frac{1}{4}g^{-2}g'' + 1\right]. \tag{9}$$

Let $\varphi = -g^{-3/4}\frac{d^2}{dx^2}g^{-1/4}$. By $g = -\lambda\sigma$, we have

$$\begin{aligned}
\varphi &= \frac{-5}{16}g^{-3}\left(g'\right)^2 + \frac{1}{4}g^{-2}g'' \\
&= \frac{5\left(\sigma'\right)^2}{16\lambda\sigma^3} - \frac{\sigma''}{4\lambda\sigma^2}.
\end{aligned}$$

Therefore, (9) becomes

$$\frac{d^2W}{d\xi^2} = \left(1+\varphi\right)W. \tag{10}$$

Now, if $\lambda \gg 1$, then φ is negligible. The approximated equation of (10) is

$$\frac{d^2W}{d\xi^2} \approx W.$$

Therefore, the approximated solution to W is

$$W \approx Ae^{\xi} + Be^{-\xi},$$

where A and B are arbitrary constants. By $\xi(x) = \int_0^x g^{1/2}(s)\,ds$ and $\phi(x) = (\xi'(x))^{-1/2} W$, we obtain

$$\phi(x) \approx A g^{-\frac{1}{4}} e^{\int_0^x g^{1/2} ds} + B g^{-\frac{1}{4}} e^{-\int_0^x g^{1/2} ds}.$$

By $g = -\lambda\sigma$, we get

$$\phi(x) \approx \tilde{A}\sigma^{-\frac{1}{4}} e^{\int_0^x (-\lambda\sigma)^{1/2} ds} + \tilde{B}\sigma^{-\frac{1}{4}} e^{-\int_0^x (-\lambda\sigma)^{1/2} ds},$$

where $\tilde{A} = A(-\lambda)^{-1/4}$ and $\tilde{B} = B(-\lambda)^{-1/4}$. The proof is complete. □

From the result of Theorem 1,

$$\phi(x) \approx \tilde{A}\sigma^{-\frac{1}{4}} e^{i\lambda^{1/2}\int_0^x \sigma^{1/2} ds} + \tilde{B}\sigma^{-\frac{1}{4}} e^{-i\lambda^{1/2}\int_0^x \sigma^{1/2} ds}.$$

We use sine and cosine functions instead of exponential function, it leads to

$$\phi(x) \approx C_1\sigma^{-\frac{1}{4}} \cos\left(\lambda^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds\right) + C_2\sigma^{-\frac{1}{4}} \sin\left(\lambda^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds\right),$$

where C_1 and C_2 are arbitrary constants. Substitute $x = 0$ in the above expression and by the boundary condition in (4), $\phi(0) = 0$, it gives

$$C_1\sigma^{-\frac{1}{4}} \approx 0,$$

which implies $C_1 = 0$. Therefore,

$$\phi(x) \approx C_2\sigma^{-\frac{1}{4}} \sin\left(\lambda^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds\right).$$

Substitute $x = L$ in the above expression and by another boundary condition in (4), $\phi(L) = 0$, we have

$$C_2\sigma^{-\frac{1}{4}} \sin\left(\lambda^{\frac{1}{2}} \int_0^L \sigma^{\frac{1}{2}} ds\right) \approx 0.$$

As $C_2 \neq 0$, otherwise $\phi(x) \equiv 0$, we obtain

$$\lambda_n^{\frac{1}{2}} \int_0^L \sigma^{\frac{1}{2}} ds \approx n\pi \text{ for } n = 1, 2, 3....$$

Let

$$\hat{\lambda}_n = \left(\frac{n\pi}{\int_0^L \sigma^{\frac{1}{2}} ds}\right)^2 \tag{11}$$

be the approximation of λ_n. We note that $\left\{\hat{\lambda}_n\right\}_{n=1}^{\infty}$ is an increasing sequence. That is,

$$\hat{\lambda}_1 < \hat{\lambda}_2 < ... < \hat{\lambda}_n < ....$$

Therefore,

$$\phi_n(x) \approx C_{2_n} \sigma^{-\frac{1}{4}} \sin\left(\hat{\lambda}_n^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds\right).$$

We choose C_{2_n} such that $\{\phi_n\}$ forms an orthonormal set, that is,

$$\int_0^L \sigma \phi_n \phi_m dx = \begin{cases} 0 \text{ if } n \neq m, \\ 1 \text{ if } n = m. \end{cases}$$

Thus, if $n = m$, then

$$(C_{2_n})^2 \int_0^L \sigma^{\frac{1}{2}} \sin^2\left(\hat{\lambda}_n^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds\right) dx \approx 1.$$

Let $z = \hat{\lambda}_n^{1/2} \int_0^x \sigma^{1/2} ds$, then $dz = \hat{\lambda}_n^{1/2} \sigma^{1/2} dx$. By (11), we have

$$(C_{2_n})^2 \int_0^{n\pi} \hat{\lambda}_n^{-\frac{1}{2}} \sin^2 z dz \approx 1.$$

It implies

$$\begin{aligned} (C_{2_n})^2 &\approx \frac{\hat{\lambda}_n^{\frac{1}{2}}}{\int_0^{n\pi} \sin^2 z dz} \\ &= \frac{2}{\hat{\lambda}_n^{-\frac{1}{2}} n\pi}. \end{aligned}$$

From (11), we obtain

$$(C_{2_n})^2 \approx \frac{2}{\int_0^L \sigma^{\frac{1}{2}} ds}.$$

This gives our approximated eigenfunction of (7)

$$\hat{\phi}_n(x) = \left(\frac{2}{\int_0^L \sigma^{\frac{1}{2}} ds}\right)^{\frac{1}{2}} \sigma^{-\frac{1}{4}} \sin\left(n\pi \frac{\int_0^x \sigma^{\frac{1}{2}} ds}{\int_0^L \sigma^{\frac{1}{2}} ds}\right). \tag{12}$$

3. Examples

In this section, we let $L = 1$. We want to look at the behavior of the approximated solution to the problem (1)-(2) for some $\sigma(x)$ and $f(x)$. In the first example, we consider $\sigma(x) = 1$ and $f(x) = x(1-x)$. In the second example, we let $\sigma(x) = 1 + 50x$ and $f(x) = x(1-x)$.

Example 1

Let $\sigma(x) = 1$ and $f(x) = x(1-x)$. From (4),

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \ \phi(0) = 0 = \phi(1).$$

From the results of Nagle, Saff, and Snider [4, p. 582], the eigenvalue of the above equation is $\lambda_n = (n\pi)^2$, and the corresponding eigenfunction is

$$\phi_n(x) = \sin(n\pi x). \tag{13}$$

By (6), we obtain

$$a_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx. \tag{14}$$

By (5), the exact solution to problem (1)-(2) is

$$u(x,t) = 2\sum_{n=1}^{\infty}\left[\int_0^1 x(1-x)\sin(n\pi x)\,dx\right]\sin(n\pi x)\,e^{-(n\pi)^2 t}. \tag{15}$$

To obtain the graph of the solution $u(x,t)$, we use the finite sum

$$v(x,t) = 2\sum_{n=1}^{100}\left[\int_0^1 x(1-x)\sin(n\pi x)\,dx\right]\sin(n\pi x)\,e^{-(n\pi)^2 t}$$

to approximate the solution in (15). Use `Maple`®[2] version 9.03 to graph $v(x,t)$. Figures 1, 2, and 3 below show the graphs of $v(x,t)$ at $t = 0$, 0.5, and 2 respectively.

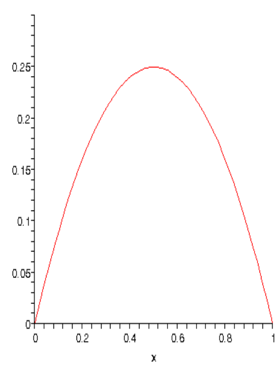

Figure 1: $v(x,0)$

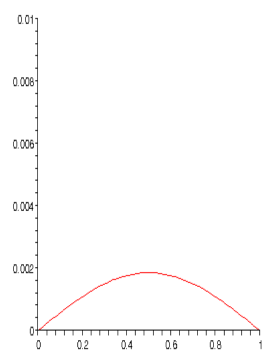

Figure 2: $v(x,0.5)$

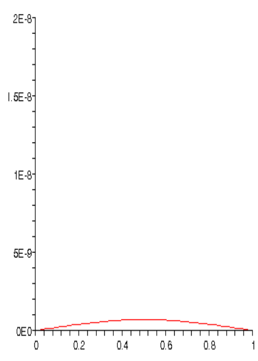

Figure 3: $v(x,2)$

Now, we use (11)

$$\hat{\lambda}_n = \left(\frac{n\pi}{\int_0^1 1ds}\right)^2 = (n\pi)^2,$$

[2] `Maple`® is a registered trademark of Waterloo Maple Inc., Waterloo, Ontario, Canada

which is equal to λ_n. (12) gives the following approximated eigenfunction

$$\hat{\phi}_n(x) = \left(\frac{2}{\int_0^1 1^{\frac{1}{2}} ds}\right)^{\frac{1}{2}} 1^{-\frac{1}{4}} \sin\left(n\pi \frac{\int_0^x 1^{\frac{1}{2}} ds}{\int_0^1 1^{\frac{1}{2}} ds}\right)$$
$$= \sqrt{2}\sin(n\pi x).$$

By (6), the approximated value of a_n is given by

$$\hat{a}_n = \frac{\int_0^1 x(1-x)\sqrt{2}\sin n\pi x dx}{\int_0^1 2\sin^2 n\pi x dx}$$
$$= \frac{\sqrt{2}\int_0^1 x(1-x)\sin n\pi x dx}{\int_0^1 1-\cos 2n\pi x dx}$$
$$= \sqrt{2}\int_0^1 x(1-x)\sin n\pi x dx.$$

It is noticed that $\hat{\phi}_n(x)$ and $\hat{a}_n$ are different from $\phi_n(x)$ and a_n in (13) and (14). If we replace $\phi_n(x)$ and a_n in (5) by $\hat{\phi}_n(x)$ and $\hat{a}_n$. The approximated solution to the problem (1)-(2) is

$$\hat{u}(x,t) = \sum_{n=1}^{\infty}\left(\sqrt{2}\int_0^1 x(1-x)\sin n\pi x dx\right)\sqrt{2}\sin(n\pi x)e^{-n\pi t}.$$

$\hat{u}(x,t)$ is equal to the solution given by (15). The graphs of the finite sum (from $n = 1$ to 100) of the series of $\hat{u}(x,t)$ at $t = 0$, 0.5, and 2 are the same as figures 1, 2, and 3 respectively.

Example 2

Let $\sigma(x) = 1 + 50x$ and $f(x) = x(1-x)$. From (4),

$$\frac{d^2\phi}{dx^2} + \lambda(1+50x)\phi = 0,\ \phi(0) = 0 = \phi(1).$$

Using (11), we obtain

$$\hat{\lambda}_n = \left(\frac{75n\pi}{51^{3/2}-1}\right)^2. \tag{16}$$

According to the book of Haberman [2, p. 182], $\hat{\lambda}_n$ given by (11) is reasonably accurate even when n is not very large. An example in there [2, p. 183] shows that $\hat{\lambda}_n$ is close to λ_n. From (12), the approximated eigenfunction is

$$\hat{\phi}_n(x) = \left(\frac{150}{51^{3/2}-1}\right)^{1/2}(1+50x)^{-\frac{1}{4}}\sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}} ds\right). \tag{17}$$

Let $n = 10$, then

$$\hat{\phi}_{10}(x) = \left(\frac{150}{51^{3/2}-1}\right)^{1/2}(1+50x)^{-\frac{1}{4}}\sin\left(\frac{750\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}} ds\right).$$

The graph of this approximated eigenfunction is shown below.

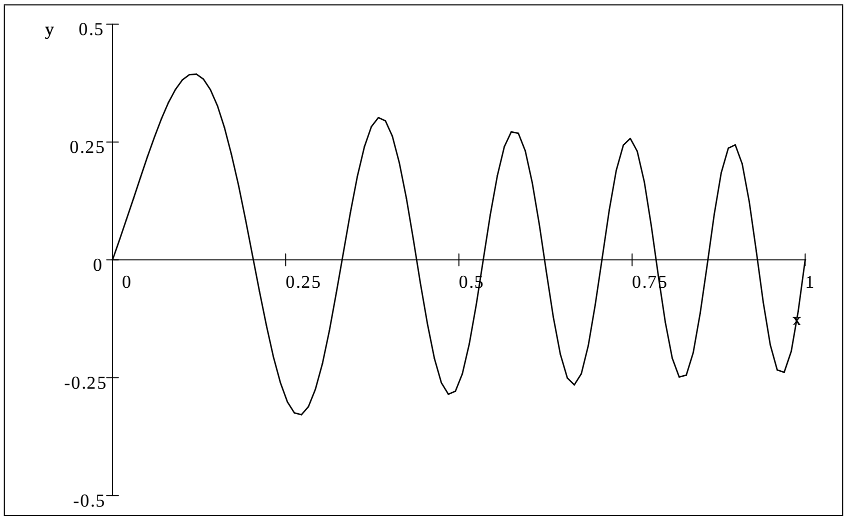

Figure 4

In the above graph, there are exactly 9 roots in the interval $(0,1)$. This matches to the result of the following theorem (cf. Haberman [2, p. 135]).

Theorem 2. *The eigenfunction $\phi_n(x)$ has exactly $n-1$ roots in the interval $0<x<L$.*

Substitute (16) and (17) into (5), we obtain the approximated solution to the problem (1)-(2)

$$u(x,t) \approx \sum_{n=1}^{\infty} a_n \left(\tfrac{150}{51^{3/2}-1}\right)^{1/2} (1+50x)^{-\frac{1}{4}} \sin\left(\tfrac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right) e^{-\left(\frac{75n\pi}{51^{3/2}-1}\right)^2 t}.$$

Rewrite the above expression, it yields

$$u(x,t) \approx \sum_{n=1}^{\infty} \tilde{a}_n (1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right) e^{-\left(\frac{75n\pi}{51^{3/2}-1}\right)^2 t},$$

where $\tilde{a}_n = a_n \left(\frac{150}{51^{3/2}-1}\right)^{1/2}$. From (6), $\tilde{a}_n$ is given by

$$\tilde{a}_n \approx \frac{\int_0^1 (1+50x)\,x(1-x)(1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right)dx}{\int_0^1 (1+50x)\left[(1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right)\right]^2 dx}.$$

Simplify the above expression, we have

$$\tilde{a}_n \approx \frac{\int_0^1 (1+50x)^{\frac{3}{4}}\,x(1-x)\sin\left(\frac{\left[(1+50x)^{3/2}-1\right]n\pi}{51^{3/2}-1}\right)dx}{\int_0^1 (1+50x)^{\frac{1}{2}} \sin^2\left(\frac{\left[(1+50x)^{3/2}-1\right]n\pi}{51^{3/2}-1}\right)dx}.$$

Since the analytic solution of the right-hand side of the above expression cannot be determined, we use numerical method to compute the approximated solution. Adaptive Gaussian Quadrature is adopted to compute the value of the numerator and denominator, this numerical integration method is built in `Maple`®. We use the finite sum

$$w(x,t) = \sum_{n=1}^{100} \tilde{a}_n (1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\, ds\right) e^{-\left(\frac{75n\pi}{51^{3/2}-1}\right)^2 t}$$

to approximate $u(x,t)$. The table below shows that the values of $f(x) = x(1-x)$ and $w(x,0)$ are close to each other at $x = 0,\ 0.1,\ 0.2,\ 0.3, ..., 1$.

x	$f(x)$	$w(x,0)$
0	0	0
0.1	0.09	0.08997841586
0.2	0.16	0.1599906913
0.3	0.21	0.2100016569
0.4	0.24	0.2399981029
0.5	0.25	0.2499991097
0.6	0.24	0.2400001991
0.7	0.21	0.2100001255
0.8	0.16	0.1599993234
0.9	0.09	0.08999867687
1	0	$-9.377258464 \times 10^{-12}$

The graphs of $w(x,t)$ when $t = 0,\ 0.5$, and 2 are as seen in figures 5, 6, and 7.

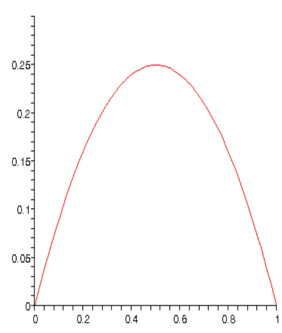

Figure 5: $w(x,0)$

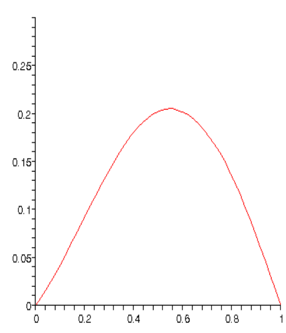

Figure 6: $w(x,0.5)$

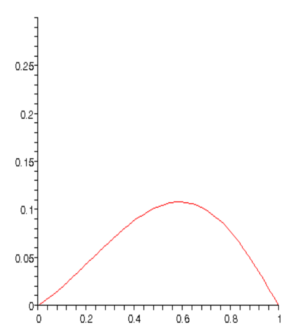

Figure 7: $w(x,2)$

From figures 5, 6, and 7, $w(x,t)$ is a non-increasing function in t for $x \in [0,1]$. Physically, it represents that the rod is cooling. It is due to the fact that the temperature outside the rod is lower than the temperature inside. The heat is leaving from the rod through the endpoints. Similarly, we can explain that

$v(x,t)$ is a non-increasing function in t too. Also, it is noticed that the graphs $w(x,t)$ skew to the right-hand side as t increases. It is because at $x=0$ and $x=1$, the thermal diffusivities of the rod are $1/\sigma(0)=1$ and $1/\sigma(1)=1/51$ respectively. According to the book of Incropera and DeWitt [3, p. 46], if any kinds of material have a larger value of thermal diffusivity, they are more effective in transferring heat by conduction. Since the thermal diffusivity of the rod at $x=1$ is smaller than the one at $x=0$, the heat is more difficult to escape from $x=1$. That is, the rod near $x=1$ keeps the heat for a longer time. Thus, the graphs of $w(x,t)$ skew to the right-hand side. On the other hand, the thermal diffusivity in Example 1 is a constant. The heat transferred at $x=0$ and $x=1$ are the same, and the initial temperature is symmetric with respect to $x=0.5$. Thus, $v(x,t)$ is symmetric with respect to the line $x=0.5$ for $t \geq 0$. In addition, the thermal diffusivity of the rod in Example 2 is smaller than or equal to the one in Example 1 for $x \in [0,1]$. It will take a longer time to withdraw heat from the rod in Example 2. When we compare figures 5, 6, and 7 to figures 1, 2, and 3, it shows that the rate of decrease of $w(x,t)$ is lower than the one of $v(x,t)$.

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