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Isoperimetric Regions in Gauss Sectors

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Isoperimetric Regions in Gauss Sectors

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Abstract

We consider the free boundary isoperimetric problem in sectors of the Gauss plane. The solution is not always a circular arc as in sectors of the Euclidean plane. We prove that the solution is sometimes a ray and we conjecture that the solution is sometimes a ″rounded n-gon″ which we discovered computationally using Mathematica.

1 Introduction

The Gauss plane is the Euclidean plane with Gaussian density $(1/2\pi)e^{-r^2/2}$, used to weight area and length. Gauss space is of much interest to probabilists (see e.g. [LT] or [S]; or [Bo1], [Bo2] for applications to Brownian motion and to stock option pricing). Borell and Sudakov-Tsirel'son proved independently that in the Gauss plane for prescribed area lines minimize perimeter. Carlen and Kerce ([CK]) proved uniqueness. We consider the free boundary isoperimetric problem in α-sectors of the Gauss plane ($0 \leq \theta \leq \alpha$). We prove that solutions are sometimes rays and conjecture that in other cases they are circular arcs or "rounded n-gons," which we discovered computationally (see Definition 3.2 and Figure 1). These new curves are noncircular, nonlinear curves from one boundary of the sector to the other, with constant generalized curvature and strictly increasing (or decreasing) distance from the origin within the sector.

> **Conjecture (see 3.3)**. *The least-perimeter way to enclose a given area in an α-sector of the Gauss plane is either a circular arc, a half-edge of a rounded n-gon (see Figure 1), a ray orthogonal to the boundary, or a ray emanating from the origin. The minimizer depends on the measure of α and on the area enclosed. Circular arcs are minimizing for small areas in small sectors, half-edges of rounded n-gons are minimizing for large areas in small sectors and rays are minimizing for large areas in large sectors.*

We support this conjecture with partial results and computer experiments. In addition, we prove that complete, embedded, noncircular, nonlinear, constant-curvature curves exist in the Gauss plane (Proposition 3.26).

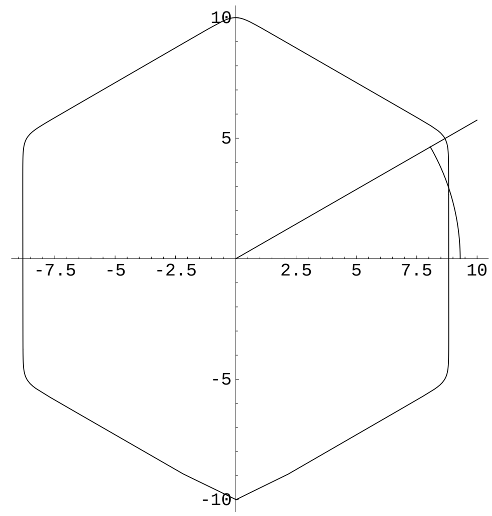

Figure 1: Computationally discovered constant-curvature curve in the Gauss plane, which we conjecture sometimes solves the isoperimetric problem in sectors (beating the circular arc).

Somewhat analogous to our rounded n-gons, unduloids rather than cylinders are sometimes solutions to the isoperimetric problem in high-dimensional slabs of Euclidean space (Pedrosa and Ritoré [PR], [Ros], Thm. 4).

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2 Gauss Space and the Isoperimetric Problem

Definitions 2.1. *Gauss space* G^m is R^m endowed with density

$$e^{\phi} = (2\pi)^{-m/2} e^{-r^2/2},$$

where $\phi = -r^2/2 + c$, used to weight area and length. Specifically, the Gauss plane has density $e^{\phi} = (1/2\pi)e^{-r^2/2}$. In terms of the underlying Euclidean area dA and length dL, the new weighted area and length are given by

$$\begin{aligned} dA_{\phi} &= e^{\phi} dA, \\ dL_{\phi} &= e^{\phi} dL. \end{aligned}$$

The constant $(2\pi)^{-m/2}$ scales the total measure of Gauss space to 1. For example, a circle with radius r has length $re^{-r^2/2}$ and encloses area

$$A = 1 - e^{-r^2/2}.$$

The line $\{x = h\}$ has length $e^{-h^2/2}/\sqrt{2\pi}$ and encloses area

$$A = \frac{1}{\sqrt{2\pi}} \int_h^\infty e^{-t^2/2} dt.$$

An *α-sector* in the Gauss plane is the closed region $0 \leq \theta \leq \alpha$ bounded by two rays from the origin. We allow any positive α, including multiple covers ($\alpha > 2\pi$). Take two identical α-sectors and identify the boundaries so that the vertices are identified. This process yields a *2α-cone*.

Note that Gaussian density is *log-concave*, i.e., that ϕ is concave. (By rotational symmetry, it suffices to check that on horizontal lines, $d^2\phi/dr^2$ is negative.)

For a given area A, an *isoperimetric region* is a region of area A with minimum perimeter.

Remark 2.2. Since enclosing an area A in an α-sector of Gauss space is equivalent to enclosing its complement (of area $\alpha/2\pi - A$), it suffices to consider areas between 0 and $(1/2)(\alpha/2\pi) = \alpha/4\pi$, or areas between $\alpha/4\pi$ and $\alpha/2\pi$, at our convenience.

Remark 2.3. The unique minimizer in the Gauss plane is a line, by Borell [Bo1] or Sudakov-Tsirel'son [ST] and Carlen-Kerce [CK] for uniqueness.

Remark 2.4. In the Gauss line or plane, perimeter-minimizing regions exist by compactness ([M1], 5.5, 9.1) and are regular: locally finite collections of intervals in the Gauss line or smooth regions in the Gauss plane ([M3], Section 3.10). Moreover, in sectors of the Gauss plane, the curves must meet the boundary orthogonally; if $\alpha < \pi$ the curve cannot meet the origin (Proposition 2.6).

Proposition 2.5. *There is a one-to-one correspondence between symmetric minimizers (minimizers with reflectional symmetry) in a 2α-cone and minimizers in an α-sector.*

Proof. Take a symmetric minimizer M in the 2α-cone. Suppose that one of the halves of M is not a minimizer in the α-sector. Then some other region R in the α-sector with the same area has less perimeter. But then R and its reflection will be more efficient than M, a contradiction.

Conversely, take a minimizer R in the α-sector. Suppose that R and its reflection are not minimizing in the 2α-cone. Then some competing region M in the 2α-cone has less perimeter than R and its reflection. There are antipodal rays dividing the total area of M into halves. The cheaper half will have less perimeter than R, contradicting the fact that R is a minimizer. □

Proposition 2.6. *In an α-sector of the Gauss plane, minimizing curves that do not pass through the origin must meet the boundaries orthogonally. If $\alpha < \pi$, a minimizer cannot meet the origin.*

Proof. Let M be a minimizer in an α-sector. By Proposition 2.5, M and its reflection minimize in the 2α-cone. By Remark 2.4, M and its reflection must meet smoothly. Therefore, M must meet the boundary of the α-sector orthogonally. If $\alpha < \pi$ and the curve enters the origin, it cannot be minimizing since it will meet the boundaries at angles less than $\pi/2$. □

Remark 2.7. A limit of minimizing curves is a minimizing curve, essentially because an improvement of the limit could be modified to an improvement of latter terms of the sequence.

Definition 2.8. In R^2 with density e^ψ, the *ψ-curvature* κ_ψ of a curve with unit normal vector $\mathbf{n}$ is defined as

$$\kappa_\psi = \kappa - \frac{\partial\psi}{\partial\mathbf{n}},$$

where κ is the Euclidean curvature of the curve. Proposition 2.11 justifies this definition.

Example 2.9. In the Gauss plane, a circle of radius r with inward normal has constant ϕ-curvature $(1-r^2)/r$.

By Definition 2.8,

$$\kappa_\phi = \kappa - \frac{\partial\phi}{\partial\mathbf{n}} = \frac{1}{r} - r = \frac{1-r^2}{r}.$$

Example 2.10. A ray at a distance h from the origin with downward normal has constant ϕ-curvature $-h$.

By Definition 2.8,

$$\kappa_\phi = \kappa - \frac{\partial\phi}{\partial\mathbf{n}} = 0 - h = -h.$$

Proposition 2.11 (Variation formulae, [B], [Co], [M2])**.** *The first variation $\delta^1(v) = dL_\psi/dt$ of the length of a smooth curve in a smooth Riemannian surface with smooth density e^ψ under a smooth, compactly supported variation with initial velocity v satisfies*

$$\delta^1(v) = \frac{dL_\psi}{dt} = -\int \kappa_\psi v ds_\psi.$$

If κ_ψ is constant then $\kappa_\psi = dL_\psi/dA_\psi$, where dA_ψ denotes the weighted area on the side of the compactly supported normal. It follows that an isoperimetric curve has constant curvature κ_ψ.

The second variation $\delta^2(v) = d^2L_\psi/dt^2$ of a curve Γ in equilibrium in R^n with density e^ψ for a compactly supported normal variation with initial velocity v and $dA_\psi/dt = 0$ satisfies

$$\delta^2 L(v,v) = -\int_\Gamma v(\frac{d^2}{ds^2}v + \kappa^2 v) - \int_\Gamma v\frac{d\psi}{ds}\frac{dv}{ds} + \int_\Gamma v^2\frac{\partial^2\psi}{\partial n^2}$$

where κ is the Euclidean curvature, s is the Euclidean arc length, and integrals are taken with respect to weighted length.

If $\delta^2 L(v,v)$ is nonnegative, the curve is *stable*.

Proposition 2.12 ([RCBM]). *In R^2 endowed with strictly log-concave density, compact minimizing curves are connected.*

Proof. Suppose that a minimizer has two components, Γ_1 and Γ_2. Choose nonzero, constant initial velocities v_1, v_2 on Γ_1 and Γ_2 such that $A' = 0$. By the second variation formula,

$$\delta^2 L(v,v) = -\int_{\Gamma_1} \kappa^2 v_1^2 + \int_{\Gamma_1} v_1^2 \frac{\partial^2 \psi}{\partial n^2} - \int_{\Gamma_2} \kappa^2 v_2^2 + \int_{\Gamma_2} v_2^2 \frac{\partial^2 \psi}{\partial n^2} < 0$$

because ψ is strictly concave, a contradiction. □

Corollary 2.13. *For $\alpha < \pi$, minimizing curves in Gauss α-sectors are connected.*

Proof. By Proposition 2.5 it is sufficient to show that symmetric minimizers in cones are connected under symmetric variations. Suppose that a symmetric minimizer in a 2α-cone has two components, Γ_1 and Γ_2. Since $\alpha < \pi$, neither curve can pass through the vertex by Remark 2.4. Therefore, the second variation formula applies. Choose nonzero, constant initial velocities v_1 and v_2 on Γ_1 and Γ_2 such that $A' = 0$. Although such a variation might not be compactly supported, the second variation formula still holds because Gaussian density goes to 0 exponentially. Since Gaussian density is strictly log-concave, by Proposition 2.12, minimizing curves must be connected. □

3 Isoperimetric curves in sectors of Gauss space

Theorem 3.1 states that every isoperimetric curve in the Gauss halfplane is a ray perpendicular to the boundary. Our main Conjecture 3.3 describes likely minimizers in other sectors of Gauss space, followed by partial results for some cases. Theorem 3.12 shows that in α-sectors, $\alpha < \pi$, minimizers must be monotonic in their distance from the origin. Results 3.18, 3.19, 3.32−3.34 give upper and lower bounds for minimum perimeter. Theorem 3.26 proves the existence of complete, embedded, noncircular, nonlinear, constant-curvature curves in the Gauss plane.

Theorem 3.1. *In the Gauss halfplane, the unique minimizer is a ray perpendicular to the boundary.*

Proof. Suppose that there exists some other smooth curve C enclosing area A with no greater length than the ray perpendicular to the boundary enclosing area A. Then C with its reflection across the boundary encloses an area of $2A$ with no greater length than a line enclosing $2A$, contradicting Remark 2.3. □

This proof holds for all halfspaces of G^m for $m > 1$ but a different proof is necessary in the case of $m = 1$:

Definition 3.2. A *rounded n-gon* is a Euclidean-concave constant-ϕ-curvature curve in the Gauss plane such that the distance to the origin is strictly increasing on a (π/n)-sector between two consecutive rays from the origin perpendicular to the curve, and continuation to successive such sectors is by reflection across the boundary. These curves were discovered computationally using Mathematica (see Section 4). For examples, see Figures 1, 10, 13, 14, and 16.

Conjecture 3.3. *In the Gauss quarterplane, there exists $A_0 \approx .075$ such that the unique minimizer for a given area A is a ray for $0 < A < A_0$. At A_0 a quartercircle is also a minimizer and becomes the unique minimizer for $A_0 < A < 1/8$ (see Figures 2 and 3).*

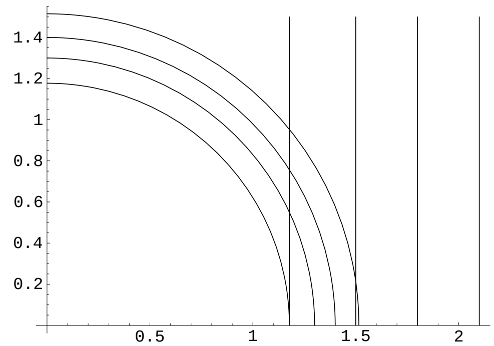

Figure 2: We enclose regions outside circular arcs and to the right of vertical rays. The leftmost circular arc encloses exactly half the total area, and circular arcs are shorter than rays until the rightmost circle, which encloses the same area with the same length as the leftmost ray. For smaller areas, the rays are shorter than the circles. We conjecture that these curves are minimizers.

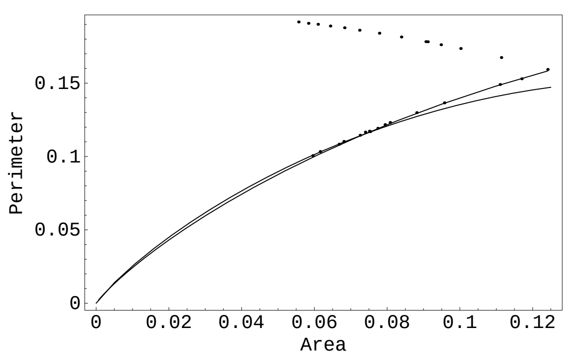

Figure 3: This graph shows the three competitors in the $\pi/2$-sector. The curve that ends up higher represents the circular arc; the other curve represents the ray. The ray is more efficient than the circular arc for areas up to $A_0 \approx 0.075$; the circle is more efficient than the ray for areas between A_0 and 0.125. The points represent bigons that we have found experimentally, which apparently are not minimizing.

For α-sectors of the Gauss plane with $\alpha < \pi/2$, there exists an A_α such that, for given area $\alpha/4\pi \leq A < A_\alpha$, the unique minimizer is a circular arc. At A_α a half-edge of a regular, rounded π/α-gon meeting the boundaries of the sector perpendicularly (see Figure 4) is also a minimizer. For $A_\alpha < A < \alpha/2\pi$, it becomes the unique minimizer.

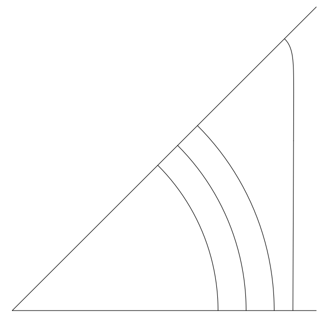

Figure 4: We conjecture that half of an edge of a regular, rounded n-gon is minimizing for large (and complementary, small) areas and that the circular arc is minimizing for intermediate areas.

For α-sectors of the Gauss plane with $\alpha > \pi/2$, there exists $\alpha_0 \approx 0.58\pi$ such that for $\pi/2 < \alpha < \alpha_0$, the unique minimizer for a given area A is a circular arc until the ray becomes more efficient. When $\alpha = \alpha_0$, the circular arc and the ray are both minimizers at a single point enclosing half the area, after which the ray always minimizes (see Figure 5). For $\alpha > \alpha_0$, the ray always minimizes. When $\alpha > \pi$, the minimizer to enclose an area A with $1/4 < A < \alpha/4\pi$ is a ray from the origin. For $A < 1/4$, the minimizer is a ray perpendicular to the boundary of the sector.

Of course the same curves are minimizing for complementary areas.

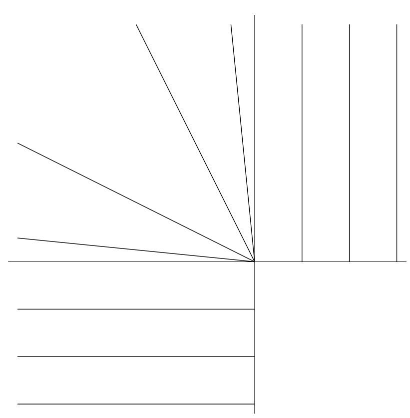

Figure 5: We conjecture that the ray is minimizing for all areas in this 1.5π-sector or in any sector greater than $\alpha_0 \approx 0.58\pi$.

The following results culminate in Theorem 3.12, which shows that in sectors minimizers must be monotonic in their distance from the origin.

Proposition 3.4. *A closed curve cannot be a minimizer in a sector.*

Proof. Suppose C is a closed curve that is minimizing for area A in an α-sector of the Gauss plane. Then C must be smooth (Remark 2.4). Maintaining area and perimeter, we rotate C about the origin until C is tangential to the boundary of the sector, a contradiction of regularity (Proposition 2.6). □

Proposition 3.5. *A curve from infinity to infinity cannot be a minimizer in a sector.*

Proof. Take a curve from infinity to infinity. Rearrange the radial slices of the enclosed region from shortest to longest. The density at the endpoint of each slice remains the same, but the new curve has less tilt and hence less length. □

Lemma 3.6. *A noncircular minimizer in a Gauss sector cannot intersect a circular arc three or more times.*

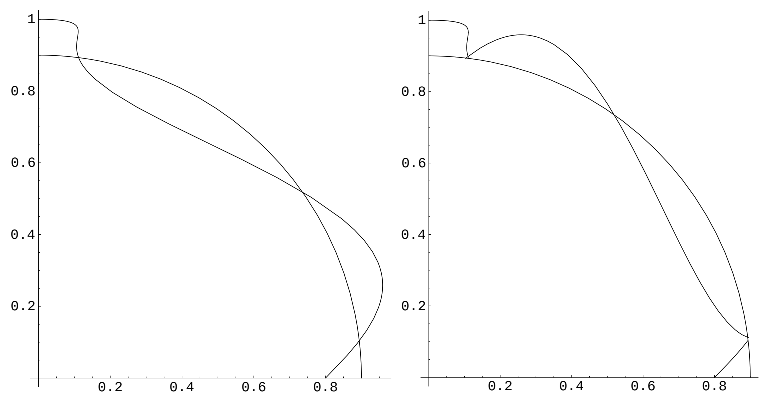

Figure 6: By Lemma 3.6, a circular arc cannot intersect a minimizer three times because flipping the portion of the curve between the two outer intersection points about a ray from the origin maintains area and perimeter, yet creates sharp points on the curve.

Proof. Suppose a smooth, noncircular curve C intersects a circular arc at least three times. Take any three consecutive intersection points. If the curve intersects the circular arc tangentially at the two outer intersection points, then pick a circular arc slightly bigger or smaller such that it still intersects the curve at least three times but not tangentially. Now, take any three consecutive non-tangential intersection points. Then we can rearrange C by flipping the portion of the curve between the two outer intersection points across a ray from the origin through their midpoint to obtain a new curve C' (see Figure 6). This operation maintains both area and length. C', however, has sharp corners, so it cannot be minimizing (Remark 2.4). Therefore C cannot be minimizing. □

Lemma 3.7. *Any constant-curvature curve in a Gauss sector that is perpendicular to a ray from the origin must be symmetric about the ray.*

Proof. Since the density of the Gauss plane is symmetric under reflection across a line through the origin, this result follows from the uniqueness of solutions to differential equations. □

Lemma 3.8. *In a Gauss sector, if a minimizer goes from one boundary to infinity, its distance from the origin must be monotonic.*

Proof. By Lemma 3.6, such a minimizer can intersect a circular arc at most twice. If the curve is not monotonic, then there exists at most one point where the curve's distance from the origin is a strict local minimum. At this point, the curve is orthogonal to a ray from the origin, and therefore by Lemma 7, it must be symmetric about this ray. Reflect the portion of the curve that meets the boundary about this line. If this curve does not hit the other boundary, then at the end of the curve it is again tangential to a circular arc, so reflect again. Repetition of this process yields a curve from one boundary to the other, a contradiction. □

Lemma 3.9. *In a Gauss sector, if a minimizer goes from one boundary to the other, its distance from the origin must be monotonic.*

Proof. By Lemma 3.6, there is at most one strict local extremum. If such a point exists, by Lemma 3.7 the curve must be symmetric about the ray from the origin through this point. At the edges of the sector, there exist slices of the enclosed region with rays through the origin that consist of only one component (see Figure 7). By symmetry, at least some of these slices are repeated. Rearrange all repeated slices at one side of the curve in order of increasing length, thereby decreasing the total tilt of the curve and reducing length, a contradiction. Therefore, there are no strict local extrema and the distance from the origin is monotonic. □

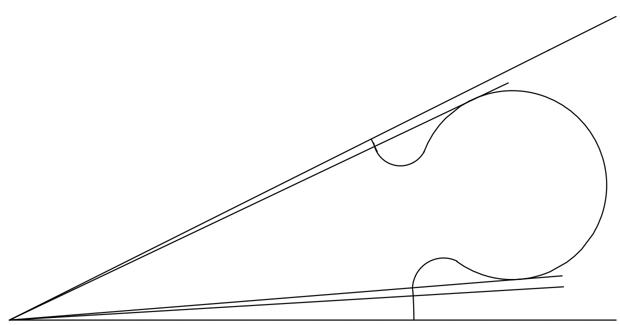

Figure 7: By Lemma 3.9, the sliced portions of this curve are symmetric and can be rearranged in order of length to decrease perimeter.

Lemma 3.10. *In a Gauss sector, if a minimizer begins and ends on the same boundary, then its distance from the origin must be monotonic.*

Proof. By Lemma 3.6, we may assume that the point P farthest or closest to the origin is not an endpoint. Then at P the curve is tangent to a circular arc. So by Lemma 3.7 the curve must symmetric about the ray from the origin through P. Reflect the curve about the ray from the origin through P. If that curve does not hit the other boundary then at the end of that curve it is tangent again to a circular arc. So reflect that new curve again around the ray from the origin to the endpoint. Repetition of this process yields a curve that goes from one boundary to the other, a contradiction. □

Proposition 3.11. *If a minimizer begins and ends on the same boundary, it must be concave.*

Proof. By Lemma 3.10, the distance from the minimizing curve to the origin must be monotonic. Suppose the minimizer from a boundary to itself has a portion of convexity. Then there is a point in the concave region and a point in the convex region that are tangent to rays from the origin (see Figure 8). At these points, $d\phi/d\mathbf{n}$ is zero, so the ϕ-curvature is just the Euclidean curvature. The curvature is positive in the concave region and negative in the convex region, a contradiction since a minimizer must have constant curvature. □

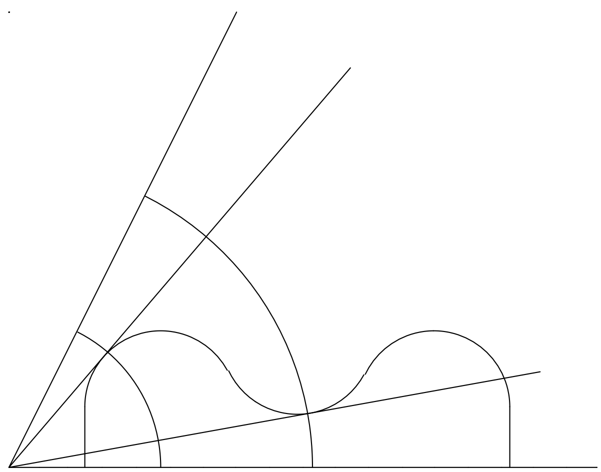

Figure 8: If a minimizer begins and ends on the same boundary, it must be concave, because otherwise at points tangent to rays from the origin the curvature will not be the same.

Theorem 3.12. *In an α-sector of the Gauss plane, $0 < \alpha < \pi$, a minimizer's distance from the origin is monotonic.*

Proof. Since $\alpha < \pi$, a minimizer must be connected by Proposition 2.13. By Propositions 3.4 and 3.5, neither a closed curve nor a curve from infinity to infinity can be minimizing. The three remaining possibilities are curves from a boundary to infinity, a boundary to the other boundary, or a boundary to the same boundary. By Lemmas 3.8, 3.9, and 3.10, in each case the curve must be monotonic in its distance from the origin. □

Remark 3.13. To further support Conjecture 3.3 curves from one boundary to itself and curves from one boundary to infinity, not including the ray, still need to be ruled out (see Figure 9). Then, the only remaining candidates would be the circular arc, the half-edge of a rounded n-gon, and the ray.

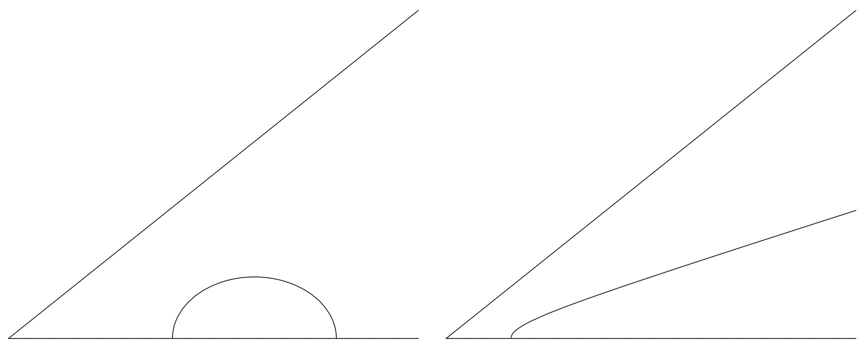

Figure 9: Curves from one boundary to itself and curves from one boundary to infinity still need to be ruled out.

Proposition 3.14. *In an α-sector, with $\pi/2 \leq \alpha < \pi$, a minimizer must be either a ray perpendicular to a boundary or a curve that touches both boundaries.*

Proof. Otherwise the same curve would do as well as the ray in a π-sector, contradicting Theorem 3.1. □

Proposition 3.15. *In an α-sector of the Gauss plane with $\alpha \geq \pi$, the unique minimizer dividing the sector in half is a ray from the origin bisecting the angle α.*

Proof. Suppose there is some other minimizer M dividing the α-sector in half. When shrinking this sector to a π-sector by scaling α and M by a factor of π/α, the ray does not shrink at all. So after shrinking, M has no more length than the ray in a π-sector. This contradicts Theorem 3.1. □

Proposition 3.16. *For area less than or equal to $1/4$, if a ray perpendicular to the boundary is minimizing in both an α-sector and a β-sector, then it is also minimizing in the $(\alpha + \beta)$-sector.*

Proof. Suppose there exists some other minimizer M in an $(\alpha + \beta)$-sector that is not a ray. In the α-sector, and equivalently in the β-sector, we can replace the contained portion of M with a ray enclosing the same area with minimum perimeter. Further replace these two rays with a single ray, which maintains area and strictly reduces perimeter since the total area can be contained in a π-sector, where the ray is known to be minimizing (Theorem 3.1). □

Corollary 3.17. *In all πk-sectors, where k is a positive integer, the length-minimizing curve for areas less than or equal to $1/4$ is a ray perpendicular to the boundary.*

Proof. For the case where $k = 1$, the ray perpendicular to the boundary is length-minimizing by Theorem 3.1. Now assume the ray is minimizing for a πk-sector. Then the $\pi(k+1)$-sector can be divided into a π-sector and a πk-sector. The ray is minimizing in each of these regions. By Proposition 3.16, a ray is the length-minimizing curve. □

Proposition 3.18. *In an α-sector, minimum perimeter is less than 0.1996.*

Proof. A ray from the origin can enclose all possible areas. □

Proposition 3.19. *In an α-sector with $\alpha \geq \pi/2$, an upper bound for minimum perimeter for area less than $1/4$ is the length of a ray perpendicular to the boundary, given by $(1/2\sqrt{2\pi})e^{-h^2/2}$, where h is the distance from the origin to the ray.*

Proof. In an α-sector with boundary, $\alpha \geq \pi/2$, there always exists a ray perpendicular to the boundary enclosing any area less than $1/4$. □

Corollary 3.20. *A whole line is never minimizing in a sector.*

Proof. We may assume $\alpha \geq \pi$.

Case 1: Suppose that the enclosed area is greater than $1/4$. A line enclosing such areas has perimeter greater than or equal to 0.3177, so by Proposition 3.18 it cannot be minimizing.

Case 2: Suppose that the enclosed area is less than or equal to $1/4$. Then the minimum is at most the length of a ray perpendicular to the boundary (Proposition 3.19). This is always less than the length of a line enclosing equal area, since this is the case in the π-sector (Theorem 3.1). □

Proposition 3.21. *Two rays are never minimizing in a sector.*

Proof. For $\alpha < \pi$, minimizers must be connected (Proposition 2.13). For $\alpha = \pi$, minimizers are always single rays perpendicular to the boundary (Theorem 3.1). Thus we need only consider sectors where $\alpha > \pi$.

Case 1: Suppose we have two rays enclosing area A with one of the rays through the origin. Then remove the other ray and change the angle of the ray from the origin until it encloses A on one side. These operations maintain area and reduce perimeter.

Case 2: Suppose there are two rays, each perpendicular to a boundary of the sector, where total area enclosed is less than $1/2$. If total area $A \leq 1/4$, we can enclose the total area with a single ray perpendicular to one of the boundaries, which is shorter by Theorem 3.1. At area $1/4$, this ray is also a ray from the origin. Now take $1/4 < A \leq 1/2$, enclosed outside two rays perpendicular to the boundaries using the shortest combination of two such rays. Move one ray away from the origin until the area enclosed outside is equal to $1/4$, reducing length. But we know that a ray from the origin is shorter than these two rays. Therefore a ray from the origin, which can enclose any area, also is shorter than our two original rays.

Therefore, two rays are never minimizing in any Gauss sector. □

Lemma 3.22. *In the Gauss plane, the length $L(r)$ of a circle of radius r is strictly increasing for $0 < r < 1$ and strictly decreasing for $r > 1$.*

Proof. Since

$$L(r) = re^{-r^2/2},$$

then

$$L'(r) = e^{-r^2/2}(1 - r^2).$$

The result follows. □

Proposition 3.23. *In an α-sector of the Gauss plane, to enclose an area A with $\alpha/4\pi < A < \alpha/2\pi$ it is more efficient to use a larger circular arc enclosing A inside than a smaller circular arc enclosing A outside. For complementary areas, a larger circular arc enclosing A outside is more efficient than a smaller circular arc enclosing A inside.*

Proof. By computation, a circular arc enclosing area $A = \alpha/4\pi$ has radius $r > 1$. So to enclose $\alpha/4\pi < A < \alpha/2\pi$, a larger circular arc with area inside is shorter than a smaller circular arc with area outside by Lemma 3.22. A similar argument shows the result for complementary areas. □

Lemma 3.24. *If a circular arc is minimizing for a particular α-sector, it is minimizing for all smaller sectors.*

Proof. Suppose that the minimizer in an α_0-sector is a circular arc and that some curve C encloses an area A in an α-sector with less perimeter than a circular arc enclosing area A. When we stretch the sector out to an angle of α_0, the circular arc gains more perimeter than C because all of its length is in the direction of the stretching, so the stretched circular arc (which is still a circular arc) is longer than the stretched curve C. This contradicts the hypothesis that the circular arc is minimizing in the α_0-sector. □

Proposition 3.25. *In an α-sector of the Gauss plane, a circular arc is unstable if and only if the radius $r > \sqrt{(\pi/\alpha)^2 - 1}$. Consequently, for small or large areas, minimizers are not circular.*

Proof. The second variation formula applies to curves that do not pass through the origin in the α-sector because it applies to curves that do not pass through the vertex in the 2α-cone. For a circular arc centered at the origin with initial velocity $v(\theta)$,

$$\begin{aligned}\delta^2 L(v,v) &= -\int_\Gamma v(v'' + \frac{1}{r^2}v) - \int_\Gamma v^2 \\ &= -\int_\Gamma v(v'' + v(1+r^2)).\end{aligned}$$

In the α-sector any $v(\theta)$ has a Fourier cosine series $A_0 + \sum_{n=1}^{\infty} A_n cos(n\pi\theta/\alpha)$.

$$\begin{aligned}A'(0) &= -\int_\Gamma v(\theta) \\ &= -(1-e^{-r^2/2})(A_0\alpha + \sum_{n=1}^{\infty} A_n(sin(n\pi) - sin(0))) \\ &= -(1-e^{-r^2/2})(A_0\alpha).\end{aligned}$$

The condition that $A'(0) = 0$ means exactly that $A_0 = 0$. So $v(\theta) = \sum_{n=1}^{\infty} A_n cos(n\pi\theta/\alpha)$. Then

$$\begin{aligned}\delta^2 L(v,v) &= -\int_\Gamma \sum_{n=1}^{\infty} A_n cos(\frac{n\pi\theta}{\alpha})[-\sum_{n=1}^{\infty}(\frac{n\pi}{\alpha})^2 A_n cos(\frac{n\pi\theta}{\alpha}) + (1+r)\sum_{n=1}^{\infty} A_n cos(\frac{n\pi\theta}{\alpha})] \\ &= -\sum_{n=1}^{\infty} A_n^2 \int_\Gamma cos^2(\frac{n\pi\theta}{\alpha})(-(\frac{n\pi}{\alpha})^2 + 1 + r^2).\end{aligned}$$

(The cross terms cancel because the $cos(n\pi\theta/\alpha)$ terms are orthogonal.) Clearly if $1 + r^2 < (\pi/\alpha)^2$, $\delta^2 L(v,v) \geq 0$. On the other hand, if $1 + r^2 > (\pi/\alpha)^2$, $\delta^2 L(v,v) < 0$ when $A_1 = 1$ and $A_i = 0$, $i \neq 1$, proving the instability condition.

To prove the final statement, note that to enclose large area, by Proposition 3.23, it is more efficient to enclose area inside a circular arc, and large circular arcs are unstable. □

Proposition 3.26. *There exist complete, embedded, noncircular, nonlinear, constant-curvature curves in the Gauss plane.*

Proof. Consider a π/n sector with integer $n \geq 3$. By Remark 2.4, a minimizer for small areas is a smooth, constant-curvature, noncircular, connected curve which does not pass through the origin. If it goes from a boundary to itself, the curve with its reflection is a complete, embedded, noncircular, nonlinear, constant-curvature curve. (It is smooth at the points of reflection by Proposition 2.6). If it goes from one boundary to another, its repeated reflection yields the desired curve. □

Remark 3.27. It is interesting to note that Ritoré has proven that the circles of revolution are the only simple closed curves of constant geodesic curvature in a Riemannian surface of revolution of strictly decreasing Gauss curvature ([MHH] Section 3.10, [Rit]). In the Gauss plane, the appropriate analog of Gauss curvature is constant (see [M2]).

Lemma 3.28. *For a smooth, closed curve enclosing the origin, there are two critical points for distance from the origin not on the same line through the origin.*

Proof. Suppose there is a smooth, closed curve enclosing the origin with all the critical points for distance from the origin on the same line through the origin. Since the curve encloses the origin, the critical point furthest from the origin and the critical point closest to the origin must lie on opposite sides of the origin. Start at one of these critical points and move along the curve. Near the critical point the angles between the ray and the curve are no longer equal; one of them is less than $\pi/2$ and the other is greater than $\pi/2$. Take the angle less than $\pi/2$ and continue travelling along the curve with rays from the origin to the curve. Near the other critical point this angle will be greater than $\pi/2$. So by the Intermediate Value Theorem, there is a ray from the origin that meets the curve perpendicularly at some point in between, a contradiction. $\square$

Proposition 3.29. *Any closed, constant-curvature curve that encloses the origin has center of mass at the origin.*

Proof. By Lemma 3.28, there are two lines from the origin that meet the curve perpendicularly. By Lemma 3.7, the curve is symmetric under reflection about these two lines. Therefore the center of mass must be at the point where the two lines meet, the origin. $\square$

Conjecture 3.30. *Every compact constant-curvature curve encloses the origin.*

Proposition 3.31. *Let C be a closed curve of constant curvature κ_ϕ in the Gauss plane. Then the centers of mass $\bar{m}_C$ and $\bar{m}_R$ of the curve and the enclosed region satisfy*

$$\bar{m}_C = \kappa_\phi \bar{m}_R.$$

In particular if C is a geodesic, then $\bar{m}_C = 0$.

Proof. Under constant initial velocity in the horizontal direction, the first variations of length and area satisfy

$$\delta^1(L) = \int_C x$$
$$\delta^1(A) = \int_A x.$$

Since curvature is dL/dA and the center of mass is the average of the position vector, $\bar{m}_C = \kappa_\phi \bar{m}_R$. $\square$

Proposition 3.32. *Let $P_\alpha(A)$ denote the minimum perimeter in an α-sector. For $k \geq 1$,*

$$P_{k\alpha}(kA) \leq kP_\alpha(A)$$

Proof. Stretching an α-sector to a $k\alpha$-sector stretches area by k and length by at most k. $\square$

Corollary 3.33. *Any region of area A in an α-sector, where $\alpha < \pi$, has perimeter*

$$P \geq \frac{\alpha}{(2\pi)^{\frac{3}{2}}} e^{-h^2/2},$$

where h is the signed Euclidean distance from the origin of a ray perpendicular to the boundary enclosing area $A' = (\pi/\alpha)A$, which satisfies

$$A' = \frac{1}{2\sqrt{2\pi}} \int_h^\infty e^{-t^2/2} dt.$$

Proof. This follows immediately from Theorem 3.1 and Proposition 3.32. □

Proposition 3.34. *Let $P_\alpha(A)$ denote minimum perimeter in an α-sector. Suppose $P_\alpha(A)$ is concave. Then for any positive integer k, $P_{k\alpha}(A) \geq P_\alpha(A)$.*

Proof. Divide the $k\alpha$-sector enclosing area A into k α-sectors. In each α-sector, the curve encloses an area $A_1, A_2, ..., A_k$ such that $A_1 + A_2 + ... + A_k = A$, with total perimeter $P_{k\alpha}(A)$. So for some $A_1 + A_2 + ... + A_k = A$, $P_{k\alpha}(A) \geq P_\alpha(A_1) + P_\alpha(A_2) + ... + P_\alpha(A_k)$. Furthermore, $P_\alpha(A_1) + P_\alpha(A_2) + ... + P_\alpha(A_k) \geq P_\alpha(A)$ by concavity. □

We conjecture that the above result will hold for any real $k \geq 1$.

Corollary 3.35. *If a ray is minimizing in an α-sector for up to half the total area, it is minimizing in the 2α-sector.*

Proof. The ϕ-curvature dL/dA of a ray in the Gauss plane is $-h$, where h is the distance from the origin (Example 2.10). Since we enclose area outside the ray, as the enclosed area increases, dL/dA is negative and increasing, so $P_\alpha(A)$ is concave. The corollary now follows from Proposition 3.34. □

4 Experimental results

In the Gauss plane, a curve bounding an isoperimetric region must have constant ϕ-curvature by Proposition 2.11. Therefore when we look for isoperimetric curves in the Gauss plane and sectors thereof, we restrict our search to curves with constant ϕ-curvature.

In the Euclidean plane, the circle and the line are the only curves with constant curvature. In the Gauss plane, the circle and the line still have constant generalized curvature as in Examples 2.9 and 2.10, but there are also many other constant-curvature curves. We conjecture (Conjecture 3.3) that for some area fractions in some sectors of the Gauss plane, some of these other curves are minimizing. Supporting evidence comes from the fact that in a sector of the Gauss plane less than $\pi/2$, neither the (unstable) circular arc nor the (nonexistent) ray can be a minimizer for large enclosed areas (Proposition 3.25).

Additionally, we conjecture (Conjecture 3.3) that a ray perpendicular to the boundary is uniquely minimizing for some areas in the $\pi/2$-sector. The ray does not exist for sectors smaller than $\pi/2$, and minimizers in the $(\pi/2-\epsilon)$-sectors must approach a minimizer in the $\pi/2$-sector. Thus, in sectors less than $\pi/2$ there must be (nonlinear) minimizers approaching the ray.

We use the generalized curvature equation in Definition 2.8 to find a differential equation for constant-curvature curves in the Gauss plane. We set the curvature equal to a constant κ_ϕ to find concave constant-curvature curves, which are the most likely candidates for minimizers, since the circle and the line are both concave. We plot solutions to this differential equation.

Proposition 4.1. *All concave constant-curvature curves in Gauss space leaving the y-axis perpendicularly with downward unit normal satisfy the following differential equation:*

$$\sqrt{x''(s)^2+y''(s)^2}+x'(s)y(s)-x(s)y'(s)=\kappa_\phi \tag{1}$$

where s is the arc length, with the constraints

$$\begin{aligned} x'(s)^2+y'(s)^2 &= 1\\ x(0) &= 0\\ x'(0) &= 1\\ y'(0) &= 0\\ x''(0) &= 0\\ y''(0) &= -y(0)-\kappa_\phi \leq 0. \end{aligned}$$

Proof. Equation 1 follows from the definition of ϕ-curvature (Definition 2.8), and the fact that the Euclidean curvature squared equals $x''(s)^2+y''(s)^2$. Note that Euclidean curvature is positive because the curve is concave with downward unit normal. The constraints specify that the parameterization is by arc length, that the curve leaves the y-axis orthogonally, and that the equation holds for the inital condition $s=0$. □

We arbitrarily choose values for $y(0)$ and κ_ϕ to generate constant-curvature curves. Using Mathematica, we plot solutions to the differential equation. Examples appear in Figures 1, 10, 13, 14, and 16.

The constant-curvature curves of most interest to us resemble polygons with rounded vertices. Those that have two edges (Figure 10) will intersect the boundaries of the quarterplane perpendicularly. These so-called "bigons" incorporate features of both the circle and the ray. Although it might seem plausible that one quarter of a bigon centered at the origin would be more efficient than the ray to enclose large areas in the quarterplane, experimentally it appears that the taller a bigon becomes, the thinner it becomes, so that the area enclosed outside the bigon is never small, as necessary for a minimizer.

Examples 4.2. We computed with Mathematica that one quarter of a bigon beginning at a height of 1 with $\kappa_\phi=-0.91$, enclosing an area inside of approximately 0.16187, has a

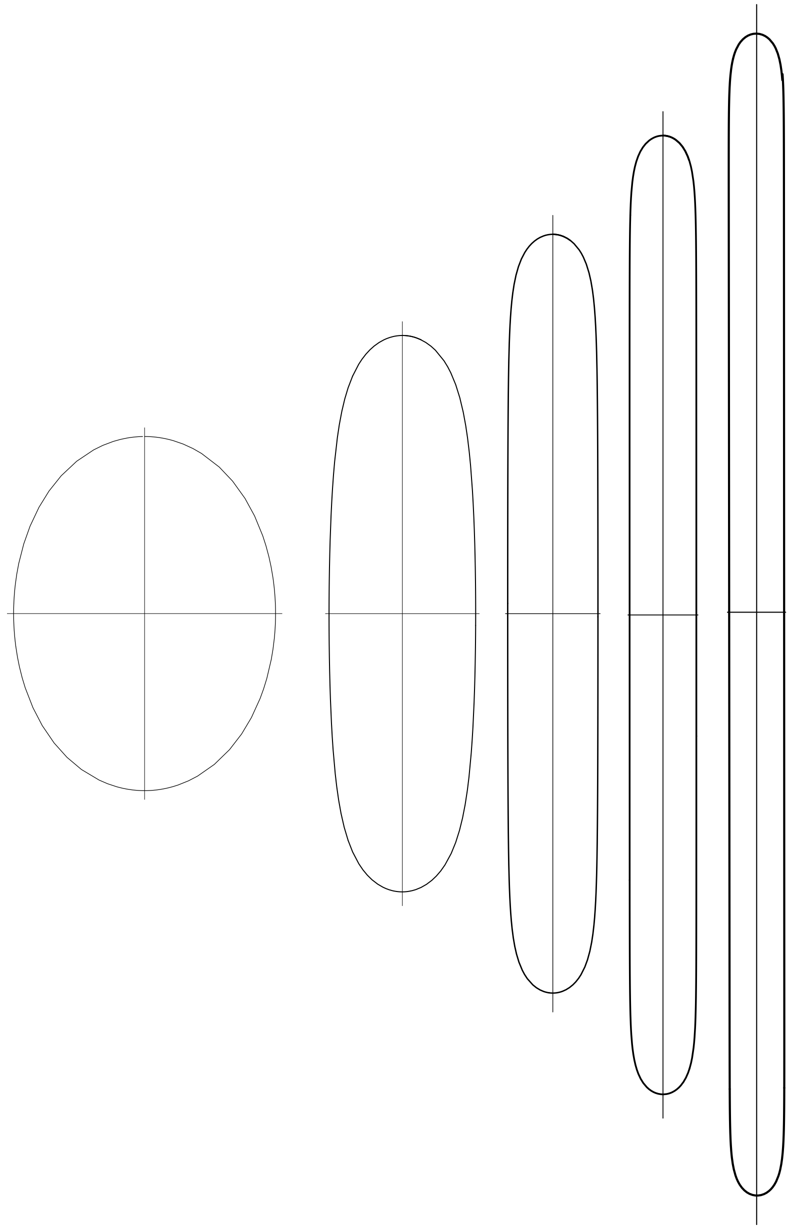

Figure 10: Examples of constant-curvature bigon curves at heights 2, 3, 4, 5, and 6.

perimeter of approximately 0.1299, and a circular arc enclosing the same area has a perimeter of approximately 0.1273. Thus, in this case, the circular arc beats the bigon.

One quarter of a bigon beginning at a height of 2 with $\kappa_\phi = -1.126$, enclosing an area inside of approximately 0.1939, has a perimeter of approximately 0.1030, and a ray enclosing the same area has a perimeter of approximately 0.0954. Thus, in this case, the ray beats the bigon.

One quarter of a bigon beginning at a height of 1.16 with $\kappa_\phi = -1.0047$, enclosing an area inside of approximately 0.175, has a perimeter of approximately 0.1173. For this area the ray and the circular arc have the same perimeter, approximately 0.1164. Thus, both the circular arc and ray beat the bigon at the point where they tie.

Conjecture 4.3. *In the Gauss plane, for area $0 < A < A_2 = 1 - e^{-3/2} \approx .777$, there is an unstable bigon enclosing area A, unique up to rotation. As A approaches A_2, the bigons approach a circle. For $A > A_2$, there are no bigons.*

A similar result holds for $1 < n < 2$, with $A_n = 1 - e^{-(n^2-1)/2}$. For $n \leq 1$, there are no rounded n-gons.

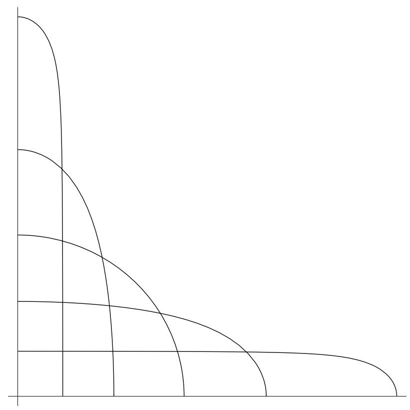

Figure 11: We conjecture (Conjecture 4.3) that as the circular arc becomes unstable, it bifurcates into two equivalent bigons near the circular arc. As the bigon becomes less circular, the area enclosed decreases.

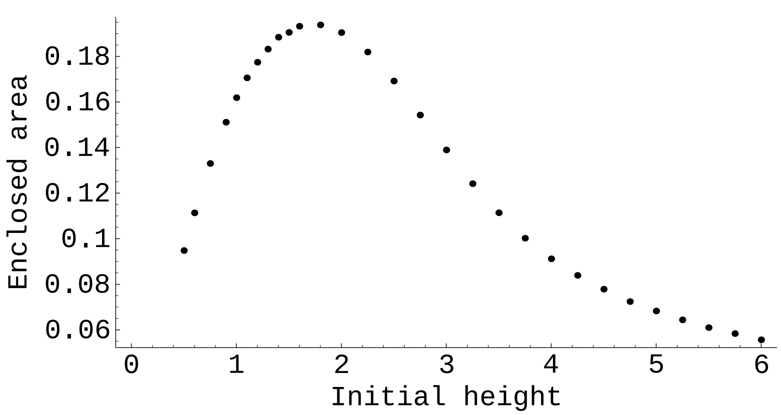

Figure 12: This graph depicts the relationship between the initial height of a bigon and the area enclosed by one quarter of the bigon in the quarterplane. At the maximum, the bigons are circular.

In the $\pi/2$-sector each bigon that we have found experimentally (Figure 10) corresponds to another bigon reflected over the line $y = x$ that encloses the same area (see Figure 11). These correspond to the two bigons of Figure 12 of the same area but different initial heights. The critical area A_0 is the area of the last circular arc (of radius $\sqrt{3}$) stable in the $\pi/2$-sector (Proposition 3.25).

Conjecture 4.4. *In the Gauss plane, for $n > 2$, for area $a_n < A < A_n = 1 - e^{-(n^2-1)/2}$, there are up to rotation precisely two rounded n-gons, one stable and one unstable. As A approaches A_n, the unstable n-gons approach a circle. For $A > A_n$, there remains just the stable n-gon. As A approaches a_n, the stable and unstable rounded n-gons coalesce. For $A < a_n$, there are no rounded n-gons.*

The stable rounded n-gons for $n > 2$ play the role of the rays for $n \leq 2$. The critical area A_n is the area of the last circle stable in the π/n-sector.

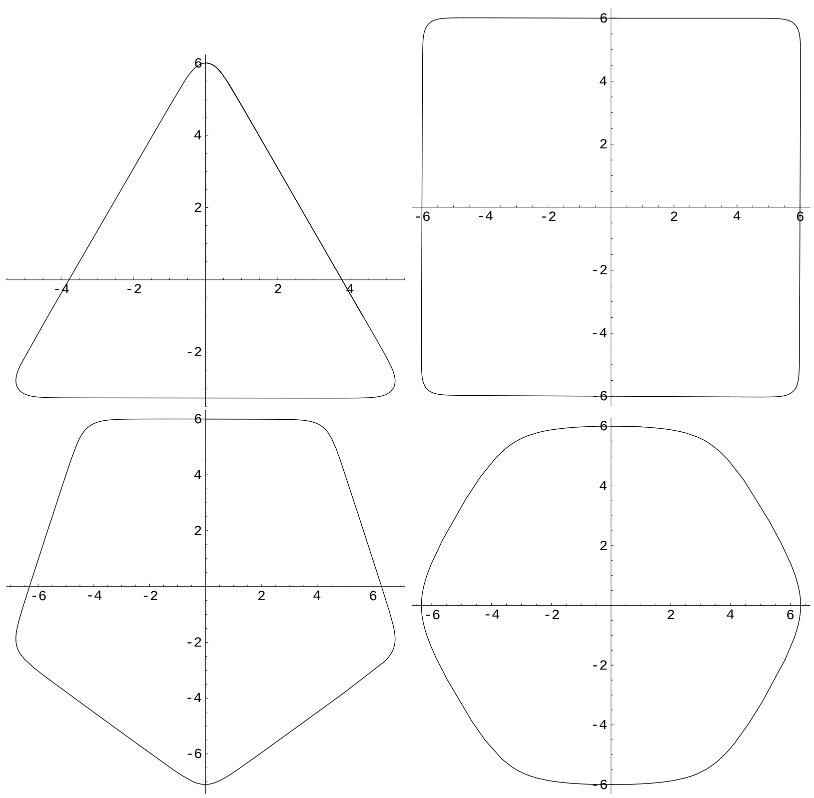

Figure 13: Examples of constant-curvature curves beginning at a height of 6.

We found many other curves approximating polygons with rounded vertices that have constant curvature in the Gauss plane (Figure 13). These curves, including the bigons, are not minimizing and are presumably unstable in the whole Gauss plane, but half of a side of one of these so-called n-gons (Definition 3.2) may be stable in a (π/n)-sector. Furthermore, this half-edge is an approximation of a ray, which we know to be minimizing in some sectors, specifically in the π-sector, and thus we conjecture that the half-edge is minimizing for some areas in α-sectors for $\alpha < \pi$. This conjecture is similar to our conjecture for the $\pi/2$-sector (Conjecture 3.3), and limits to it.

We also observe that we have found many n-gons for small values of n if the starting height is small (Figure 13), and we have found many n-gons with large values of n when the starting height is large (Figure 14). However, we are unable to find bigons for starting heights over 7, and we cannot find hexagons for starting heights below 6. Still, Table 2, shows that n-gons with large n can be found at a height of 6. Based on this data, the behaviors of n-gons are still not completely understood.

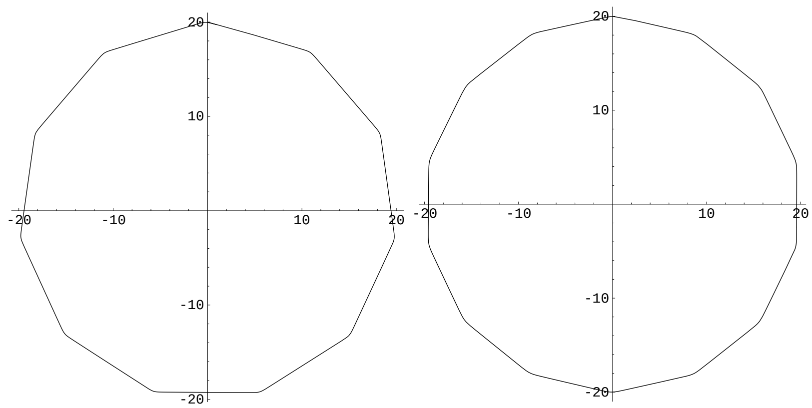

Figure 14: Examples of rounded polygons with 11 and 14 sides beginning at a height of 20.

Of course, not every rounded n-gon has an integer number of sides, or even a rational number of sides (Figures 15 and 16).

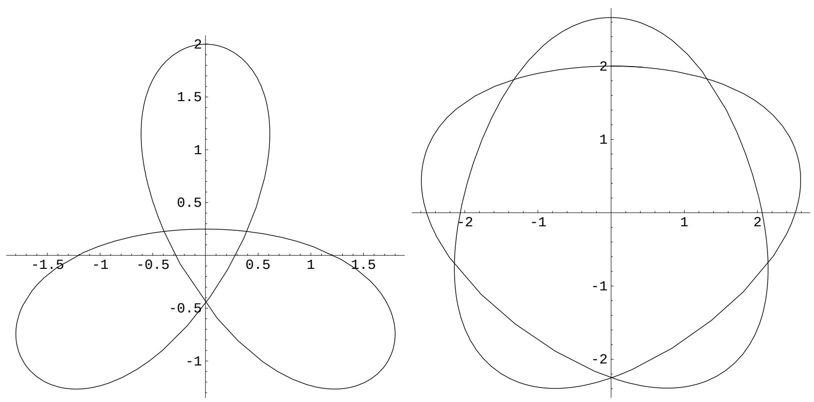

Figure 15: Examples of rational, non-integer, constant-curvature curves (a 1.5-gon and a 2.5-gon).

Through experimentation using Mathematica, we have found many examples of rounded

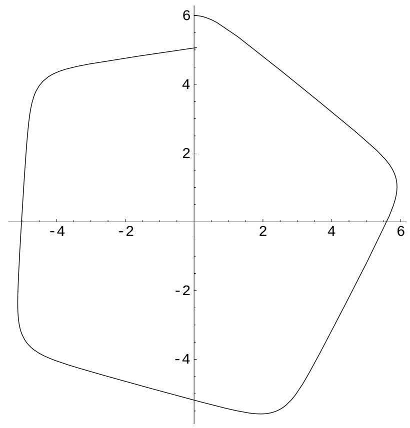

Figure 16: An example of a constant-curvature curve that does not close.

n-gons for integer or rational values of n (see Table 1). In this table, "height" is the height at which the curve crosses the positive y-axis; "n" is the number of sides of the approximate n-gon, and "κ_ϕ" is the generalized curvature in the Gauss plane of the curve. The value for n has a prime after it (e.g., $2'$) if the rounded n-gon has a vertex at the top (such as the triangle in Figure 13) and has no prime if it has an edge at the top (such as the square in Figure 13). Note however that, for instance, a $4'$-gon at a height of 5 is congruent to a 4-gon at a height of approximately $2.5\sqrt{2}$.

From Table 1 we can determine several patterns (see values for a height of 10). First, as the magnitude of the ϕ-curvature increases, we find n-gons with a vertex at the top (those with n'), and the n for these rounded polygons increases until the ϕ-curvature reaches that of a circle. Then we find n-gons with an edge at the top, and the n decreases until the ϕ-curvature reaches that of a ray perpendicular to the y-axis (see Figure 17).

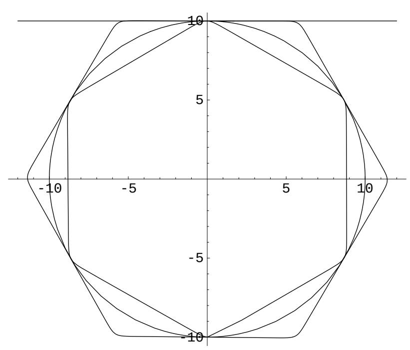

Figure 17: The horizontal line has a ϕ-curvature of -10; the hexagon with a horizontal edge at the top has a ϕ-curvature of -9.9999996; the circle has a ϕ-curvature of -9.9; and the hexagon with a vertex at the top has a ϕ-curvature of -8.82357.

height	n	κ_ϕ
0.5	2	-0.49793
1	2	-0.91
1.5	$2'$	-1.13
2	$2'$	-1.126
2.5	$2'$	-0.96
	3	-2.47
3	$2'$	-0.7565
3.5	$2'$	-0.589
4	$2'$	-0.4745
	$4'$	-3.61
	4	-3.9897
4.5	$2'$	-0.3995
5	$2'$	-0.348
	$4'$	-3.9755
	$5'$	-4.67
	4	-4.999925

height	n	κ_ϕ
6	$2'$	-0.282
	$3'$	-3.272
	$5'$	-5.25
	6	-5.954
	5	-5.99915
	4	-5.9999997471
7	$2'$	-0.235
8	6	-7.171
10	$6'$	-8.82357
	$7'$	-9.19
	$8'$	-9.45
	circle	-9.9
	8	-9.999
	7	-9.999948
	6	-9.9999996
	ray	-10

Table 1: This table shows starting heights of constant-curvature rounded n-gons, the number of sides of the rounded n-gon, and the ϕ-curvature required to achieve that curve. The table is arranged with monotonically increasing starting heights, and within each starting height, the magnitude of the ϕ-curvature κ_ϕ is monotonically increasing. (See Appendices A and B for a more complete table.)

This pattern of monotonically increasing values of n for increasing magnitudes of ϕ-curvature is not always true, though it appears to always be true for the integer values that we have explored. As we experiment with values of κ_ϕ very close to that of the ray, it appears that n does not change monotonically (see Table 2). However, since the values of α and thereby the value of n are not easy to obtain and are imprecise, it is difficult to determine exactly what occurs. At this point, around the value for the ray, the data exhibits very peculiar behavior that may well just be numerical noise.

height	κ_ϕ	α	n
6	-5.999999747	0.787342	3.990124495
6	-5.99915	0.628552	4.998142715
6	-5.95	0.522044	6.017869375
6	-5.9	0.514324	6.108197556
6	-5.89	0.511018	6.147714171
6	-5.88	0.512172	6.133862452
6	-5.85	0.515204	6.097764381
6	-5.84	0.515931	6.089172002
6	-5.835	0.51636	6.084113022
6	-5.8334	0.516494	6.08253455
6	-5.83334	0.516666	6.080509652
6	-5.833	0.516528	6.082134173
6	-5.83333	0.514667	6.104126746
6	-5.833333	0.00583333	538.5590392
6	-5.8333333	0.0868333	36.17958318
6	-5.833333333	0.102	30.79992745

Table 2: This table demonstrates the behavior of the number of sides of a rounded n-gon for values of κ_ϕ close to that of a ray, for a starting height of 6. In this table, we obtain a value for α by using Mathematica to determine the angle between the y-axis and the first ray from the origin that is perpendicular to the curve. We find n by the relationship $n = \pi/\alpha$. Note that there is a sudden jump in n between $\kappa_\phi = -5.83333$ and $\kappa_\phi = -5.833333$.

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APPENDIX A

height	n	κ_ϕ
1	circle	0
	1.5	-0.22
	2.125	-0.9796
	2.167	-0.9942
	ray	-1
2	1.5′	0.035
	circle	-1.5
	2.5	-1.82
	2.667	-1.985
	ray	-2
2.5	2.5	-1.84
	circle	-2.1
	3	-2.47
2.75	2.5′	-1.8
4	4′	-3.61
	circle	-3.75
	4	-3.9897
	3.667	-3.99931
	ray	-4
5	4′	-3.9755
	4.5′	-4.36
	5′	-4.67
	circle	-4.8
	4	-4.999925
	ray	-5
6	3′	-3.272
	4.5′	-4.93
	5′	-5.25

height	n	κ_ϕ
6	circle	-5.8333
	6	-5.954
	5	-5.99915
	4	-5.9999997471
	ray	-6
7	3.333′	-4.332
	circle	-6.85714
	5.5	-6.9999996
8	6	-7.171
10	6′	-8.82357
	7′	-9.19
	8′	-9.45
	circle	-9.9
	8	-9.999
	7	-9.999948
	6	-9.9999996
	ray	-10
20	9′	-19.0098902
	11′	-19.257
	12′	-19.4
	13′	-19.5
	14′	-19.585
	15′	-19.658
	16′	-19.716
	17′	-19.77
	18′	-19.824
	19′	-19.875
	20′	-19.94

Table 3: This table is an extension of Table 1 showing the starting heights of constant-curvature rounded n-gons, the number of sides of the rounded n-gon, and the ϕ-curvature required to achieve that curve. The bigons are omitted from this table and appear in Appendix B.

APPENDIX B

height	n	κ_ϕ	area inside	perimeter
0.5	2	-0.49793	0.0948189	0.176176
0.75	2	-0.72345	0.133003	0.15298
1	2	-0.91	0.16187	0.15298
1.1	2	-0.9717	0.170572	0.121738
1.2	2	-1.025	0.177425	0.114483
1.3	2	-1.069	0.183192	0.108416
1.4	2	-1.1075	0.188402	0.103376
1.5	2	-1.13	0.190513	0.100014
1.6	2	-1.147	0.193226	0.0976955
1.8	2	-1.15	0.193802	0.0971153
2	$2'$	-1.126	0.190436	0.100606
2.25	$2'$	-1.056	0.181905	0.110265
2.5	$2'$	-0.96	0.16918	0.123193
2.75	$2'$	-0.858	0.154258	0.136576
3	$2'$	-0.7565	0.138952	0.149073
3.25	$2'$	-0.666	0.124144	0.159327
3.5	$2'$	-0.589	0.111389	0.16746
3.75	$2'$	-0.5255	0.100222	0.173606
4	$2'$	-0.4745	0.0911854	0.178208
4.25	$2'$	-0.433	0.0839131	0.181456
4.5	$2'$	-0.3995	0.0778688	0.18403
4.75	$2'$	-0.3715	0.0724249	0.186048
5	$2'$	-0.348	0.0682948	0.187749
5.25	$2'$	-0.3275	0.0643909	0.188934
5.5	2	-0.31	0.0610003	0.190103
5.75	$2'$	-0.295	0.0583733	0.190761
6	$2'$	-0.282	0.0556667	0.191706

Table 4: This table shows the starting heights of constant-curvature rounded bigons, the ϕ-curvature required to achieve that curve, the area enclosed inside, and the perimeter of the bigon.

APPENDIX C

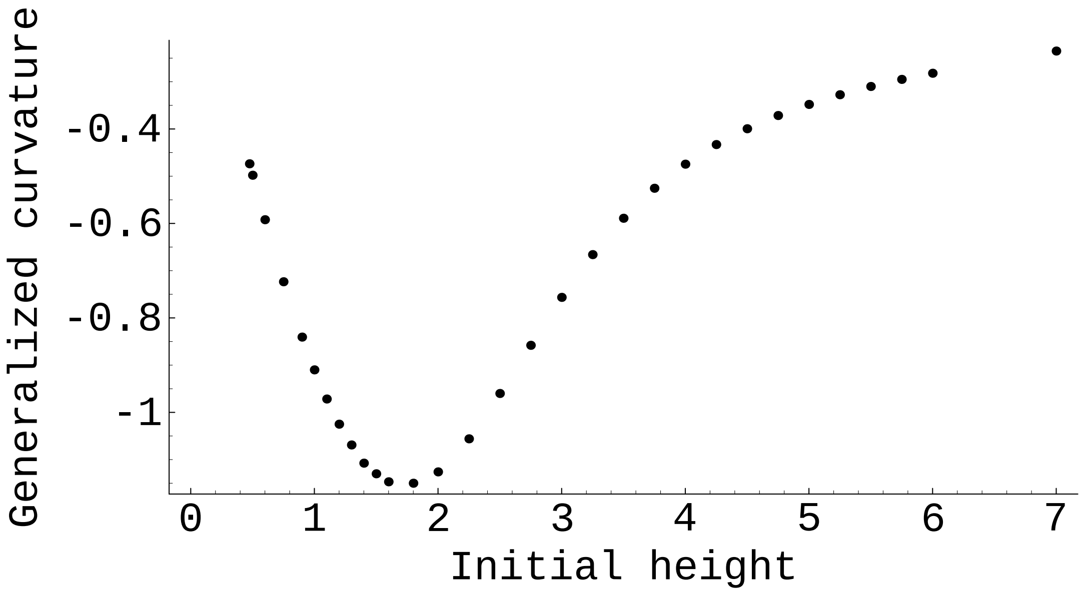

Figure 18: This graph depicts the relationship between the generalized curvature and the initial height of a bigon.

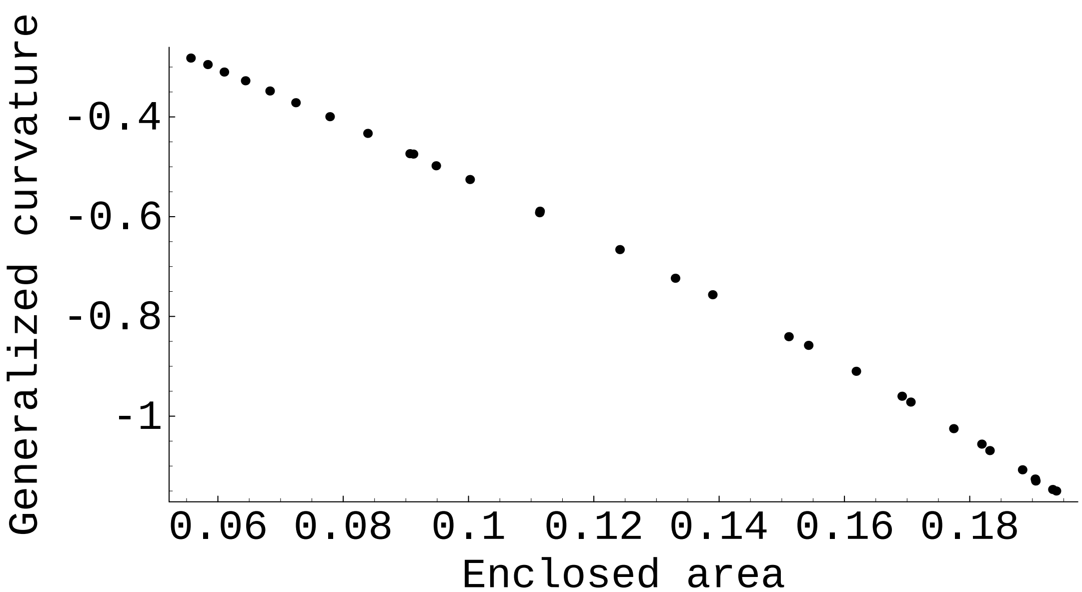

Figure 19: This graph depicts the relationship between the generalized curvature and the area enclosed by the bigon.