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# On Involutions With Many Fixed Points in Gassmann Triples

Jim Stark
*University of Virginia*, jstarx@gmail.com

# ON INVOLUTIONS WITH MANY FIXED POINTS IN GASSMANN TRIPLES

JIM STARK

Abstract. We show that in a non-trivial Gassmann triple $(G, H, H')$ of index $n$ there does not exist an involution $\tau \in G$ such that the value of the permutation character on $\tau$ is $n-2$. In addition we describe a GAP program designed to search for examples of Gassmann triples and give a brief summary of the results of this search.

## Introduction

Adolf Hurwitz was a German mathematician, a student of Felix Klein and a teacher of David Hilbert. When Frobenius left the Eidgenössische Technische Hochschule Zürich it was Hurwitz who took his chair and he remained there until his death in 1919. Fritz Gassmann, a student at the ETH Zürich under George Pólya and Hermann Weyl, was asked to look at a notebook of Hurwitz's containing unpublished work. In 1926 Gassmann published a section of this notebook along with an article explaining what he believed to be the point of Hurwitz's work [Gas26].

In his article Gassmann introduced the following condition on two subgroups $H$ and $H'$ of a group $G$: Each conjugacy class of $G$ intersects $H$ and $H'$ in the same number of elements, that is for every $g \in G$ we have $|g^G \cap H| = |g^G \cap H'|$. Today we say two such subgroups are *Gassmann equivalent* and call $(G, H, H')$ a *Gassmann triple.*

The motivation for studying Gassmann triples comes from several fields of mathematics. Most recently Terras and Stark showed in [TS00] that Gassmann equivalent subgroups can be used to create non-isomorphic graphs whose Ihara zeta functions are equal. In [Sun85] Sunada described how Gassmann equivalent subgroups of a group $G$ can be used to construct Riemannian manifolds that are isospectral but not isometric. Lastly Perlis has shown in [Per77] that two algebraic number fields $F$ and $F'$ share the same Dedekind zeta function precisely when the Galois groups $H = \mathrm{Gal}(L/F)$ and $H' = \mathrm{Gal}(L/F')$ are Gassmann equivalent subgroups of $G = \mathrm{Gal}(L/\mathbb{Q})$ where $L \subseteq \mathbb{C}$ is a common normal extension. It is this last topic in which we find the motivation for this paper.

Given an automorphism in $G$ we can construct an embedding $F \hookrightarrow \mathbb{C}$ by restricting the domain of the automorphism to $F$. The automorphisms in the group

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$H = \mathrm{Gal}(L/F)$ fix elements of the field $F$; therefore, given an $x \in G$, each automorphism in the coset $xH$ gives the same embedding $F \hookrightarrow \mathbb{C}$. This is in fact a one-to-one correspondence between embeddings $F \hookrightarrow \mathbb{C}$ and cosets in the coset space $G/H$. The group $G = \mathrm{Gal}(L/\mathbb{Q})$ of automorphisms of $L$ acts on these embeddings by left multiplication $G \curvearrowright G/H$. If $\tau \in G$ is the restriction of complex conjugation on $\mathbb{C}$ to the subfield $L$ then a fixed point of $\tau$ under the action $G \curvearrowright G/H$ corresponds to a real embedding $F \hookrightarrow \mathbb{R}$. We define $\chi_{G/H}(\tau)$ to be the number of fixed points of $\tau$ and call an element of order 2, such as $\tau$, an *involution*. Then in Section 3 we prove the following theorem.

**Main Theorem.** *If $(G, H, H')$ is a Gassmann triple of index $n$ and if there exists an involution $\tau \in G$ such that $\chi_{G/H}(\tau) = n-2$ then the triple $(G, H, H')$ is trivial.*

We call $(G, H, H')$ a *trivial* Gassmann triple when $H$ and $H'$ are conjugate in $G$. The two fields $F$ and $F'$ are isomorphic if and only if their Galois groups $H$ and $H'$ are conjugate in $G$. Thus this main theorem proves that an algebraic number field $F$ with only 2 non-real embeddings is uniquely determined up to isomorphism by its Dedekind zeta function. We also give an example of a non-trivial Gassmann triple $(G, H, H')$ and an involution $\tau \in G$ such that $\chi_{G/H}(\tau) = n - 4$; thus the number $n-2$ of fixed points of an involution cannot be reduced in a linear fashion.

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## 1. Definitions

Let $G$ be a group and $H \leq G$ a subgroup of index $n$. The group $G$ acts on the coset space $G/H$ by left multiplication. The *fixed point character* of this action is the function $\chi_{G/H} \colon G \to \mathbb{N}_0$ defined by $g \mapsto |\mathrm{Fix}(g)|$ where

$$\mathrm{Fix}(g) = \{xH \in G/H \mid gxH = xH\}$$

is the set of cosets fixed under the action of the element $g$. The following proposition gives us a formula for the fixed point character. We will need this to prove Proposition 5 in Section 3.

**Proposition 1** (Proposition 1.6 in [Bea91])**.** *Let $H \leq G$ be any subgroup. For all $g \in G$ we have $\chi_{G/H}(g) = \frac{|C_G(g)| \cdot |g^G \cap H|}{|H|}$.*

Here $C_G(g)$ is the *centralizer* of $g$ in $G$ given by $C_G(g) = \{x \in G \mid xg = gx\}$. It is the set of all elements in $G$ that commute with $g$. The connection between the permutation characters $\chi_{G/H}$ and $\chi_{G/H'}$ and the Gassmann equivalence of $H$ and $H'$ is given in the following proposition.

**Proposition 2** (Lemma 1.9 in [Bea91])**.** *Let $H, H' \leq G$. The following conditions are equivalent:*

(i) $\chi_{G/H} = \chi_{G/H'}$.
(ii) $|g^G \cap H| = |g^G \cap H'| \quad \forall g \in G$.
(iii) $\exists \phi \in \mathrm{Bij}(H, H')$ *satisfying* $\phi(g) \in g^G \quad \forall g \in G$.

Note that we always have $\chi_{G/H}(1) = [G:H]$. Thus if condition (i) holds then $\chi_{G/H}(1) = \chi_{G/H'}(1)$ which gives us $[G:H] = [G:H']$. So $H$ and $H'$ have the same index in $G$. We now expand the definition of Gassmann triple that was given in the introduction.

**Definition 1.** Let $H$ and $H'$ be subgroups of $G$. The triple $(G, H, H')$ is called a *Gassmann triple* provided that any of the three equivalent conditions given in Proposition 2 hold. Alternatively we may say that $H$ and $H'$ are *Gassmann equivalent subgroups* of the group $G$. As per the above note $[G:H] = [G:H']$, we define the *index* of the Gassmann triple $(G, H, H')$ to be this common index.

Note that if $H$ and $H'$ are conjugate in $G$, then there is an inner automorphism of $G$ taking $H$ to $H'$. The restriction of this automorphism to $H$ is a bijection satisfying condition (iii). Thus given any subgroup $H \leq G$, the triple $(G, H, H^g)$ is Gassmann for all $g \in G$.

**Definition 2.** Let $(G, H, H')$ be a Gassmann triple. We call $(G, H, H')$ a *trivial* Gassmann triple if $H$ and $H'$ are conjugate in $G$. We call $(G, H, H')$ a *faithful* Gassmann triple if the left multiplication action $G \curvearrowright G/H$ is a faithful action.

A *faithful action* is an action whose kernel is trivial. The kernel $K$ of the left multiplication action $G \curvearrowright G/H$ is exactly the set of elements that fix all $[G:H]$ cosets; that is, the elements $g \in G$ such that $\chi_{G/H}(g) = [G:H]$. In a Gassmann triple $(G, H, H')$ we know that $\chi_{G/H} = \chi_{G/H'}$ thus the kernel $K$ of $G \curvearrowright G/H$ is equal to the kernel of $G \curvearrowright G/H'$. We call $K$ the *common kernel* of these actions and note that as elements in $K$ fix both $H$ and $H'$ we have $K \subseteq H \cap H'$.

**Example 1.** Consider the ring $\mathbb{Z}_8$. The group of units $\mathbb{Z}_8^*$ acts on the additive group $\mathbb{Z}_8$ by multiplication. Thus we can form the semidirect product $G = \mathbb{Z}_8^* \ltimes \mathbb{Z}_8$ whose group operation is $(a_1, b_1)(a_2, b_2) = (a_1a_2, a_2b_1 + b_2)$. Let $H$ and $H'$ be the subgroups $\{(1, 0), (3, 0), (5, 0), (7, 0)\}$ and $\{(1, 0), (3, 4), (5, 4), (7, 0)\}$ respectively. We claim that $(G, H, H')$ is a non-trivial faithful Gassmann triple.

*Proof of claim.* Define the map $\phi \colon H \to H'$ as follows.

$$\begin{array}{cccll} (1,0) & \overset{\phi}{\longmapsto} & (1,0) & & \\ (3,0) & \longmapsto & (3,4) & = & (3,0)^{(1,2)} \\ (5,0) & \longmapsto & (5,4) & = & (5,0)^{(1,1)} \\ (7,0) & \longmapsto & (7,0) & & \end{array}$$

The map $\phi$ satisfies (iii) of Theorem 2, thus $(G, H, H')$ is Gassmann.

Assume that $H$ and $H'$ are conjugate, that is there is some element $(a, b) \in G$ such that $H^{(a,b)} = H'$. Conjugation in $G$ fixes the first factor of any element, thus we must have $(3, 0)^{(a,b)} = (3, 4)$ and $(7, 0)^{(a,b)} = (7, 0)$. Note that $(a, b)^{-1} = (a, -ab)$. First we compute $(3, 4) = (a, b)(3, 0)(a, -ab) = (3, 2ab)$ which gives us $2ab \equiv 4 \pmod 8$. But then we see that $(a, b)(7, 0)(a, -ab) = (7, 6ab) = (7, 4) \neq (7, 0)$. Thus the assumption that $H$ and $H'$ are conjugate is false. The Gassmann triple is non-trivial.

Finally observe that the only non-identity element in the intersection $H \cap H'$ is $(7, 0)$ so the kernel of $G \curvearrowright G/H$ is either $\{1\}$ or $\{1, (7, 0)\}$. But $(7, 0)(1, 1)H = (7, 1)H$ and we see that $(1, 1)H \neq (7, 1)H$ because $(7, 1)^{-1}(1, 1) = (7, 2) \notin H$. Thus

$(7,0)$ moves $(1,1)H$ and so is not in the kernel. Hence the kernel is $\{1\}$; the triple is faithful. □

We are interested in the involutions of $G$, that is the elements of order 2.

**Example 2.** Note that conjugation in a group $G$ is an automorphism of $G$ therefore every conjugate of an involution is also an involution. Hence when listing all involutions in a particular group it suffices to list them up to conjugation. Let $G$, $H$, and $H'$ be as in example 1. The involutions in $G$ up to conjugation are $(1,4)$, $(3,0)$, $(5,0)$, $(7,0)$, and $(7,1)$. The index is $n = 8$ and the reader can easily compute that $\chi_{G/H}(5,0) = 4$.

Consider a group $G$ containing an involution $\tau \in G$ and a subgroup $H \leq G$ of index $n$. By associating the cosets in $G/H$ with the integers $\{1, 2, \ldots, n\}$ we obtain a permutation representation of $\tau$. This permutation sends $i$ to $j$ if $\tau$ sends the coset associated with $i$ to the coset associated with $j$. As $\tau$ is an involution its permutation representation will either be the identity or a product of some number of disjoint 2-cycles and some number of 1-cycles. The number of fixed points of $\tau$, that is the number of 1-cycles, will be the index $n$ minus the number of moved points of those 2-cycles. Thus the possible values of $\chi_{G/H}(\tau)$ are $n$, $n-2$, $n-4$, and so on. If $\chi_{G/H}(\tau) = n$ then $\tau$ is in the kernel of the action thus $(G, H, H')$ is a non-faithful triple. We have just seen in Example 2 an involution $\tau$ in a non-trivial faithful triple satisfying $\chi_{G/H}(\tau) = n - 4$. We will investigate the consequences when $\chi_{G/H}(\tau) = n - 2$.

## 2. A structure theorem for $\langle \tau^G \rangle$

In this section we will show that strict conditions are put on the structure of the group generated by the conjugacy class of an involution with $n-2$ fixed points. For the remainder of the section we make the following definitions. Let $G$ be a group, $H \leq G$ a subgroup of index $n$, and let $\tau \in G$ be an involution such that $\chi_{G/H}(\tau) = n - 2$. Define $N = \langle \tau^G \rangle$.

For the remainder of this paper there will only be one type of action. This action is $G$ acting on the coset space $G/H$ by left multiplication. We will however consider the orbits and stabilizers of this action when it is restricted to various subgroups of $G$. Given a subgroup $J \leq G$ we will refer to the orbits of the restricted action $J \curvearrowright G/H$ as $J$-orbits and will use $\mathrm{Stab}_J(gH)$ to denote the stabilizer of a coset $gH$ under the action of $J$.

We define $k$ to be the number of distinct $N$-orbits and will denote these orbits $\mathcal{O}_1$, $\mathcal{O}_2$, …, $\mathcal{O}_k$. Finally define $K$ to be the kernel of the restricted action $N \curvearrowright G/H$. We begin with a lemma.

**Lemma 1.** *Let $J \leq G$ be any subgroup. Given any two distinct cosets $xH, yH \in G/H$, if there exists an element $a \in \langle \tau^G \cap J \rangle$ such that $axH = yH$ then there exists an element $a' \in \tau^G \cap J$ such that $a'xH = yH$.*

*Proof.* We will prove that if $w_1, w_2 \in \tau^G \cap J$ and $(w_1 w_2)xH = zH$ where $zH \neq xH$ then there exists a $w_3 \in \tau^G \cap J$ such that $w_3 xH = zH$. This suffices to prove the lemma because the element $a \in \langle \tau^G \cap J \rangle$ can be written as a word $a = w_1 w_2 \cdots w_s$ with letters $w_i \in \tau^G \cap J$. We will have shown that we can reduce the length of this word without altering where the coset $xH$ is sent. This reduction can be continued inductively until the word consists of a single letter $a' \in \tau^G \cap J$.

Thus let $w_1, w_2 \in \tau^G \cap J$ with $(w_1 w_2)xH = zH$. Define $uH = w_2 xH$ so that $w_1 uH = zH$. If $uH = zH$ then set $w_3 = w_2$. Similarly if $uH = xH$ then set $w_3 = w_1$. All that is left is the case when $uH$, $xH$, and $zH$ are 3 distinct cosets.

We have $w_1, w_2 \in \tau^G$ so these elements are involutions that move exactly two cosets. As $w_1 uH = zH$ we know that $w_1$ moves $uH$ and $zH$ thus fixes $xH$. Set $w_3 = w_2^{w_1}$; this element is in the conjugacy class of $\tau$ and is a product of elements in $J$ thus $w_3 \in \tau^G \cap J$. As desired we have $w_3 xH = w_1 w_2 w_1 xH = w_1 w_2 xH = zH$. $\square$

We now focus our attention on the restricted action $N \curvearrowright G/H$ with orbits $\mathcal{O}_1$, $\mathcal{O}_2$, ..., $\mathcal{O}_k$ in order to determine the structure of $N$.

**Proposition 3.** *Given any two $N$-orbits $\mathcal{O}_i$ and $\mathcal{O}_j$ we have $|\mathcal{O}_i| = |\mathcal{O}_j|$.*

*Proof.* Choose two elements $x, y \in G$ such that $xH \in \mathcal{O}_i$ and $yH \in \mathcal{O}_j$. Every coset in $\mathcal{O}_i$ can be written in the form $axH$ for some $a \in N$. Define the map $\phi \colon \mathcal{O}_i \to \mathcal{O}_j$ in terms of the chosen representatives $x$ and $y$ by $axH \overset{\phi}{\mapsto} a^{yx^{-1}} yH$. Note that $\phi$ maps into $\mathcal{O}_j$ because $N$ is normal so $a^{yx^{-1}} \in N$. Also $\phi$ is well defined; it does not depend on the choice of element $a \in N$. For assume $axH = bxH$ for some $a, b \in N$; then $axy^{-1}yH = bxy^{-1}yH \Rightarrow yx^{-1}axy^{-1}yH = yx^{-1}bxy^{-1}yH \Rightarrow a^{yx^{-1}}yH = b^{yx^{-1}}yH \Rightarrow \phi(axH) = \phi(bxH)$.

I claim that $\phi$ is onto. Every coset in $\mathcal{O}_j$ can be written in the form $ayH$ for some $a \in N$. Then $a^{xy^{-1}}xH \in \mathcal{O}_i$ and $\phi(a^{xy^{-1}}xH) = (a^{xy^{-1}})^{yx^{-1}}yH = ayH$. Thus $\phi$ is onto giving $|\mathcal{O}_i| \geq |\mathcal{O}_j|$. By symmetry we have $|\mathcal{O}_j| \geq |\mathcal{O}_i|$ thus $|\mathcal{O}_i| = |\mathcal{O}_j|$. $\square$

We now define $m = |\mathcal{O}_1|$. By the previous proposition $m$ is the order of *any* $N$-orbit. There are $k$ such orbits thus $n = km$. Note that $\tau$ is an element in $N$ that moves two cosets in some $N$-orbit so $m > 1$.

We've just seen that $G/H$ is the union of $k$ orbits under $N$. For $i \in \{1, 2, \ldots, k\}$ the action of $N$ on the $N$-orbit $\mathcal{O}_i$ is transitive on $m$ points. By associating the cosets in $\mathcal{O}_i$ to the integers $\{1, 2, \ldots, m\}$ we obtain a homomorphism $\pi_i \colon N \to S_m$ called the *permutation representation of $N$ acting on the $i^{th}$ $N$-orbit.*

**Example 3.** To see how this representation is constructed consider the Gassmann triple $(G, H, H')$ from Example 1. The quotient group $G/H$ contains the cosets

$$\begin{aligned}
(1,0)H &= \{(1,0),(3,0),(5,0),(7,0)\}\\
(1,1)H &= \{(1,1),(3,3),(5,5),(7,7)\}\\
(1,2)H &= \{(1,2),(3,6),(5,2),(7,6)\}\\
(1,3)H &= \{(1,3),(3,1),(5,7),(7,5)\}\\
(1,4)H &= \{(1,4),(3,4),(5,4),(7,4)\}\\
(1,5)H &= \{(1,5),(3,7),(5,1),(7,3)\}\\
(1,6)H &= \{(1,6),(3,2),(5,6),(7,2)\}\\
(1,7)H &= \{(1,7),(3,5),(5,3),(7,1)\}.
\end{aligned}$$

We will associate the coset $(1, i)H$ with the integer $i + 1$ and let $\pi_{G/H} \colon G \to S_8$ be the map that sends an element in $G$ to the permutation representation of $G$ acting on $G/H$.

Now take any element $g \in G$, for instance $g = (3, 4)$. It is easy to check that $(3, 4)(1, 0)H = (3, 4)H = (1, 4)H$. The cosets $(1, 0)H$ and $(1, 4)H$ are associated with the integers 1 and 5 respectively, thus the permutation representation of $g$ sends 1 to 5. In fact we find that $\pi_{G/H}(g) = (1\ 5)(2\ 8)(4\ 6)$.

Note that each element of $\tau^G$ moves exactly two cosets in $G/H$. Thus if $t \in \tau^G$ transposes two cosets in $\mathcal{O}_i$ then it fixes the cosets in every other $N$-orbit giving $t \in \ker \pi_j$ for all $j \neq i$. That is, each element of $\tau^G$ is in the support of $\pi_i$ for exactly one $i \in \{1, 2, \ldots, k\}$ (the *support* of a homomorphism is the set of elements in the domain that are not in the kernel).

**Proposition 4.** *Let $I = \{1, 2, \ldots, k\}$, then we have the following.*

(i) *Each $t \in \tau^G$ is in the support of $\pi_i$ for exactly one $i \in I$.*
(ii) *For each $t \in \tau^G$ and each $i \in I$, the permutation $\pi_i(t)$ is either the identity or a 2-cycle.*
(iii) *For each $i \in I$ every 2-cycle in $S_m$ is contained in the set $\pi_i(\tau^G)$.*
(iv) *$\pi_i$ maps $N$ onto $S_m$.*

*Proof.*

(i) This is given in the paragraph above.
(ii) Let $t \in \tau^G$ and $i \in I$. If $\pi_i(t)$ is not the identity then $t$ moves cosets in the $i^{\text{th}}$ $N$-orbit. As $t \in \tau^G$ we know that $t$ is an involution that moves exactly 2 cosets in $G/H$. Thus $\pi_i(t)$ is a permutation that moves exactly 2 points so $\pi_i(t)$ is a 2-cycle.
(iii) Let $i \in I$ and let $\sigma \in S_m$ be any 2-cycle, $\sigma = (a\ b)$. The points $a$ and $b$ correspond to two cosets in the $i^{\text{th}}$ $N$-orbit, call them $xH$ and $yH$. Because they are in the same $N$-orbit there exists an element $g \in N$ such that $gxH = yH$. Lemma 1 with $J = N$ states that there exists a $t \in \tau^G$ such that $txH = yH$. Then $\pi_i(t)$ is a two cycle that moves the points associated to $xH$ and $yH$, that is $\pi_i(t) = \sigma$.
(iv) A standard generating set for $S_m$ is the set of all 2-cycles that move the point 1. By (iii) these 2 cycles are all contained in $\pi_i(\tau^G) \subset \pi_i(N) \leq S_m$ thus $\pi_i(N) = S_m$.

□

Now we are in a position to say something about the structure of $N$.

**Theorem 1.** *There exists an epimorphism $\psi : N \to (S_m)^k$ and the kernel of this epimorphism is $K$, the kernel of $N \curvearrowright G/H$.*

*Proof.* Define $\psi\colon N \to (S_m)^k$ by $g \mapsto (\pi_1(g), \pi_2(g), \ldots, \pi_k(g))$. Each $\pi_i$ is a homomorphism thus $\psi$ is a homomorphism. By (iv) in Proposition 4 every $\sigma_i \in S_m$ has a $\pi_i$ pre-image, $g_i' \in N$. As $N = \langle \tau^G \rangle$, we can express $g_i'$ as a word in the letters $\tau^G$. Then define $g_i$ by taking the expression for $g_i'$ and throwing out any letters in the kernel of $\pi_i$. As $g_i$ and $g_i'$ only differ by elements in the kernel of $\pi_i$ we have $\pi_i(g_i) = \pi_i(g_i') = \sigma_i$. By (i) in Proposition 4 we have chosen a $g_i$ such that $\pi_j(g_i) = 1$ for all $j \neq i$.

Now given any element $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in (S_m)^k$ we choose a pre-image $g_i \in N$ for each $\sigma_i \in S_m$ as described above and form the product $g = \prod_i g_i$. Then we have

$$\begin{aligned}\psi(g) &= (\pi_1(g), \pi_2(g), \ldots, \pi_k(g)) \\ &= \left(\pi_1\left(\prod_i g_i\right), \pi_2\left(\prod_i g_i\right), \ldots, \pi_k\left(\prod_i g_i\right)\right) \\ &= \left(\prod_i \pi_1(g_i), \prod_i \pi_2(g_i), \ldots, \prod_i \pi_k(g_i)\right) \\ &= (\pi_1(g_1), \pi_2(g_2), \ldots, \pi_k(g_k)) \\ &= (\sigma_1, \sigma_2, \ldots, \sigma_k) \\ &= \sigma\end{aligned}$$

Thus $\psi$ is onto. Finally $K$, the kernel of $N \curvearrowright G/H$, consists of exactly those elements that are in the kernel of the action $N \curvearrowright \mathcal{O}_i$ for every $i$. Thus $K = \bigcap_i \ker \pi_i = \ker \psi$. □

The previous theorem along with the first isomorphism theorem gives that $N/K \cong (S_m)^k$. If $G \curvearrowright G/H$ is faithful we have that $N \cong (S_m)^k$.

## 3. The structure of Gassmann triples

In this section we will continue the analysis of the previous section and so will retain the definitions of the group $G$, subgroup $H$, and involution $\tau$. We add the condition that there is a subgroup $H' \leq G$ and the triple $(G, H, H')$ is faithful Gassmann.

We also wish to define the subgroup $N_{H'} = \langle \tau^G \cap H' \rangle$. The $N_{H'}$-orbits of the left multiplication action of $N_{H'}$ on $G/H$ will be essential in investigating the relationship between $H$ and $H'$. We begin this investigation by determining the cardinality of some of the sets that interest us.

**Proposition 5.** *The number of elements in $G$ conjugate to $\tau$ is $|\tau^G| = \frac{1}{2}n(m-1)$. The number of conjugates of $\tau$ in $H'$ is $|\tau^G \cap H'| = \frac{1}{2}(m-1)(n-2)$.*

*Proof.* As the action is faithful, $\psi$ of Theorem 1 is an isomorphism. So to determine $|\tau^G|$ we count its image under $\psi$. By (i)-(iii) of Proposition 4, $\psi(\tau^G)$ is the set of all 2-cycles in all factors of $(S_m)^k$. There are $\frac{1}{2}m(m-1)$ 2-cycles in $S_m$ and $k$ factors of $S_m$, thus $|\tau^G| = k \cdot \frac{1}{2}m(m-1) = \frac{1}{2}n(m-1)$.

To determine $|\tau^G \cap H'|$ note that $\chi_{G/H'}(\tau) = n-2$. Proposition 1 then gives $|\tau^G \cap H'| = \frac{(n-2)|H'|}{|C_G(\tau)|}$. The orbit-stabilizer relation can be applied to the conjugation action of $G$ on itself to obtain the formula $|\tau^G| \cdot |C_G(\tau)| = |G|$. Using this we have $|\tau^G \cap H'| = \frac{(n-2)|H'| \cdot |\tau^G|}{|G|} = \frac{(n-2)}{n}|\tau^G| = \frac{1}{2}(m-1)(n-2)$. □

What will be of particular interest to us are the number of elements in $\tau^G$ that are not in $H'$. This can easily be computed from Proposition 5. We have $|\tau^G - H'| = |\tau^G| - |\tau^G \cap H'| = \frac{1}{2}n(m-1) - \frac{1}{2}(m-1)(n-2) = m-1$.

Now consider the subgroup $N_{H'} = \langle \tau^G \cap H' \rangle$. The generating set $\tau^G \cap H'$ of $N_{H'}$ is contained in the generating set $\tau^G$ of $N$ thus $N_{H'} \leq N$. This implies that $N$-orbits are disjoint unions of one or more $N_{H'}$-orbits. The question we now ask is when moving from the action $N \curvearrowright G/H$ to the action $N_{H'} \curvearrowright G/H$ how do the $N$-orbits decompose into the $N_{H'}$-orbits?

It cannot be the case that the $N_{H'}$-orbits are exactly the $N$-orbits. As noted below Proposition 3 we have $m-1>0$, thus there are elements in $\tau^G$ that are not in $H'$. If $t \in \tau^G - H'$ moves $xH$ and $yH$ then those two cosets must be in different $N_{H'}$-orbits. For suppose there exists an $a \in N_{H'}$ such that $axH = yH$. Then Lemma 1, with $J = H'$, states that there exists an $a' \in \tau^G \cap H'$ such that $a'xH = yH$. In this section we have assumed that the Gassmann triple $(G, H, H')$ is faithful; that is, the action on the coset space $G/H$ is faithful. In a faithful action no two distinct elements act identically. As $a'$ and $t$ act identically we must have $a' = t$. But $t \notin H'$ so this is a contradiction.

Thus at least one $N$-orbit must decompose into two or more $N_{H'}$-orbits. We will see that exactly one of the $N$-orbits decomposes into exactly two $N_{H'}$-orbits of length 1 and $m-1$.

**Theorem 2.** *The $N_{H'}$-orbits consist of $k-1$ orbits of length $m$, 1 orbit of length $m-1$, and 1 orbit of length 1.*

*Proof.* For $A = N$ or $N_{H'}$ define the undirected graph $\Gamma(A)$ as follows. The vertex set of $\Gamma(A)$ is the coset space $G/H$. We associate $\tau^G \cap A$ with the edge set, that is we join two vertices if there is a conjugate of $\tau$ in $A$ that transposes the cosets associated with those vertices. Note that because the action on $G/H$ is faithful no two distinct elements of $\tau^G$ transpose the same two cosets, thus the association of $\tau^G \cap A$ with the edge set is a one-to-one correspondence.

Lemma 1 with $J = N$ shows that if two cosets are in the same $N$-orbit then they are connected by an edge in $\Gamma(N)$. Similarly Lemma 1 with $J = H'$ shows that if two cosets are in the same $N_{H'}$-orbit then they are connected by an edge in $\Gamma(N_{H'})$. Thus the graph $\Gamma(A)$ represents the $A$-orbits of the action on the coset space $G/H$. An edge connects two vertices in $\Gamma(N)$ if and only if the cosets that those vertices represent are in the same $N$-orbit and an edge connects two vertices in $\Gamma(N_{H'})$ if and only if the cosets that those vertices represent are in the same $N_{H'}$-orbit.

There are $k$ $N$-orbits thus $\Gamma(N)$ is a disconnected graph with $k$ components. Each component has $m$ vertices and is complete (every vertex in the component is connected to every other vertex in that component). As $N$-orbits decompose into $N_{H'}$-orbits the graph $\Gamma(N_{H'})$ is obtained from the graph $\Gamma(N)$ by removing a number of edges. As discussed above, elements in $\tau^G - H'$ transpose cosets that are in the same $N$-orbit but not in the same $N_{H'}$-orbit. Thus the edges removed from $\Gamma(N)$ to obtain $\Gamma(N_{H'})$ are precisely the $m-1$ edges associated with elements in $\tau^G - H'$.

As stated above at least one of the $N$-orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$ decomposes into two or more $N_{H'}$-orbits. Let $\mathcal{O}_i$ be such an $N$-orbit and let $\mathcal{A}$ be one of the $N_{H'}$-orbits that $\mathcal{O}_i$ decomposes into. We have $\mathcal{O}_i \neq \mathcal{A}$ because $\mathcal{O}_i$ decomposes thus define $\mathcal{B} = \mathcal{O}_i - \mathcal{A}$ to be the complement of $\mathcal{A}$ in $\mathcal{O}_i$.

We determined in paragraph 3 of this proof that the component of $\Gamma(N)$ corresponding to $\mathcal{O}_i$ is the complete graph on $m$ vertices. Let $a = |\mathcal{A}|$ and $b = |\mathcal{B}|$ so that the component of $\Gamma(N_{H'})$ corresponding to $\mathcal{A}$ is the complete graph on $a$ vertices and $a + b = m$. Now when passing from $\Gamma(N)$ to $\Gamma(N_{H'})$ we must disconnect the component of $\Gamma(N)$ corresponding to $\mathcal{A}$ from the vertices in $\mathcal{B}$. Each of the $a$ vertices in $\mathcal{A}$ is connected to each of the $b$ vertices in $\mathcal{B}$, thus we must remove $e = ab$ edges. Given $e = ab$, $a + b = m$, and $1 \le a, b \le m-1$ it is a simple calculus problem to find the minimum possible value of $e$ when considered as a function of

$a$ and $b$. This value is $e = m - 1$ and occurs when $a$ and $b$ take the values 1 and $m - 1$.

Thus the minimal case of disconnecting only a single component of $\Gamma(N_{H'})$ requires the removal of $m-1$ edges. As only $m-1$ edges are to be removed it cannot be the case that the component of $\Gamma(N)$ that corresponds to $\mathcal{O}_i$ decomposes into more than two $\Gamma(N_{H'})$ components for that would require the removal of additional edges. It must decompose into exactly 2 components of sizes 1 and $m - 1$. It also cannot be the case that any component of $\Gamma(N)$ other than the one that corresponds to $\mathcal{O}_i$ decomposes because that would again involve removing more than $m - 1$ edges. Thus exactly one of the $N$-orbits decomposes into two $N_{H'}$-orbits of length 1 and $m - 1$. The other $k - 1$ $N$-orbits do not decompose. Hence the $N_{H'}$-orbits consist of $k - 1$ orbits of length $m$, 1 orbit of length $m - 1$, and 1 orbit of length 1. □

Now that we know something about the $N_{H'}$-orbits we ask how these orbits behave under the action of $H'$. It turns out that $H'$ takes $N_{H'}$-orbits to $N_{H'}$-orbits.

**Proposition 6.** *If $xH$ and $yH$ are in the same $N_{H'}$-orbit, then for every $h \in H'$ the cosets $hxH$ and $hyH$ are also in the same $N_{H'}$-orbit.*

*Proof.* Let $xH$ and $yH$ be two cosets in the same $N_{H'}$-orbit and let $h$ be an arbitrary element of $H'$. There exists an $a \in N_{H'}$ such that $axH = yH$. Note that $N_{H'} \trianglelefteq H'$; thus we may define $b = a^h$ and we have $b \in N_{H'}$. Then $bhxH = hah^{-1}hxH = haxH = hyH$. This proves that $hxH$ and $hyH$ are in the same $N_{H'}$-orbit. □

We now must consider 2 cases, when $m = 2$ and when $m > 2$. When $m > 2$ we have $m - 1 > 1$ so Theorem 2 gives us that there is a unique $N_{H'}$-orbit of length 1. As $H'$ sends $N_{H'}$-orbits to $N_{H'}$-orbits it must send this orbit to itself, that is it must stabilize the coset in this orbit. On the other hand when $m = 2$ we have $m - 1 = 1$ so Theorem 2 gives us that there are 2 $N_{H'}$-orbits of length 1. Thus in this case it is not immediately obvious that some coset of $H$ is stabilized.

We wish to show that it must always be the case that $H'$ stabilizes some coset of $H$ and that this implies that the Gassmann triple $(G, H, H')$ is trivial. The fact that a non-trivial faithful Gassmann triple of index $n$ cannot contain in involution with $n-2$ fixed points is a key result of this paper. The main theorem given in the introduction will follow easily from this.

**Theorem 3.** *If $(G, H, H')$ is a faithful Gassmann triple of index $n$ and if there exists an involution $\tau \in G$ such that $\chi_{G/H}(\tau) = n - 2$ then the triple $(G, H, H')$ is trivial.*

*Proof.* We begin by showing that if $H'$ stabilizes any coset of $H$, then the triple is trivial. The stabilizer of a coset $xH \in G/H$ is simply the $H$-conjugate $xHx^{-1}$. Assume $xH$ is a coset stabilized by $H'$. Then we have $H' \leq \mathrm{Stab}_G(xH) = xHx^{-1}$. But $|H'| = |H| = |xHx^{-1}|$ thus $H' = xHx^{-1}$. This proves that the triple $(G, H, H')$ is trivial.

Now we wish to show that $H'$ always stabilizes some coset in $G/H$. We have already shown above that if $m > 2$ then $H'$ stabilizes the unique $N_{H'}$-orbit of length 1. What remains is only the case when $m = 2$.

Thus assume that $m = 2$ and no coset in $G/H$ is stabilized by $H'$. We intend to derive a contradiction from this assumption by constructing an element $h' \in H'$

such that $\chi_{G/H}(h') = 0$. By condition (iii) in Proposition 2 the element $h'$ is conjugate to some element $h \in H$. Since $\chi_{G/H}$ is a character of $G$ it is constant on conjugacy classes. This gives $\chi_{G/H}(h) = 0$, but we know $h$ fixes the coset $H$. This is a contradiction.

Recall that $n = mk = 2k$; Proposition 5 then gives $|\tau^G| = k$. Each $N$-orbit contains cosets that are transposed by some conjugate of $\tau$ and there are $k$ $N$-orbits, thus there is exactly one such conjugate of $\tau$ for each $N$-orbit. Let $\tau^G = \{t_1, t_2, \ldots, t_k\}$ where $t_j$ is the conjugate of $\tau$ that transposes the two cosets in the $j^{\text{th}}$ $N$-orbit $\mathcal{O}_j$.

By Theorem 2 exactly 1 $N$-orbit decomposes into 2 $N_{H'}$-orbits of length 1. Let $\mathcal{O}_i$ be this $N$-orbit. The element $t_i$ transposes the two cosets in $\mathcal{O}_i$; since these cosets are not in the same $N_{H'}$-orbit we have $t_i \notin N_{H'}$. The set $\tau^G \cap H'$ generates $N_{H'}$ and $t_i \in \tau^G$ so $t_i \notin H'$. Proposition 5 gives $|\tau^G \cap H'| = k - 1$; thus $t_i$ is the only element of $\tau^G$ that does not lie in $H'$.

Let $xH, yH \in \mathcal{O}_i$ be the two cosets that are transposed by $t_i$. These two cosets form the two length 1 $N_{H'}$-orbits. By hypothesis $H'$ does not stabilize any coset; therefore there exists some $g \in H'$ such that $g$ moves $xH$. We must have $gxH = yH$ because $H'$ takes $N_{H'}$-orbits to $N_{H'}$-orbits and $yH$ is the only other orbit of length 1. So the coset $yH$ is also moved by $g$.

For $j \in \{1, 2, \ldots, k\}$ note that $g$ either fixes both cosets in $\mathcal{O}_j$ or moves both cosets. This is because $g$ moves both cosets in $\mathcal{O}_i$ and for $j \neq i$ if $g$ moves the orbit $\mathcal{O}_j$ then it obviously must move both cosets in the orbit. If $g$ does not move the orbit $\mathcal{O}_j$ then it can only move one coset in the orbit by sending it to the other coset in that same orbit, thus $g$ either transposes the two cosets or fixes the two cosets.

Define the set $F = \{j \in \{1, 2, \ldots, k\} \mid g \text{ fixes both cosets in } \mathcal{O}_j\}$ and let

$$\eta = \prod_{j \in F} t_j.$$

Note that $i \notin F$ so $\eta$ is a product of conjugates of $\tau$ and none of these conjugates are equal to $t_i$, thus $\eta \in H'$. Finally we define $h' = \eta g$.

Observe that a coset is moved by $\eta$ if and only if the coset belongs to an orbit $\mathcal{O}_j$ whose index $j$ is in $F$. Also a coset is moved by $g$ if and only if the coset belongs to an orbit $\mathcal{O}_j$ whose index $j$ is not in $F$. Thus every coset is moved by $h' = \eta g \in H'$ as desired. □

We end the theoretical section of this paper by using two lemmas of Beaulieu to prove our final result, that the previous theorem can be lifted to the case when $(G, H, H')$ is not faithful.

**Main Theorem.** *If $(G, H, H')$ is a Gassmann triple of index $n$ and if there exists an involution $\tau \in G$ such that $\chi_{G/H}(\tau) = n - 2$ then the triple $(G, H, H')$ is trivial.*

**Lemma 2** (Lemma 1.13 in [Bea91])**.** *Let $H \leq G$ be a subgroup and let $K \leq H$ be normal in $G$. Then $\chi_{G/H}(g) = \chi_{(G/K)/(H/K)}(gK)$ for all $g \in G$.*

**Lemma 3** (Lemma 1.14 in [Bea91])**.** *Let $(G, H, H')$ be a non-trivial Gassmann triple. Let $K$ be the common kernel of the actions $G \curvearrowright G/H$ and $G \curvearrowright G/H'$. Then $(G/K, H/K, H'/K)$ is a non-trivial faithful Gassmann triple.*

*Proof of Main Theorem.* Let $(G, H, H')$ be a Gassmann triple of index $n$ and let $\tau \in G$ be an involution satisfying $\chi_{G/H}(\tau) = n - 2$. Let $K$ be the common kernel of the coset actions. Assume that $(G, H, H')$ is non-trivial. By Lemma 3 $(G/K, H/K, H'/K)$ is a non-trivial faithful Gassmann triple and by the Third Isomorphism Theorem the index remains $n$. As $\tau$ moves two cosets $\tau \notin K$ but $\tau^2 = 1 \in K$. We conclude that $\tau K \in G/K$ is an involution and by Lemma 2 we have $\chi_{(G/K)/(H/K)}(\tau K) = n - 2$. This contradicts Theorem 3, thus the original assumption that the triple $(G, H, H')$ is non-trivial is false. □

## 4. Searching for Gassmann triples

In order to further study Gassmann triples we wish to generate a large number of non-trivial faithful examples. The computer algebra system GAP was chosen to perform a brute force search for Gassmann triples. Some of the computations involved in this search are time consuming so we have written a collection of GAP functions that perform the search and save the resulting triples to file for future study.

These functions look for non-trivial faithful Gassmann triples up to isomorphism. Given two Gassmann triples $(G_1, H_1, H_1')$ and $(G_2, H_2, H_2')$ we say the two triples are isomorphic when there exists an isomorphism $\phi\colon G_1 \to G_2$ that satisfies either $\phi(H_1) = H_2$ and $\phi(H_1') = H_2'$ or $\phi(H_1) = H_2'$ and $\phi(H_1') = H_2$. We allow the second condition because given two Gassmann equivalent subgroups $H$ and $H'$ of a group $G$ we wish to consider $(G, H, H')$ and $(G, H', H)$ to be equivalent triples.

There are two methods implemented to search for Gassmann triples. The first method uses GAP's library of transitive subgroups of the symmetric groups. This method was intended to be able to find all Gassmann triples of a given index $n$ but memory overflow problems occurred when attempting a complete search of $S_{10}$. The symmetric groups of degree 9 and less have 90 transitive subgroups [GAP06]. A total of 143 Gassmann triples were found.

The second method uses GAP's small groups library. This method searches for all Gassmann triples $(G, H, H')$ in which the group $G$ has a given order. Due to time constraints we have only searched all groups of order 200 and less. We have encountered no memory problems and believe that searching through larger orders is quite feasible. There are 6065 groups of order less than or equal to 200 [GAP06]. A total of 531 Gassmann triples were found.

Accounting for the overlap between the two methods 6088 groups were searched. In total 657 non-trivial faithful non-isomorphic Gassmann triples were found. The source code for the GAP functions is fully commented and is available online at http://www.math.lsu.edu/~jstarx/gassmann.g. The data file containing the results of the search described above is at http://www.math.lsu.edu/~jstarx/gassmann.dat.

## 5. Further Questions

Here we give two questions that may motivate further research on the topic of Gassmann triples and involutions in Gassmann triples.

5.1. **What is the minimum number of prime factors in the order of a group containing a non-trivial faithful Gassmann triple?** Let $G$ be a group whose order has the prime factorization $|G| = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Define $P(G) = \sum_{i=1}^{k} a_i$, the number of primes in $|G|$ counting duplicates. In each of the 657 examples of non-trivial faithful Gassmann triples that we have found it is always the case that $P(G) \geq 5$. Does a non-trivial faithful Gassmann triple $(G, H, H')$ with $P(G) < 5$ exist?

5.2. **Can we search for Gassmann triples by index efficiently?** Currently the program can only search for Gassmann triples within a given group. To systematically search for Gassmann triples we rely on a comprehensive library of groups. If $(G, H, H')$ is a faithful Gassmann triple of index $n$ then using the action $G \curvearrowright G/H$, we can see that $G$ is a transitive subgroup of $S_n$; thus we use the GAP library of transitive permutation groups.

Theoretically an algorithm such as the one used in the first method for finding Gassmann triples could search through $S_n$ to find all such triples of index $n$. Unfortunately our implementation fails due to memory problems at $n = 10$. With more resources we may be able to achieve a full search of $S_{10}$; but, given any finite amount of memory we can easily imagine a large enough $n$ for the current method to exhaust it. Additionally this method is slow to execute and the complexity is very high. Larger values of $n$ will quickly run into problems with processing time.

Thus the current algorithm is simply not practical. Given an arbitrary index $n$ it is not known if there is a practical method of constructing a list of all non-trivial faithful Gassmann triples of index $n$.

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Department of Maths, Louisiana State University, Baton Rouge, LA 70803

*E-mail address*: JStarx@gmail.com