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Structure of Number Theoretic Graphs

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Abstract

The tools of graph theory can be used to investigate the structure imposed on the integers by various relations. Here we investigate two kinds of graphs. The first, a square product graph, takes for its vertices the integers 1 through n, and draws edges between numbers whose product is a square. The second, a square product graph, has the same vertex set, and draws edges between numbers whose sum is a square.

We investigate the structure of these graphs. For square product graphs, we provide a rather complete characterization of their structure as a union of disjoint complete graphs. For square sum graphs, we investigate some properties such as degrees of vertices, connectedness, hamiltonicity, and planarity.

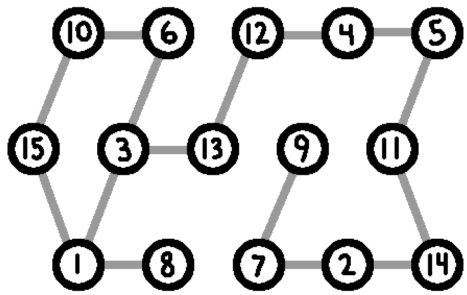

Figure 1: Hamiltonian path in the square sum graph of order 15, $S^2(15)$

1 Introduction

In Matt Parker's "Things to Make and Do in the Fourth Dimension," the question of whether the integers 1 through n can be arranged in a sequence so that the sum of each pair of adjacent elements is a square is investigated briefly (Parker 2014, 187 ff). For n from 1 to 14, n from 18 to 22, and $n = 24$, it can't be done. For all other n less than 92, it can be done, as shown in examples found algorithmically. Whether it's possible for all n greater than 25 is suspected but not certain. For $n = 15$, the smallest n for which it is possible, the sequence looks like this:

$$9, 7, 2, 14, 11, 5, 4, 12, 13, 3, 6, 10, 15, 1, 8$$

This question can be reframed as a question about graphs. We call a graph whose vertices are the integers 1 through n and whose edges indicate pairs of integers whose sums are squares, a square sum graph. The question is now a question about whether a square sum graph has a hamiltonian path. Figure 1 shows the relevant graph when $n = 15$.

This is a graph whose structure is informed by a number theoretic relation. Because questions about the structure of this graph involve both multiplication and addition, they can be tricky number theoretic questions. We also investigate a similar graph with the same vertex set but whose edges indicate pairs of integers whose products are squares. We call this a square product graph. Because this graph's structure is all about multiplication with no addition, it is easier to completely characterize.

Section 2 contains the graph theoretic and number theoretic definitions needed to do the analysis we would like to do. Section 3 contains analysis of square product graphs: their components, planarity, sizes of components, and degrees of vertices. Section 4 contains analysis of square sum graphs: the degrees of their vertices, their connectedness and planarity, and attempts to determine their hamiltonicity with some known theorems. Those attempts are without success. Nonetheless, this indicates that application of those particular theorems to this particular problem need not be attempted further.

2 Definitions

Because we will be using graphs to answer our questions about relationships between numbers, we need a number of definitions related to graphs.

Definition 1 (Chartrand and Zhang 2021, p. 2)**.** *A* ***graph*** $G = (V, E)$ *consists of a set* V *of* ***vertices*** *and a set* E *of* ***edges****. An edge is a set of two distinct vertices from* V*. If it may be unclear which graph* V *and* E *belong to, they may be written* $V(G)$ *and* $E(G)$*.*

Definition 2 (Chartrand and Zhang 2021, p. 3)**.** *The* ***order*** *of a graph* $G = (V, E)$ *is the number of vertices in the graph,* $n = |V|$*, and the* ***size*** *of* G *is the number of edges in the graph,* $m = |E|$*.*

The graphs we work with (square sum graphs and square product graphs) form families where smaller ordered graphs are always subgraphs of all larger ordered graphs in the family. Square product graphs turn out to be made up of many disjoint components. Working with subgraphs and components allows us to investigate properties of graphs such as their planarity.

Definition 3 (Chartrand and Zhang 2021, p. 10)**.** *Graph* H *is a* ***subgraph*** *of graph* G *if* $V(H) \subset V(G)$ *and* $E(H) \subset E(G)$*.*

Definition 4 (Chartrand and Zhang 2021, p. 13)**.** *A graph is* ***connected*** *if there is a path between every pair of vertices in the graph. A* ***component*** *of a graph* G *is a connected subgraph of* G *which is not a proper subgraph of any other connected subgraph of* G*.*

Square product graphs turn out to be made up of complete components. This will be useful for investigating their planarity.

Definition 5 (Chartrand and Zhang 2021, p. 19)**.** *A graph is* ***complete*** *if every pair of vertices in the graph share an edge.*

Many useful things about graphs are shown by working with degrees of vertices, for example some of the theorems regarding hamiltonicity rely on degrees of vertices.

Definition 6 (Chartrand and Zhang 2021, p. 31)**.** *The* ***neighborhood*** *of a vertex* v *is the set* $N(v) = \{u \in V : \{u, v\} \in E\}$ *and the* ***degree*** *of the vertex is* $\deg(v) = |N(v)|$*.*

Definition 7 (Chartrand and Zhang 2021, p. 31)**.** $\Delta(G)$ *and* $\delta(G)$ *denote the greatest and least degrees of all vertices in* G*, respectively.*

We are interested in whether certain types of graphs have hamiltonian cycles or paths.

Definition 8 (Chartrand and Zhang 2021, p. 12)**.** *A **path** $P = (v_1, v_2, ..., v_k)$ in a graph G is a sequence of distinct vertices such that for $1 \leq i \leq k$, $v_i \in V(G)$, and for $1 \leq i < k$, $\{v_i, v_{i+1}\} \in E(G)$. The length of the path is $k - 1$.*

Definition 9 (Chartrand and Zhang 2021, p. 12)**.** *A **cycle** $C = (v_1, v_2, ..., v_k)$ in a graph G is a sequence of vertices in which $v_1 = v_k$, but all other vertices are distinct from each other and v_1. As in a path, for $1 \leq i \leq k$, $v_i \in V(G)$, and for $1 \leq i < k$, $\{v_i, v_{i+1}\} \in E(G)$. The length of the cycle is $k - 1$.*

Definition 10 (Chartrand and Zhang 2021, p. 141)**.** *A path or cycle in a graph G is **hamiltonian** if it contains every vertex in $V(G)$. A graph is **hamiltonian** if it contains a hamiltonian cycle.*

If a graph is hamiltonian, and therefore contains a hamiltonian cycle, it necessarily contains a hamiltonian path as a subset of that cycle.

Definition 11 (Chartrand and Zhang 2021, p. 228)**.** *A graph is **planar** if it can be drawn in the plane such that no edges overlap. Such a drawing is called a **planar embedding**.*

If all components of a graph are planar, the graph is planar.

Definition 12 (Weisstein n.d.(a),Weisstein n.d.(b))**.** *A positive integer is **square-free** if it is divisible by no square other than 1. The **square-free part** of a positive integer x is the part of a x left after all square factors are divided out.*

3 Square Product Graphs

We will define and investigate graphs defined by the relationship of two numbers having a square product.

Definition 13. *If $G = P^2(n)$ is the **square product graph** of order n, then $G = (V, E)$ where*

$$V = \mathbb{N} \cap [1, n], \text{ and}$$
$$E = \{(x, y) \in V \mid x \neq y \text{ and } \exists a \in \mathbb{N}(xy = a^2)\}$$

We note that the square product graph of order n is a subgraph of the square product graph of order $n + 1$.

3.1 Components and Planarity

We discuss the general structure of square product graphs. First, we note that square product graphs have a fairly well defined structure:

Theorem 3.1. *Every component of a square product graph is complete.*

Proof. Let G be $P^2(n)$. Consider a component C of G. Suppose toward contradiction that C is not complete, that is, there is some pair of vertices in G, v and u, which do not share an edge. Since a component is connected, there is a path between v and u, and therefore there is a shortest path between v and u. Consider such a path, P. Since v and u share no edge, there is at least one vertex between them on this path, so there is at least one sequence of three consecutive vertices on P. Let these vertices be a, b, c. (Note that a and c are permitted to be u and v, but aren't necessarily.) Now, since a and b share an edge, ab is square. Similarly, bc is square. Because squares are closed under multiplication, ab^2c is square. By prime factorization, and $ab^2c/b^2 = ac$ is square. So, a and c must share an edge. But then, there is a path P' which is the same as P but skips over vertex b via the edge between a and c, and this path is one edge shorter than P. Since P was assumed to be the shortest path, this is a contradiction. So, we conclude that all components of G are complete. □

The next two theorems allow a deeper understanding of what defines each component of a square product graph.

Theorem 3.2. *A square-free number is always the smallest number in its component of a square product graph.*

Proof. Let $G = (V, E)$ be $P^2(n)$. Now consider a square-free number $u \leq n$, which is therefore a vertex in G. It is in some component, and we have shown that every component of a square product graph is complete, so every other number in the component shares an edge with u. That is, every number in the component can be multiplied by u to form a square. We represent u as its prime factorization, $\prod_{i=1}^{m} p_i$, where each prime in the factorization appears only once because u is square-free. To determine which vertices u shares an edge with, we want to know which numbers can be multiplied by u to result in a square. Since square has all even exponents in its prime factorization, we know that a number v must contain at least one of each of the primes in u's factorization in its own factorization, to bring the exponents in the product to at least 2. So the least number we can multiply u by to produce a square is u itself. Since we don't permit u to share an edge with itself, all the vertices u shares an edge with must in fact be larger than u. And since u shares an edge with every vertex in its component, all vertices in the component but be greater than u. So, u is the smallest number in its component. □

Theorem 3.3. *All numbers in a component of a square product graph share the same square-free part.*

Proof. Let $G = (V, E)$ be $P^2(n)$, and consider u, v in the same component of G. We have shown that each component of G is complete, so there is an edge between u and v. So, uv is a square. Suppose c_u and c_v are the the square-free parts of u and v respectively, and let a_u and a_v be such that $u = c_u a_u^2$ and $v = c_v a_v^2$. Then $uv = c_u c_v a_u^2 a_v^2$ is a square, and $c_u c_v$ must be square. Since c_u

and c_v are square-free, the only way their product can be a square is if they are equal. So, u and v have the same square-free part. $\square$

Since each component of a square product graph is complete and has vertices which share the same square-free part, we can consider each component to be defined by the square-free part of its vertices, and every square-free number less than or equal to n is associated to exactly one component of the square product graph of order n.

The following two theorems provide some numerical information about the structure of square product graphs.

Theorem 3.4. *The number of components in the square product graph of order n is approximately $\frac{6n}{\pi^2}$.*

Proof. The number of components in the square product graph of order n is the number of square-free numbers less than or equal to n. This is known to be approximately $\frac{6n}{\pi^2}$ (Weisstein n.d.[a]). $\square$

Theorem 3.5. *The number of isolated vertices in the square product graph of order n is approximately $\frac{9n}{2\pi^2}$.*

Proof. Every isolated vertex must be isolated because no other vertices in the graph share the same squarefree part. The smallest number with a specific squarefree part is the number itself, so the isolated vertices are are the square-free numbers greater than $n/4$. There are therefore approximately $\frac{6n}{\pi^2} - \frac{6n}{4\pi^2} = \frac{9n}{2\pi^2}$ isolated vertices in the square product graph of order n (Weisstein n.d.[a]). $\square$

Because this project was motivated by the question of whether square sum graphs are hamiltonian or not, it is worth noting that square product graphs of order greater than 1 are never hamiltonian because they are not connected.

The planarity of square product graphs is also an interesting part of their structure.

Theorem 3.6. *A square product graph is planar if and only if its order is less than* 25*.*

Proof. The square product graph of order 24 has complete components with vertices $\{1, 4, 9, 16\}$, $\{2, 8, 18\}$, $\{3, 12\}$, $\{5, 20\}$, $\{6, 24\}$, and 11 isolated vertices. Since each component is planar, the graph is planar. Figure 2 shows a planar embedding of $P^2(24)$. The square product graph of order 25 includes the complete component $\{1, 4, 9, 16, 25\}$. This component is K_5, which is known to be nonplanar. It can be seen in Figure 3. Therefore, $P^2(25)$ is nonplanar. Since this K_5 will be in every square product graph of order greater than 25 as well, no square product graph with order greater than 24 is planar.

□

3.2 Sizes of Components and Degrees of Vertices

We provide a formula and other investigation of the degrees of vertices and sizes of components in square product graphs.

Theorem 3.7. *The size of the component associated with the square-free number $c < n$ in the square product graph of order n is $\lfloor\sqrt{n/c}\rfloor$.*

Proof. Consider a square-free number c in the square product graph of order n. The component that c is in contains every number which is less than or equal to n, and which has square-free part c, that is, every number less than or equal to n which is of the form ca^2 for integer a. The size of this component is therefore $\#\{a \in \mathbb{Z}^+ \text{ s.t. } ca^2 \leq n\}$. This is equal to $\#\{a \in \mathbb{Z}^+ \text{ s.t. } a \leq \sqrt{n/c}\}$. This in turn is equal to $\lfloor\sqrt{n/c}\rfloor$. □

Theorem 3.8. *The degree of a vertex v in the square product graph of order n is $\lfloor\sqrt{n/c_v}\rfloor - 1$ where c_v is the square-free part of v.*

Proof. Because a component of a square product graph is complete, v will share an edge with every element of its component. By the previous theorem, we know that the the size of the component v is in is $\lfloor\sqrt{n/c_v}\rfloor$ where c_v is the square-free part of v. Since v shares an edge with every element of the component except itself, its degree is $\lfloor\sqrt{n/c_v}\rfloor - 1$. □

Now that we know about degrees in square product graphs, we can learn more about their structure by considering their least and greatest degrees.

Theorem 3.9. *For $G = P^2(n)$, $\Delta(G) = \lfloor\sqrt{n}\rfloor - 1$ and $\delta(G) = 0$.*

Proof. The greatest degree in G will be one less than the size of the largest component of G. The size of a component is $\lfloor\sqrt{n/c}\rfloor$ when the square-free part associated with the component is c. Component size is therefore maximized when c is minimized. The smallest square-free number less than n is 1. When $c = 1$, the component associated with c contains all square numbers less than or equal to n. There are $\lfloor\sqrt{n}\rfloor$ of these. So, the degree of all the numbers in this component is $\lfloor\sqrt{n}\rfloor - 1$, which is the greatest degree of the graph, $\Delta(G)$.

Since the set of vertices of the graph of order n is bounded above, there is a greatest square-free number in the graph. We will call it c_g. Suppose toward contradiction that its degree is not 0, that is, it has an edge. The smallest number c_g could share an edge with is $4c_g$, so $4c_g$ must be in the graph and $4c_g \leq n$. Bertrand's postulate (Hardy and Wright 2008, p. 455) provides that there is always at least one prime number between m and $2m$, so there is certainly a prime between c_g and $4c_g$. This prime would therefore be in the square product graph of order n. Since primes are square-free, this contradicts the assumption

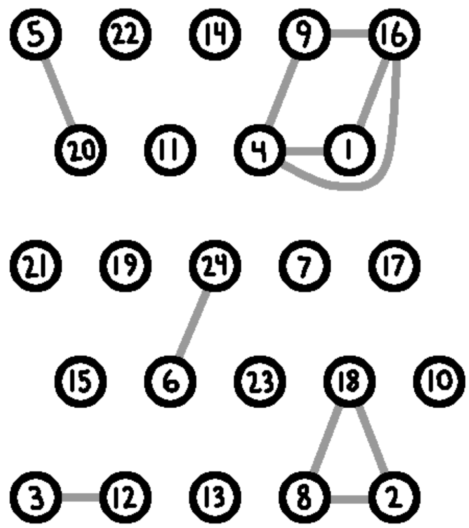

Figure 2: Planar embedding of $P^2(24)$

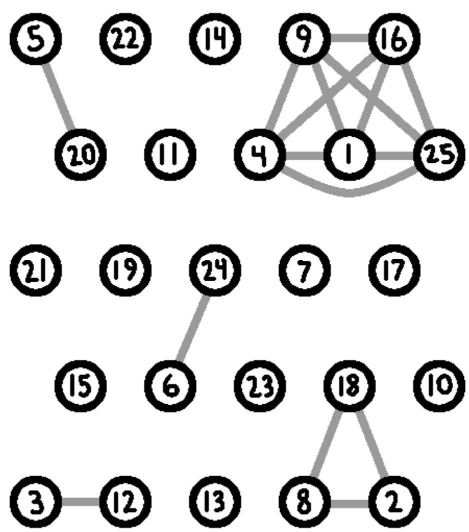

Figure 3: Complete graph of order 5 as a component of $P^2(25)$

that there is c_g is the largest square-free number in the graph of order n. So it must be that c_g does have degree 0. Since 0 is the least degree a vertex could possibly have and there is a vertex in the graph with degree 0, $\delta(G) = 0$. □

4 Square Sum Graphs

We will define and investigate graphs defined by the relationship of two numbers having a square sum.

Definition 14. *If $G = S^2(n)$ is the* ***square sum graph*** *of order n, then $G = (V, E)$ where*

$$V = \mathbb{N} \cap [1, n], \textit{ and}$$
$$E = \{(x, y) \in V \mid x \neq y \textit{ and } \exists a \in \mathbb{N}(x + y = a^2)\}$$

We note that the square sum graph of order n is a subgraph of the square sum graph of order $n + 1$.

4.1 Degrees of Vertices

We provide a formula and other investigation of the degrees of vertices in square sum graphs.

Theorem 4.1. *The degree of vertex v in $G = S^2(n)$ is $\lfloor\sqrt{v+n}\rfloor - \lfloor\sqrt{v}\rfloor$ if $2v$ is not square, and $\lfloor\sqrt{v+n}\rfloor - \lfloor\sqrt{v}\rfloor - 1$ if $2v$ is square.*

Proof. Vertex v in G shares an edge with every $u \leq n$ such that $u \neq v$ and $u + v$ is square. The least u that v could potentially share an edge with is 1, and the greatest is n. So, v has an edge for every square between $v + 1$ and $v+n$, inclusive, except for $2v$ if that is a square. So, there are $\lfloor\sqrt{v+n}\rfloor - \lfloor\sqrt{v}\rfloor$ numbers between 1 and n which, when added to v, result in a square. If v added to itself is not a square, then $\lfloor\sqrt{v+n}\rfloor - \lfloor\sqrt{v}\rfloor$ is the degree of v. If v added to itself is a square, $\lfloor\sqrt{v+n}\rfloor - \lfloor\sqrt{v}\rfloor$ overcounts edges by 1 and the degree of v is $\lfloor\sqrt{v+n}\rfloor - \lfloor\sqrt{v}\rfloor - 1$. □

We ultimately seek to understand the least and greatest degrees of vertices in square sum graphs, and to this end, we investigate how similar or different the degrees of v and $v + 1$ can be.

Theorem 4.2. *The set of differences between the degrees of vertices $u = v + 1$ and v in the square sum graph of order n is $\{2, 1, 0, -2\}$.*

Proof. The degree of v in the square sum graph of order n is the number of squares between $v + 1$ and $v + n$, inclusive. The degree of u in the same graph is the number of squares between $v + 2$ and $v + n + 1$ inclusive. Any square between $v+2$ and $v+n$ inclusive, which is not equal to $2v$ or $2v+2$, contributes an edge for both v and u. Any squares which do not contribute an edge to both v and u are of the form $v+1$, $v+n+1$, $2v$, or $2v+2$. If $v+1$ is square, it contributes an edge to v but not u. If $v+n+1$ is square, it contributes an edge to u but not v. If $2v$ is square, it contributes an edge to u but not v. If $2v+2$ is square, it contributes an edge to v but not u. Let $Q(x)$ be 1 if x is square and 0 otherwise. Then the difference between the degree of u and the degree of v is $Q(v+n+1)+Q(2v)-Q(v+1)-Q(2v+2)$. Since each of these terms is either 0 or 1, this quantity ranges between -2 and 2. Note however that for this quantity to be -2, then $v+1$ and $2(v+1)$ would both need to be square, which is not possible. So this quantity can actually only range between -1 and 2. These possible differences can be demonstrated in the square sum graphs of order 3 and 6.

In the square sum graph of order 3, the degrees of vertices 1 through 3 are $1, 0,$ and 1, respectively. Here we see differences between degrees of consecutive vertices of -1 and 1.

In the square sum graph of order 6, the degrees of vertices 1 through 6 are $1, 0, 2, 1, 1, 1$, respectively. Here we see differences between degrees of consecutive vertices of $-1, 2, -1, 0$, and 0.

Between these two graphs we see that the potential differences for degrees of consecutive vertices $-1, 0, 1$, and 2 are all achievable. □

We define a function which further allows us to understand the general behavior of degrees of larger vertices compared to smaller vertices.

Theorem 4.3. *Let $s(x, x+n-1)$ be defined for positive integers x and n such that $s(x, x+n-1)$ is the number of squares in the list of n integers: $x, x+1, x+2, ..., x+n-1$. If n is fixed, and $i > j$, then $s(i, i+n-1)$ is never more than 1 more than $s(j, j+n-1)$.*

Proof. Fix n. Consider $s(i, i+n-1)$. This is an interval with n integers in it, and some number of squares. If we add 1 to i repeatedly, we slide the interval "up" (in the positive direction), preserving its length. When doing so, we may add squares to the list when the larger end of the list reaches them. Suppose we slide this list far enough to gain k squares this way. But then, we have moved the list at least the distance of $(k-1)$ consecutive distances between squares. Since the distances between larger squares are larger (the difference between consecutive pairs of squares is always 2 more than the difference between the previous pair), this distance is certainly enough to have lost $k-1$ squares on the lower end of the sliding list. So, sliding the list up cannot result in the list having more than 1 more square than it had in a previous position. □

Now we can describe the greatest degree of a vertex in a square sum graph.

Theorem 4.4. *The greatest degree of a vertex in $G = S^2(n)$, $\Delta(G)$, is $\lfloor\sqrt{n+3}\rfloor - 1$.*

Proof. The degree of any vertex v in G is the number of squares between $v+1$ and $v+n$, inclusive, unless $2v$ is a square. That is, the number of squares in the list of n consecutive integers beginning with $v+1$, which we also now know as $s(v+1, v+n)$, unless $2v$ is a square.

Fix n and note that $s(10, 9+n)$ is less than $s(4, 3+n)$, because $s(4, 3+n)$ counts the squares 4 and 9, while $s(10, 9+n)$ does not count 4 or 9, and has only slid up 6, but differences between squares above 9 are more than 6, so it could have gained at most 1 square on the upper end of its range. So $s(10, 9+n)$ is less than $s(4, 3+n)$. Additionally, all the values $s(4+k, 3+k+n)$ for k between 1 and 5 are no more than $s(4, 3+n)$ because they do not include 4 in their count and have gained at most 1 square on the upper end of their range. Since every list of n numbers starting at 10 or greater has no more than 1 more than $s(10, 9+n)$ squares (by Theorem 4.3), and $s(4, 3+n) > s(10, 9+n)$, we can say that $s(4, 3+n)$ is greater than the number of squares in any list of n integers beginning with a larger number. It is also no less than than $s(3, 2+n)$ and $s(2, 1+n)$ since these count the same least square (4), and may lose squares on their larger end. So, the list of squares from 4 to $3+n$ is greater than the list of squares in any other list of n consecutive numbers.

The degree of every vertex v is either $s(v+1, v+n)$ or one less than that. The degree of 3 is $s(4, 3+n)$, since $2 \cdot 3$ is not square. Every other vertex has degree no greater than this, because they all have $s(v+1, v+n)$ no greater than that when $v = 3$ and their degree is no greater than $s(v+1, v+n)$. So, 3 has the greatest degree in the graph.

The degree of 3 is $\lfloor \sqrt{n+3} \rfloor - 1$. So, $\Delta(G)$, is $\lfloor \sqrt{n+3} \rfloor - 1$. □

For a few examples, the greatest degrees in the square sum graphs of order, 10, 20, 30, 40, and 100 are 2, 3, 4, 5, and 9, respectively.

Since square numbers in the list from $v+1$ to $v+n$ are less dense for higher v, the list should contain the least number of squares around $v = n$, at which point it will contain $\lfloor \sqrt{2n} \rfloor - \lfloor \sqrt{n+1} \rfloor$ squares. So, the least degree of a square sum graph, $\delta(G)$, should be around this quantity, but it may not be exact. For example, for the graphs of order 20 and 30, the minimum degree is 1 while $\lfloor \sqrt{2n} \rfloor - \lfloor \sqrt{n+1} \rfloor = 2$. On the other hand, for the graph of order 10, the minimum degree is $\lfloor \sqrt{2n} \rfloor - \lfloor \sqrt{n+1} \rfloor = 1$, and for the graph of order 40, the minimum degree is $\lfloor \sqrt{2n} \rfloor - \lfloor \sqrt{n+1} \rfloor = 2$.

4.2 Connectedness and Planarity

We discuss some more general structure of square sum graphs, their connectedness and planarity.

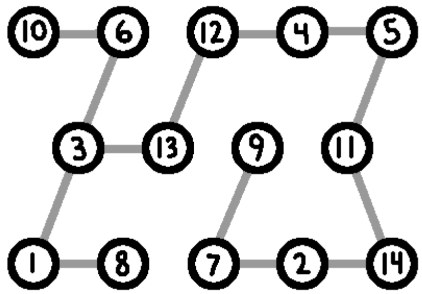

Figure 4: Connected graph $S^2(14)$

Theorem 4.5. *Square sum graphs of order* 1 *and order greater than* 14 *are connected.*

Proof. The square sum graph of order 1 is trivially connected. After that, the first connected square sum graph is that of order 14, shown in Figure 4. Now, consider vertex n in the square sum graph of order n, where $n > 14$. The degree of this vertex is either $\lfloor\sqrt{2n}\rfloor - \lfloor\sqrt{n}\rfloor$ or $\lfloor\sqrt{2n}\rfloor - \lfloor\sqrt{n}\rfloor - 1$, so it is at least $f(n) = \lfloor\sqrt{2n}\rfloor - \lfloor\sqrt{n}\rfloor - 1$. This is an increasing function, since $\lfloor\sqrt{2n}\rfloor$ grows faster than $\lfloor\sqrt{n}\rfloor$. This function's value at $n = 14$ is $f(14) = \lfloor\sqrt{28}\rfloor - \lfloor\sqrt{14}\rfloor - 1 = 5-3-1 = 1$. So this function will be at least 1 for every $n > 14$, and the degree of the vertex n in the square sum graph of order n for $n > 14$ will always be at least 1. If the square sum graph of order n is connected and the square sum graph of order $n+1$ has vertex $n+1$ with degree at least 1, then that vertex is connected at least one of the rest of the vertices, and the graph of order $n+1$ is connected. So by induction with base case at $n = 14$, square sum graphs of order greater than 14 are connected. □

Theorem 4.6. *A square sum graph is planar if and only if its order is less than* 25.

Proof. The proof that $S^2(24)$ is planar is by diagram. Figure 5 shows a planar embedding of this graph. Since square sum graphs with order less than 24 are subgraphs of $S^2(24)$, they are also all planar.

Figure 6 shows the vertex set of $S^2(25)$ partitioned into five groups. Contracting the edges within a group yields a K_5 with some edge duplication. Deleting those duplicate edges yields K_5. A minor of a graph is generated by contracting edges, deleting edges, and deleting vertices, so K_5 is a minor of $S^2(25)$. Wagner's Theorem (Chartrand and Zhang 2021, p. 250) states that a graph is planar if and only if its minors do not include K_5 nor $K_{3,3}$. Therefore, $S^2(25)$ is not planar. Since all greater-ordered square sum graphs have $S^2(25)$ as a subgraph, they are also nonplanar. □

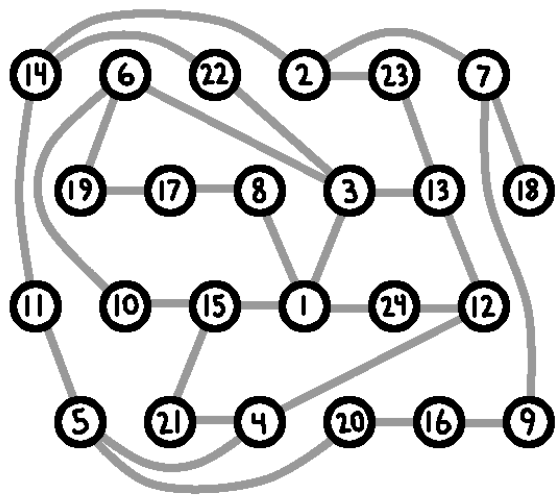

Figure 5: Planar embedding of $S^2(24)$

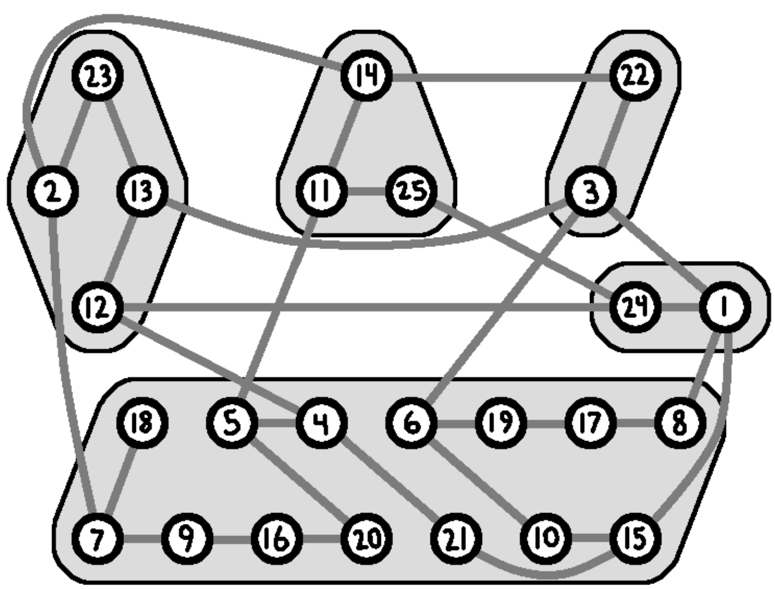

Figure 6: Partition of the vertex set of $S^2(25)$, indicating the K_5 minor of $S^2(25)$

4.3 Attempted Application of Hamiltonicity Theorems

Here we present a pair of known theorems for testing for hamiltonicity, and attempted application of them.

Theorem 4.7 (Dirac's Theorem (Chartrand and Zhang 2021, p. 148))**.** *A graph with n vertices ($n > 3$) is hamiltonian if every vertex has degree $n/2$ or greater.*

Theorem 4.8 (Ore's Theorem (Chartrand and Zhang 2021, p. 146))**.** *A graph with n vertices ($n > 3$) is hamiltonian if, for every pair of non-adjacent vertices, the sum of their degrees is n or greater.*

We know that the degree of a vertex v in the square sum graph of order n is not greater than $\lfloor\sqrt{v+n}\rfloor - \lfloor\sqrt{v}\rfloor$, which is not greater than $\sqrt{2n}$. If $n > 9$, then $n/2 > \sqrt{2n}$. So in the square sum graph of order $n > 9$, no vertices have degree $n/2$ or higher, nor do any pairs of vertices have sum n or higher. So neither Dirac's nor Ore's Theorem cannot be applied for $n > 9$.

These two theorems also cannot be applied for $n \leq 9$, as can be determined by drawing them and checking degrees. That process would also reveal that they aren't hamiltonian regardless of application of Dirac's or Ore's theorem. So these theorems are not of use in this problem.

5 Conclusion

Square sum graphs and square product graphs contain interesting structures which must be investigated using number theoretic properties. In turn, the number theoretic question of whether the integers 1 through n can be lined up so that adjacent numbers sum to squares, is best investigated using graph theory. These problems are a fine example of different fields of math being tools for answering each other's questions.

The sizes of both square product graphs and square sum graphs are not provided in this paper. The size of a graph is known to be half the sum of the degrees of the vertices, and formulas for the degrees of the vertices in both types of graphs are given. However, directly summing up the degrees of the vertices to find the size is not very informative. Further work could investigate more informative or convenient ways to describe the sizes of these graphs, or investigate the asymptotic behavior of the sizes. There are various other quantities whose asymptotic behavior could be studied.

Square sum graphs are sparse enough that it seems unlikely that an answer to the question of their hamiltonicity will come from direct application of general theorems about sizes and degrees. However when looking at the hamiltonian paths in graphs of similar orders, certain sequences of vertices often repeat themselves. This implies that a productive direction for further work may be to investigate when a hamiltonian path can be conveniently constructed from a hamiltonian path in a smaller square sum graph, thus proving hamiltonicity for all square sum graphs above a certain order by induction.

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