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CLASSIFICATION OF QUASPLATONIC ABELIAN GROUPS AND SIGNATURES

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1. Introduction

A quasiplatonic surface is a compact Riemann surface, X, which admits a group of automorphisms, G, (called a quasiplatonic group) such that the quotient space, X/G, has genus 0 and the map $\pi_G : X \to X/G$ is branched over three points. For a given genus, by using computational methods (see Theorem 2.11), we can determine all quasiplatonic groups which act on a quasiplatonic surface of that genus. Though in principle this method can be used to calculate all quasiplatonic groups, in practice it is unrealistic. Another approach is to fix a group and see what genera this group can act upon as a quasiplatonic group. In this paper we classify all Abelian groups which can act on a quasiplatonic surface.

A partial classification has previously been supplied in [2]. This partial classification provides necessary but not sufficient conditions for a group to act upon a quasipatonic surface. The classification shown in the following provides both. We accomplish this by first looking at cyclic groups with orders of a single prime power. All other Abelian groups can be built up from this case, and so can the classification of the quasiplatonic surfaces upon which they act upon.

2. Quasiplatonic Surfaces

We begin with a section that contains the definition of a quasiplatonic surface as well as the properties of these surfaces that are relevant to our classification, most of which can be found in [11].

[1]This research was completed as part of the requirements for the B.S. in Mathematics at the University of Portland under the supervision of Dr. Aaron Wootton.

Definition 2.1. If an automorphism group G acts on a surface X, then the **orbit** of a point $x \in X$ is the set of images of x by every element $g \in G$.

Definition 2.2. Consider a space X and an automorphism group G acting on X. The **quotient space**, $X/G = \{g(x)|g \in G\}$, is the set of orbits of X under the action of G.

Definition 2.3. Let G be an automorphism group acting on a surface X. Let $\pi_G : X \to X/G$ be the function where every point in X is sent to its orbit. This map is known as the **natural quotient map**.

Definition 2.4. Let G be an automorphism group acting on a surface X and let $\pi_G : X \to X/G$ be the natural quotient map. The point $x \in X$ is a **ramification point** of π_G if there exists $g \in G$ such that $g \neq e$ and $g(x) = x$. ($Note$: We denote the identity element of G as e.)

Definition 2.5. If x is a ramification point, its image $\pi_G(x)$ is called a **branch point** of π_G.

Example 2.6. Consider the surface of the sphere in $\mathbb{R}^3$, that is,

$$S^2 = \{(x, y, w) \in \mathbb{R}^3 | x^2 + y^2 + w^2 = 1\}.$$

Let $G = C_2 = \{e, g\}$, where for any $(x, y, w) \in S^2$, $e : (x, y, w) \to (x, y, w)$ and $g : (x, y, w) \to (-x, -y, w)$. By construction, G is an automorphism group acting on S^2, and all orbits not containing $n = (0, 0, 1)$ and $s = (0, 0, -1)$ are of the form $\{(x, y, w), (-x, -y, w)\}$. For n and s, $e(n) = g(n) = n$ and $e(s) = g(s) = s$. So the orbits of n and s respectively are $\{n\}$ and $\{s\}$. The set of all of these orbits is the quotient space S^2/G.

Clearly n and s are ramification points. It follows that the orbits $\{n\}$ and $\{s\}$ are branch points of π_G.

Definition 2.7. Let X be a compact Riemann surface. We call X a **quasiplatonic surface** if there exists a group G acting on X such that,

(i) X/G has genus 0, and

(ii) π_G is branched over 3 points exactly, where π_G is the natural quotient map from X to X/G.

We call the group G a **quasiplatonic group** and π_G a **quasiplatonic map**.

Remark 2.8. For the remainder of the paper, whether explicitly stated or not, all surfaces considered will be quasiplatonic surfaces.

Definition 2.9. Suppose G is a quasiplatonic group acting on a quasiplatonic surface X such that X is branched over p_1, p_2, and p_3. Suppose $x_1, x_2, x_3 \in X$ are ramification points where $\pi_G(x_1) = p_1, \pi_G(x_2) = p_2$, and $\pi_G(x_3) = p_3$. Let $n_1 = |\operatorname{Stab}(x_1)|, n_2 = |\operatorname{Stab}(x_2)|$, and $n_3 = |\operatorname{Stab}(x_3)|$, and without loss of generality assume $n_1 \leqslant n_2 \leqslant n_3$. Then, the **signature** of (G, π_G) is the triple (n_1, n_2, n_3). We call the n_i the **periods** of the signature.

Proposition 2.10. *Signature is well-defined.*

Proof. Let x and y be two ramification points such that they both map to the same branch point via the map π_G. By definition, this means that x and y lie in the same orbit. It follows that the stabilizers are conjugate in G and hence of the same order. □

The classifications that follows will make heavy use of the following theorem.

Theorem 2.11. *A group G is a quasiplatonic group for a surface X of genus $g \geqslant 2$ with signature (n_1, n_2, n_3) if and only if $n_i \geqslant 2$, the partial group presentation*

$$G = < x, y | x^{n_1} = y^{n_2} = ((xy)^{-1})^{n_3} = e, ... >$$

holds true, and

$$g(X) = 1 - |G| + \frac{|G|}{2}(3 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}).$$

(The second condition is known as the Riemann-Hurwitz formula.)

Proof. For details, see Chapter 1 of [2]. □

We finish this section with an explicit example making use of Theorem 2.11.

Example 2.12. Consider the group C_7. We can determine all surfaces for which this group is quasiplatonic. That is, we can find all signatures and corresponding genera. Since all non-identity elements of C_7 are order 7, the signature must be $(7, 7, 7)$. To see that there are corresponding elements that generate C_7, consider x and x^2 where $< x >= C_7$. Clearly $C_7 =< x, x^2 >$ where the inverse of the product x^{-3} is also order 7, so this is a generating set. Thus, $g(X) = 1 - |C_7| + \frac{|C_7|}{2}(3 - \frac{1}{7} - \frac{1}{7} - \frac{1}{7}) = 1 - 7 + \frac{7}{2}(3 - \frac{3}{7}) = 3$.

Remark 2.13. There are a limited and known number of quasiplatonic groups that can act on a surface of genus 1 or 0 quasiplatonically. A more detailed treatment of these can be found in [2]. They are as follows:

Group	Signature	Genus	Group	Signature	Genus
D_n	(2,2,n)	0	C_6	(2,3,6)	1
A_4	(2,3,3)	0	C_4	(2,4,4)	1
S_4	(2,3,4)	0	C_3	(3,3,3)	1
A_5	(2,3,5)	0			

For the remainder of this paper, unless otherwise stated, we shall assume $g(X) \geqslant 2$.

3. Previous Results

All groups to be considered in this paper will be finite. As such, the fundamental theorem of finite Abelian groups is very applicable. This, preceded with a theorem about cyclic groups, follows.

Theorem 3.1. *Let G be a finite cyclic group. Then*

$$G = C_{p_1^{n_1}} \times C_{p_2^{n_2}} \times \cdots \times C_{p_k^{n_k}}$$

where,

(i) *all* p_i *are prime numbers and*

(ii) $p_i \neq p_j$ *unless* $i = j$.

Proof. See [6], page 217. □

Theorem 3.2. (Fundamental Theorem of Finite Abelian Groups) *Let* G *be a finite Abelian group. Then*

$$G = C_{n_1} \times C_{n_2} \times \cdots \times C_{n_s}$$

where,

(i) $n_j \geqslant 2$ *for all* j, *and*

(ii) $n_{i+1} | n_i$ *for* $1 \leqslant i \leqslant s - 1$.

We call the n_i *the invariant factors of* G.

Proof. See [4], page 196. □

Besides the invariant factors of an Abelian Group, the other major aspect of Abelian groups to be used in the following classification is the concept of rank.

Definition 3.3. Given an Abelian group G, we define the **rank** of G to be equal to the minimum number of generators from G needed to generate G, or equivalently the number of invariant factors.

Referring to Theorem 3.2, this definition means a given Abelian group G satisfying this theorem has rank s.

Remark 3.4. We note that this is not the standard usage for the term rank, but since all groups we are considering are finite, this should not cause any confusion.

The following gives a list of necessary conditions for an Abelian group to act on a quasiplatonic surface as a quasiplatonic group.

Theorem 3.5. *Fix a signature* (n_1, n_2, n_3) *and let* $M = lcm(n_1, n_2, n_3)$. *There is a quasiplatonic surface* X *with quasiplatonic Abelian group* G *and signature* (n_1, n_2, n_3) *if and only if the following conditions are met:*

- (i) *G is generated by an element of order n_1 and an element of order n_2 whose product is order n_3;*
- (ii) $M = lcm(n_1, n_2) = lcm(n_1, n_3) = lcm(n_2, n_3)$*;*
- (iii) *if M is even and only one of the Abelian invariant factors of G is divisible by the maximum power of 2 dividing M, then exactly two of the periods n_i are divisible by the maximum power of 2.*

Proof. This is a modification of Theorem 9.1 in [2]. □

Example 3.6. Consider the group $G = C_{15} \times C_5$ where $< x >= C_{15}$ and $< y >= C_5$. Note that G is generated by elements x^3y and xy^4. This is apparent since their product is x^4, which generates C_{15}, and $x^3y \times x^{12} = y$, which generates C_5. The orders of these three elements are 5, 15, and 15, respectively. It follows from the above theorem that these elements generate G quasiplatonically.

The aim of the remainer of this paper is to expand upon this classification. We will be looking at (i) and finding numeric conditions that imply (i).

4. Classification of rank one Abelian quasiplatonic groups

We shall begin by classifying Quasipatonic groups that are also cyclic.

Theorem 4.1. (Harvey's Theorem) *Fix a signature* (n_1, n_2, n_3) *and let* $M = lcm(n_1, n_2, n_3)$. *There is a quasiplatonic surface* X *with quasiplatonic cyclic group* G *and signature* (n_1, n_2, n_3) *if and only if the following conditions are met:*

- (i) $|G| = M = lcm(n_1, n_2) = lcm(n_1, n_3) = lcm(n_2, n_3)$*;*
- (ii) *if M is even, then exactly 2 of the periods n_i must be divisible by the maximum power of 2 that divides $|G|$.*

Proof. This is a specific case of **Harvey's Theorem**, proved in [7]. □

The following Corollary will be useful in the Rank 2 case.

Corollary 4.2. *If $G = C_{p^k}$ and $G =< x, y >$, then p^k divides the orders of at least two of x, y, and xy.*

Example 4.3. Suppose $G = C_p$, where p is some prime $p \neq 2$. From the theorem, the signature must be (p, p, p), and it follows from the Riemann-Hurwitz formula that the genus of the surface being acted upon is $\frac{p-1}{2}$. If $p = 7$, as in Example 2.12, then the signature becomes $(7, 7, 7)$, and the genus of the surface being acted upon is $\frac{7-1}{2} = 3$, as was shown previously. If $p = 2$, by the above theorem G cannot act on a quasiplatonic surface.

Example 4.4. Suppose $G = C_{pq}$ where p and q are different primes not equal to 2. From Theorem 4.1, possible signatures are (p, q, pq), (p, pq, pq), (q, pq, pq) and (pq, pq, pq). If $p = 2$, then (p, q, pq) and (q, pq, pq) are the only possible signatures.

Example 4.5. Suppose $G = C_{45}$. Observe that $45 = 3^3 \times 5$. From the theorem, possible signatures for this group are of the form $(3^s \times 5, 3^3 \times 5^t, 3^3 \times 5)$ or $(3^s \times 5^t, 3^3 \times 5, 3^3 \times 5)$, where s could be 0, 1, 2, or 3 and t could be 0 or 1, provided that s and t are not simultaneously 0.

Example 4.6. Suppose $G = C_{30}$ where $30 = 2 \times 3 \times 5$. Let $G =< x >$, and consider $H =< x^3, x^5 >$. It follows that $((x^3)^2) = x^6 \in H$ and that $x^{-5} \in H$ by group closure. So, $x^6 \times x^{-5} = x \in H$, and $H = G$. So, G is generated by elements of orders 10 and 6. To find a potential quasiplatonic signature, we need the third element, which would be $(x^3 \times x^5)^{-1} = x^{-8} = x^{22}$, which is order 15. If the given elements are quasiplatonic generators, the signature would be $(6, 10, 15)$. When inputted into the Riemann-Hurwitz formula, we are given a value of 11. Therefore, this is a valid signature by Harvey's Theorem.

Remark 4.7. In each of the examples preceding Example 4.6, the order of the group G was one of the periods. Example 4.6 shows that this does not have to be the case if $|G|$ is divisible by at least three primes.

5. Classification of rank two Abelian quasiplatonic groups

We now move on to the rank 2 case of Abelian groups. Before proving the general case, we first need a Lemma.

Lemma 5.1. *Fix a signature* (n_1, n_2, n_3) *and let* $M = lcm(n_1, n_2, n_3)$. *There is a quasiplatonic surface* X *with quasiplatonic rank two Abelian group* $G = C_m \times C_{p^k}$ *and signature* (n_1, n_2, n_3) *where* p^k *divides* m, *only if the following conditions are met:*

(*i*) p^k *divides* n_i *for* $i = 1, 2, 3$.

(*ii*) $m = M = lcm(n_1, n_2) = lcm(n_1, n_3) = lcm(n_2, n_3)$;

(*iii*) *if* M *is even and only one of the Abelian invariant factors of* G *is divisible by the maximum power of 2 dividing* M, *then exactly two of the periods* n_i *are divisible by the maximum power of 2.*

Proof. Assume that G is a quasiplatonic group with signature (n_1, n_2, n_3). By Theorem 3.5, we know that (iii) holds and that $M = lcm(n_1, n_2) = lcm(n_1, n_3) = lcm(n_2, n_3)$ is the case. We shall first show that $m = M$. Let $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. Construct φ_i such that

$$\varphi_i : G \to C_{p_i^{k_i}}$$

where $\varphi_i(x) = x^{p_1^{k_1} p_2^{k_2} \cdots p_{i-1}^{k_{i-1}} p_{i+1}^{k_{i+1}} \cdots p_r^{k_r}}$. If we have quasiplatonic generators of orders n_1, n_2, and n_3, their images under φ_i must generate $C_{p_i^{k_i}}$. Also, observe that by construction and properties of homomorphisms, $|x| = \prod_{i=1}^{r} |\varphi_i(x)|$. By Corollary 4.2, for any i, $p_i^{k_i}$ divides two of the n_j. Thus, m divides M. Since M cannot exceed m, it follows that $m = M$.

To show (i) is the case, we shall assume otherwise, and arrive at a contradiction. Let the quasiplatonic generators of G be $x^a y^b$, $x^c y^d$ and $x^{-(a+c)} y^{-(b+d)}$ where $C_m =< x >$ and $C_{p^k} =< y >$. Without loss of generality, we shall assume that p^k does not divide the order of $x^{-(a+c)} y^{-(b+d)}$. So, p divides $a + c$ and $b + d$. Also, since these elements are generators of G, there exists integers α and β such that $(x^a y^b)^\alpha (x^c y^d)^\beta = x$. It follows that p^k divides $\alpha b + \beta d$, so, let $sp^k = \alpha b + \beta d$. Assume without loss of generality that $\alpha \geqslant \beta$ and let $\mu = \alpha - \beta$. Then, $sp^k = \beta(b+d) + \mu b$. Since p divides $\beta(b+d)$ and sp^k, it follows that p also divides μb. Suppose that p divides b. Then, p would divide d as well. Thus, the elements y^b and y^d could not possibly generate C_{p^k}. It follows that p must divide $\mu = \alpha - \beta$.

This implies $\alpha \equiv \beta \mod p$, meaning

$$1 \equiv \alpha a + \beta c \equiv \beta a + \beta c \equiv \beta(a+c) \mod p,$$

because $(x^a y^b)^\alpha (x^c y^d)^\beta = x$. Then,

$$1 \equiv \beta(a+c) \mod p.$$

By assumption, p^k does not divide the order of $x^{-(a+c)} y^{-(b+d)}$. This implies that p divides $(a+c)$, which means $(a+c) \equiv 0 \mod p$. So,

$$\beta(a+c) \equiv \beta \times 0 \equiv 0 \mod p.$$

But, this implies that

$$0 \equiv 1 \mod p,$$

which is clearly a contradiction. Thus, (i) holds.

$\square$

This Lemma allows us to generalize to the general rank two case.

Theorem 5.2. *Fix a signature (n_1, n_2, n_3) and let $M = lcm(n_1, n_2, n_3)$. There is a quasiplatonic surface X with quasiplatonic rank two Abelian group $G = C_{m_1} \times C_{m_2}$ with signature (n_1, n_2, n_3) where m_2 divides m_1 if and only if the following conditions are met:*

- (i) *m_2 divides n_i for $i = 1, 2, 3$.*
- (ii) *$m_1 = M = lcm(n_1, n_2) = lcm(n_1, n_3) = lcm(n_2, n_3)$;*
- (iii) *if M is even and only one of the Abelian invariant factors of G is divisible by the maximum power of 2 dividing M, then exactly two of the periods n_i are divisible by the maximum power of 2.*

Proof. First, assume that G is a quasiplatonic group with signature (n_1, n_2, n_3). It follows from the argument in Lemma 5.1 that $m_1 = M$. By Theorem 3.5, we need only show that (i) is the case. Let $m_1 = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and $m_2 = p_1^{h_1} p_2^{h_2} \cdots p_r^{h_r}$. Note that by assumption, $h_i \leqslant k_i$ for all i. Choose x and y such that $C_{m_1} =< x >$ and $C_{m_2} =< y >$. Let p_i be arbitrary. Define a homomorphism φ by

$$\varphi : G \to C_{m_1} \times C_{p_i^{h_i}}$$

where $\varphi(x) = x$, $\varphi(y) = y^c$ and

$$c = p_1^{k_1} p_2^{k_2} \cdots p_{i-1}^{k_{i-1}} p_i^{k_i - h_i} p_{i+1}^{k_{i+1}} \cdots p_r^{k_r} = \frac{m_1}{p_i^{h_i}}.$$

Suppose g_1, g_2, and g_3 are a quasiplatonic generating set of G. Then, $\varphi(g_1)$, $\varphi(g_2)$, and $\varphi(g_3)$ will generate $C_{m_1} \times C_{p_i^{h_i}}$. By the previous lemma, $p_i^{h_i}$ must divide the order of all three elements, $\varphi(g_1)$, $\varphi(g_2)$, and $\varphi(g_3)$. Thus, $p_i^{h_i}$ will divide the orders of g_1, g_2, and g_3. Since this will be true for every prime power that divides m_2, (i) follows.

We will now assume that the conditions listed above are true. Choose n_1, n_2, and n_3 such that the conditions are met. By Theorem 3.5, we only need to show that G is generated by elements of orders n_1 and n_2 whose product is order n_3. Let $G = C_{m_1} \times C_{m_2} =< x_1, x_2, ..., x_r > \times < y_1, y_2, ..., y_r >$ where

(i) $m_1 = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and $m_2 = p_1^{h_1} p_2^{h_2} \cdots p_r^{h_r}$;

(ii) $< x_i >= C_{p_i^{k_i}}$ and $< y_i >= C_{p_i^{h_i}}$.

We claim the elements $g_1 = \prod_{i=1}^r x_i^{a_i} y_i$, $g_2 = \prod_{i=1}^r x_i^{b_i} y_i^{p_i^{h_i - 1}}$ and $g_1 g_2 = \prod_{i=1}^r x^{a_i + b_i}$ have orders n_1, n_2 and n_3 respectively. To see that these elements do have the stated orders, first observe that for every p_i dividing $|G|$, $C_{p_i^{k_i}}$ and $C_{p_i^{h_i}}$ are subgroups of G. From (ii) we know that $p_i^{k_i}$ divides two of the n_j, and that the third is divisible by p^t but not p^{t+1} for some t such that $h_i \leqslant t \leqslant k_i$. By Corollary 4.2, we may choose two elements to generate $C_{p_i^{k_i}}$ such that of these generators and their product, two are order $p_i^{k_i}$ and the other is order p^t. We can label these three elements as $x_i^{a_i}$, $x_i^{b_i}$ and $x^{a_i+b_i}$, where $|x_i^{a_i}|$ divides n_1, $|x_i^{b_i}|$ divides n_2, and $|x^{a_i+b_i}|$ divides n_3. Since $|y_i|$ and $|y_i^{p_i^{h_i-1}}|$ divide n_1 and n_2, the claim follows.

Consider $H =< g_1, g_2 >$. We need only show that $H = G$. We know that $p_i^{k_i}$ divides one of the n_j for all i.We know that $C_{m_1} \leqslant H$. To see that $G \leqslant H$, we only need to show that $C_{m_1} \times C_{p_i^{h_i}} \leqslant H$ for all i. Choose t_i such that $p_i^{t_i}$ is the highest power of p_i dividing n_1. Then,

$$(g_1 x_i^{-a_i})^{\frac{n_1}{p_i^{t_i}}} = y_i^{\frac{n_1}{p_i^{t_i}}},$$

which has the same order as y_i, which is $p_i^{h_i}$. Thus,

$$C_{m_1} \times < y_i^{\frac{n_1}{p_i^{t_i}}} >= C_{m_1} \times C_{p_i^{h_i}} \leqslant H.$$

Since this is true for all i, then $G = H$ and the proof is complete.

□

Example 5.3. Suppose $G = C_p \times C_p$ for $p \neq 2$. Then it follows from the theorem that the signature must be (p, p, p), where the surface that G acts upon has genus $\frac{p^2-3p+2}{2}$. In the case that $p = 2$, it follows from the theorem that this group cannot generate a quasiplatonic group.

Example 5.4. Suppose $G = C_3 \times C_9$. The only prime dividing $|G|$ is 3. Thus, from the theorem, the signature can only be $(3, 9, 9)$ or $(9, 9, 9)$. The genus of the surface being acted upon is 7 or 10, respectively, according to the Riemann-Hurwitz formula.

Example 5.5. Suppose $G = C_6 \times C_{12}$. It follows from Theorem 5.2 that the signature of (G, π_G) is $(6, 12, 12)$, where the genus of the surface being acted upon is 25.

Example 5.6. Suppose $G = C_{210} \times C_5$ where $210 = 2 \times 3 \times 5 \times 7$. Let $C_{210} =< x >$ and $C_5 =< y >$, and consider $H =< y^4x^2, yx^7 >$. So, $(y^2x^2 \times yx^7)^{-1} = x^{201} \in H$. Also, $(y^4x^2)^5 = x^{10} \in H$. Thus, $x^{201} \times x^{10} = x \in H$. So, $C_{210} \subseteq H$. Then,

$yx^7 \times x^{213} = y \in H$ implies that $H = G$. The Riemann-Hurwitz formula gives a value of 496 from the signature $(30, 70, 105)$.

6. Classification of rank three and higher Abelian quasiplatonic groups

The classification of Abelian quasiplatonic groups with rank $\geqslant 3$ is simply that there are not any such groups.

Theorem 6.1. *There are no Abelian quasiplatonic groups with rank greater than or equal to 3.*

Proof. From Theorem 2.11, a quasiplatonic group can be completely generated by only two elements. If the rank of a group is greater than or equal to 3, then it cannot possibly be generated by only two elements, since by definition of rank it requires a minimum of three generators. $\square$

7. conclusion

This paper has completely classified Abelian groups based on what quasiplatonic surfaces they act upon. As one can imagine, it is not as simple to classify non-Abelian groups. While some types of non-Abelian quasiplatonic groups have been classified, it would be worth further research to attempt a complete classification.

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