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The Game of Life on the Hyperbolic Plane

Yuncong Gu

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Abstract

In this paper, we work on the Game of Life on the hyperbolic plane. We are interested in different tessellations on the hyperbolic plane and different Game of Life rules. First, we show the exponential growth of polygons on the pentagon tessellation. Moreover, we find that the Group of 3 can keep the boundary of a set not getting smaller. We generalize the existence of still lifes by computer simulations. Also, we will prove some propositions of still lifes and cycles. There exists a still life under rules $B1$, $B2$, and $S3$.

Contents

1 Introduction

The Game of Life is a famous cellular automaton devised by John Horton Conway in 1970. He initially wanted to design an interesting and unpredictable cellular automation with some configurations that can go forever, but not a cycle, and others can exist for a long time before dying. The universe of Conway's Game of Life was the tessellation of the Euclidean plane by squares. He chose a specific rule to make sure the existence of still lives, cycles, and spaceships (configurations can go forever but not a cycle). Game of Life can be used to simulate the behavior of cells. However, cells in the real world are much more complicated. Therefore it is not a helpful tool for biologists. Apart from the simulation purpose, Conway's Game of Life is still very interesting for people who are interested in mathematics and cellular automation. Here comes a question. Why are we interested in the hyperbolic plane? NP problems can be solved in polynomial-time in the space of cellular automata in the hyperbolic plane. (Margenstern, 2001)

1.1 Conway's Game of Life

First, we start with the universe of Conway's Game of Life. It is an orthogonal grid of square cells and each cell has two states, alive or dead. See Figure 1 for an example. The yellow cell in the center is alive or populated and others are

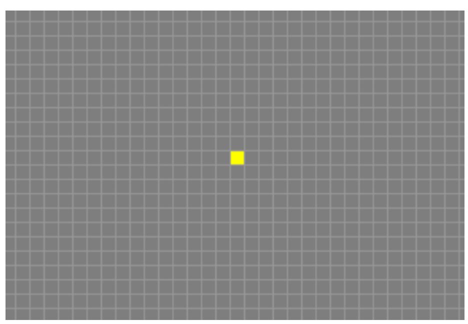


Figure 1: Universe of Conway's Game of Life (Martin, n.d.)

dead or unpopulated. Conway also carefully chose the rules of Game of Life to meet these criteria (Wolfram, 2002, page 877).

1. There should be no explosive growth.
2. There should exist small initial patterns with chaotic, unpredictable outcomes.
3. There should be potential for von Neumann universal constructors.
4. The rules should be as simple as possible, whilst adhering to the above constraints.

Then he chose the rules:

1. If an alive cell has 2 or 3 neighbors, it survives, otherwise it dies.
2. If an unpopulated cell has 3 neighbors it becomes populated.

The three most common patterns are still life, cycle, and spaceship. People are interested in finding different variations of these three patterns, especially for spaceships, which are configurations that can go forever but are not a cycle. There are several examples of these three patterns. These three patterns are essential for us to discuss on the hyperbolic plane. See Figure 2 for some examples. There are still some other cycles (oscillators) with different periods. Some oscillators have the only period of 2; others have a period of 30. People also find some huge patterns, and they combine these huge patterns to construct some crazy things. For an example, Figure 3 looks like a rocket basement keeping shooting spaceships. The thing in the middle is the spaceship. This kind of configuration is called the gun. A gun is a stationary pattern that repeatedly emits spaceships forever. If a gun can emit gliders, we call it a glider gun. (*Gun*, n.d.) Figure 4 is a billboard. On the right, there is a gun that keeps emitting the word “GOLLY”. Here the word “GOLLY” is a spaceship moving to the left.

1.2 Universal Computation

Turing machine is a hypothetical computing device capable of storing information and responding to computational questions, used in mathematical studies of computability (*Turing machine*, 2014). In computability theory, a system of data-manipulation rules (such as a computer's instruction set, a programming language, or a cellular automaton) is said to be Turing-complete or computationally universal if it can be used to simulate any Turing machine. (*Turing completeness*, n.d.) A logic gate is an idealized or physical electronic device implementing a Boolean function, a logical operation performed on one or more binary inputs that produce a single binary output. (Jaeger, 1997)
Next, we focus on the connection between the Game of Life and universal computation. There is a very special and interesting pattern in Conway's Game of Life, called a Gosper's glider gun. See Figure 5 for an example. The glider gun is on the top of Figure 5 and it can continuously shoot gliders. Gosper was the first one to find an infinitely growing pattern and won a price. This pattern is the key point of universal computation. We can use the glider gun and glider to construct all three types of logic gates: NOT gate, AND gate and OR gate. In logic, a functionally complete set of logical connectives or Boolean operators is one that can be used to express all possible truth tables by combining members of the set into a Boolean expression. (Enderton, 2001) After we construct these three logic gates, we can use them to construct all the other types of logic gates. First, we start with the NOT gate. See Figure 6. The glider gun can keep shooting gliders and two gliders can cancel each other. We can use this property to construct the NOT gate. Then the AND gate and OR gate can also be constructed. See Figure 7 and Figure 8.

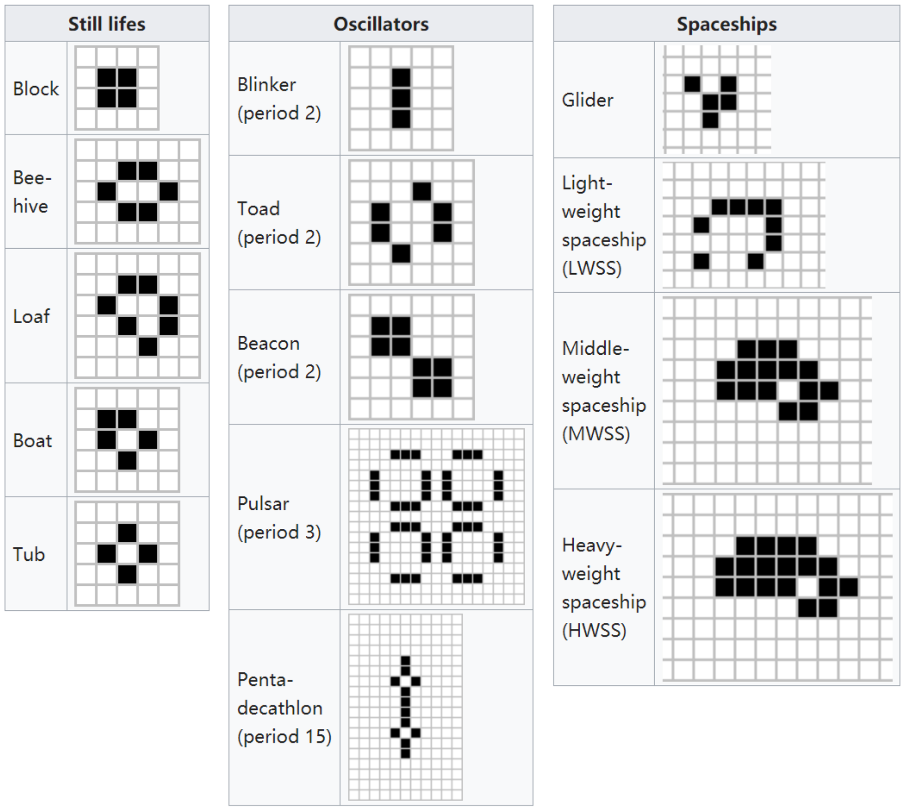


Figure 2: Examples of still lifes, cycles and spaceships (Summers, 2013)

1.3 Hyperbolic Tessellation

The hyperbolic plane is not common in real life. It exists on the surface of saddles. (*Hyperbolic geometry*, n.d.) We care about it because of some special reasons. The cellular automaton on the hyperbolic plane can help us to solve NP problems in polynomial time. The 3-SAT problem can be solved in quadratic time by constructing a cellular automaton based on the pentagonal tessellation. In this cellular automaton, the authors label each edge of the pentagon and represent each cell by two types of nodes. Then they construct the Fibonacci tree to represent the pentagonal tessellation by edges and nodes. For the rule of this cellular automaton, there are more than twenty rules and four states. We can check the propagation of signal by the transition table in their paper. This cellular automaton is quite different from Conway's Game of Life. After constructing the cellular automaton, they extend this result to solve NP problems in polynomial time. (Margenstern, 2001)

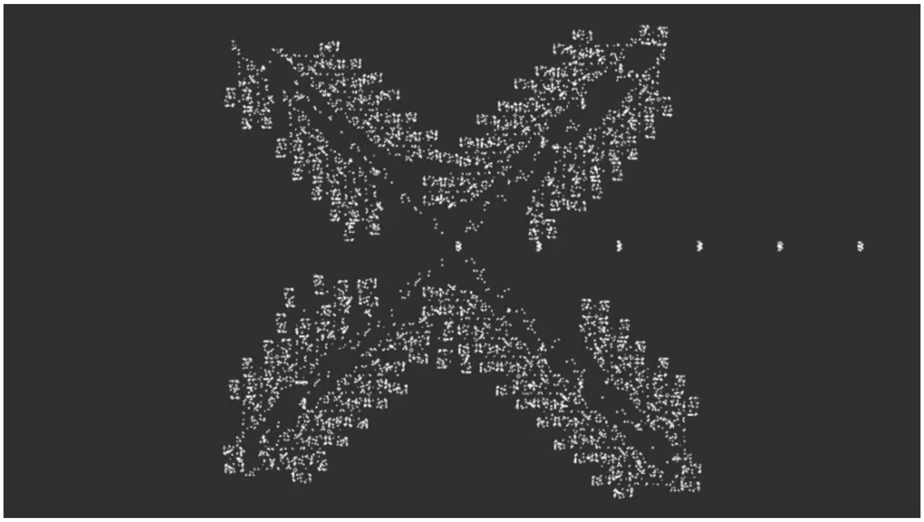


Figure 3: Rocket basement (Boingo, 2011)

Definition 1. *(Joyce, 1994) A regular* tessellation*, or tiling, is a covering of the plane by regular polygons so that the same number of polygons meet at each vertex.*

The hyperbolic geometry has a different parallel postulate. For a given line, there are at least two lines through a given point that parallel to the given line. This is the main difference between the Euclidean geometry and the hyperbolic geometry. This difference also causes some different properties of triangles.

Proposition 1.1. *(Stothers, n.d.) The sum of angles of a hyperbolic triangle is less than π.*

For the proof of this proposition, see (Stothers, n.d., Section "Hyperbolic segments and triangles").
There are three tessellations for the Euclidean Plane: Triangles meet six at each vertex ({3,6}), squares meet four at each vertex ({4,4}) and hexagons meet three at each vertex ({6,3}), where {3,6}, {4,4} and {6,3} are called Schläfli symbols. There are infinitely many tessellations on the hyperbolic plane. For a tessellation $\{n, m\}$, if $\frac{1}{n} + \frac{1}{m} < \frac{1}{2}$ then the tessellation is hyperbolic, if $\frac{1}{n} + \frac{1}{m} = \frac{1}{2}$ then the tessellation is Euclidean, if $\frac{1}{n} + \frac{1}{m} > \frac{1}{2}$ then the tessellation is elliptic. We can proof these by the sum of angles in each polygons. There are m polygons meet at a vertex for $\{n, m\}$. So each angle is $\frac{360}{m}$ degrees and the sum of angles is $\frac{360n}{m}$ degrees. We can also calculate the sum of angles by breaking polygons into triangles. The angle sum is exact $180(n-2)$ degrees in the Euclidean plane, less than this in the hyperbolic plane and more than this in the elliptic plane. After solving the inequalities between $\frac{360n}{m}$ and $180(n-2)$, we will get $\frac{1}{n} + \frac{1}{m}$ as our criteria.

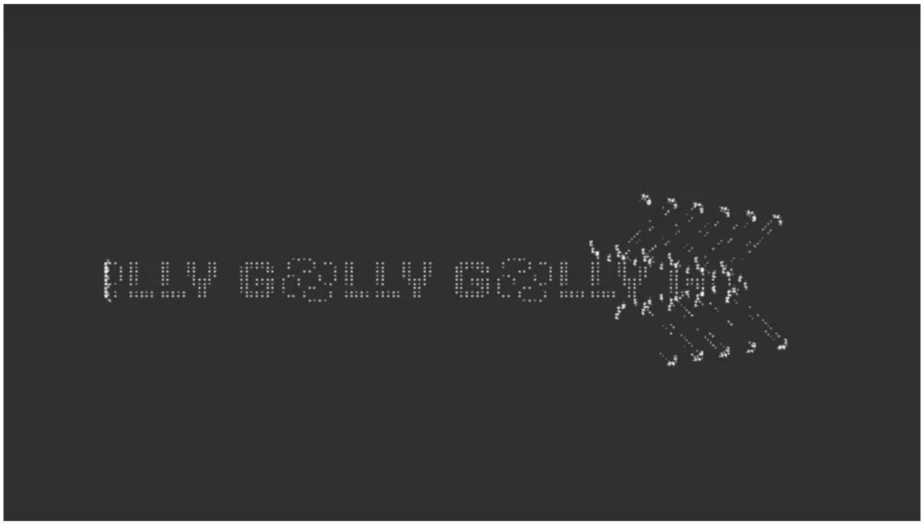


Figure 4: Billboard (Boingo, 2011)

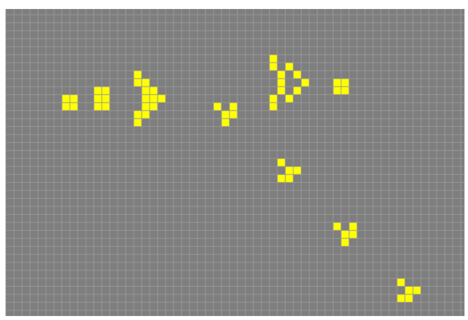
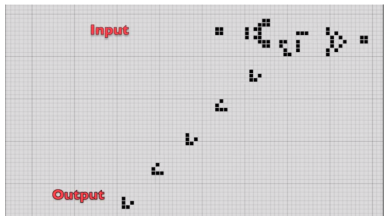
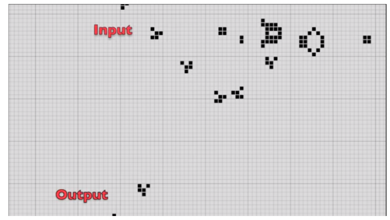


Figure 5: Gosper's glider gun (Martin, n.d.)

In this paper, the pentagon tessellation is {5,4}. We prove the main results by focusing on the geometry of tessellations. In section 2, there are many formal definitions. In section 3, we will show the exponential growth of polygons in the hyperbolic plane. In section 4, we prove by contradiction that the boundary of any cycles can never change on {5,4} with the regular Conway's Game of Life rules. We also use the pentagrid-0.2 software (Shintyakov, 2014a) to find that the group of 3 can keep the boundary of set A not getting smaller if A is not a block. In section 5, we keep working on {5,4} but with different rules. We connect 2 consecutive cells with Pascal's triangle. We prove by induction $2t_j = t_{2^i+j}$ for $0 \leq j < 2^i$ where t_n be the sum of 1's in the n_{th} level of Pascal's triangle mod 2. In section 6, we vary the tessellations and rules in order to generalize the existence of the still life. We prove for $m \geq 4$ with rule $Bi/S(m-1)$ where $i > 2$, a block is a still life. For rule $B1$ we prove if $mn - 2n - 3m + 5 \geq 1$

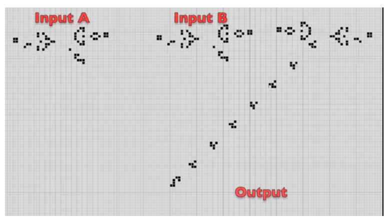


(a) Input=0, Output=1

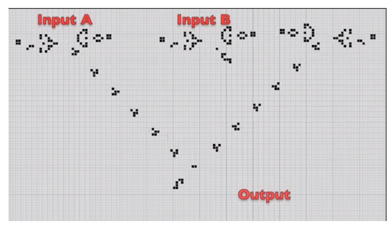


(b) Input=1, Output=0

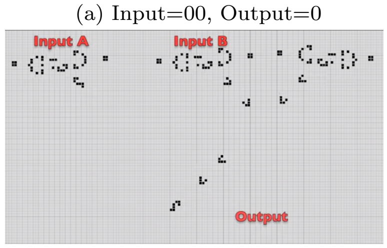
Figure 6: NOT gate (Bellos, 2014)



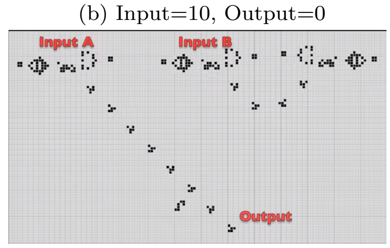
(a) Input=00, Output=0



(b) Input=10, Output=0



(c) Input=01, Output=0



(d) Input=11, Output=1

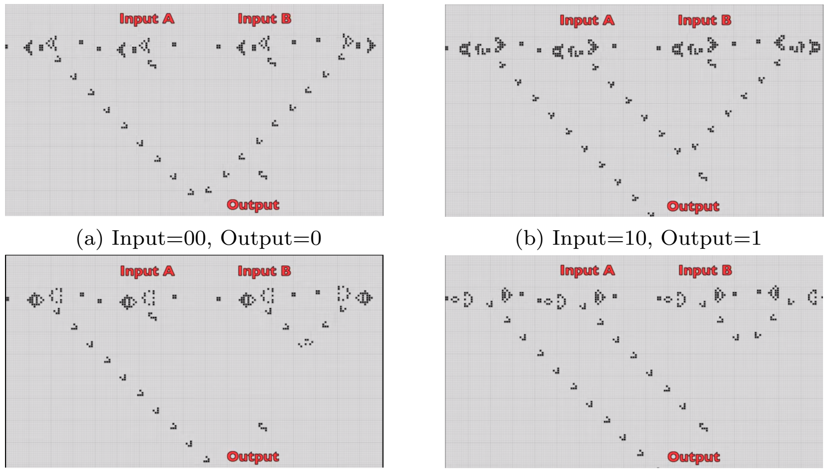
Figure 7: AND gate (Bellos, 2014)

then there is no still life. When $n = 3$, we prove there exists a still life by showing a case. For rule $S3$, we prove if $n \geq 7$, $m = 3$ then there is no still life by analyzing the geometry of the tessellation. For rule $B2$, we have a conjecture about cycles. We believe there are only two types of cycles with $n = 8, m = 3$, and rule $B2/S3$. We also provide several cases of still lifes with rule $B2$. In the conclusion section, we discuss the future works including conjectures and strobing rules.

2 Definitions

Definition 2. *n is the number of sides of the polygons in the tessellation. m is the number of the polygons that meet at each vertex.*

Definition 3. *If two cells share a common edge or vertex, we say one is the* neighbor *of the other.*



(a) Input=00, Output=0

(b) Input=10, Output=1

(c) Input=01, Output=1

(d) Input=11, Output=1

Figure 8: OR gate (Bellos, 2014)

Definition 4. *For polygon cells that are populated, each cell with i neighbors survives, otherwise, it dies. We use the notation Si to denote it. For polygon cells that are unpopulated, each cell with j neighbors becomes populated. We use notation Bj to denote it.*

See Figure 10 for an example.

Example 2.1. *We show the first two generations of a set of cells. Note, $S23$ means each cell with 2 or 3 neighbors survives, otherwise it dies. See Figure 10.*

Definition 5. *If two cells share a common edge or vertex, we say they are* connected*. Moreover, if a and b are* connected *and b and c are* connected*, then a and c are also* connected.

See Figure 11 for an example.

Definition 6. *In a set of cells A such that any two cells in A are connected, the* distance *of two cells is the minimum number of cells connect one cell to the other.*

See Figure 12 for an example. We can use $d(a, b)$ to denote the distance between a and b.
To pick a cell to be the center is important. We can use the center to define the level and it is also the starting point of our proofs in the rest of the sections.

Definition 7. *We can pick any cell on the plane to be the center i.e.* level 0. Level *n is the set of cells with distance n to the center.*

See Figure 13 for an example.

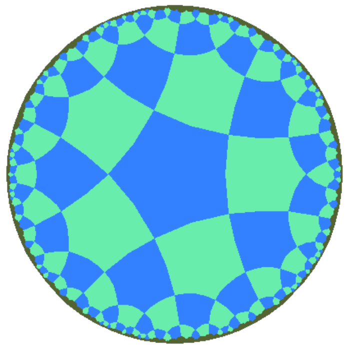
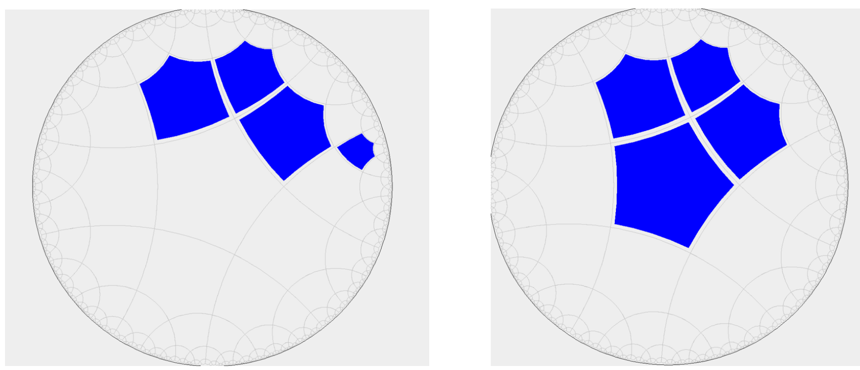


Figure 9: An example of a tessellation of hyperbolic plane with $n = 5$ and $m = 4$ (Joyce, 1994)



(a) Generation 1

(b) Generation 2

Figure 10: First two generations of a set of cells under the rule B3/S23 (Figure generated by pentagrid-0.2)

Definition 8. *The* boundary *of level n is the set of edges between level n and level $n+1$.*

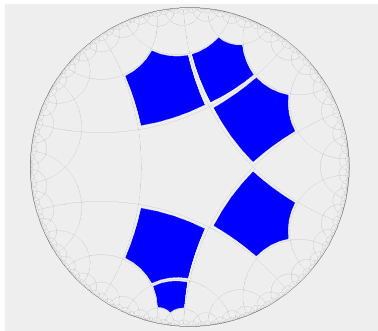
Definition 9. *For a set A, let $a \in A$ to be the center such that the maximum distance between $b \in A$ and a is k. Then the* boundary of A with center a *is set of edges between level k and level $k+1$.*

Example 2.2. *Suppose we have a set of cells A, such that the maximum distance between any cells and center in A is 3, then the boundary of set A with a given center is the set of edges between level 3 and level 4.*

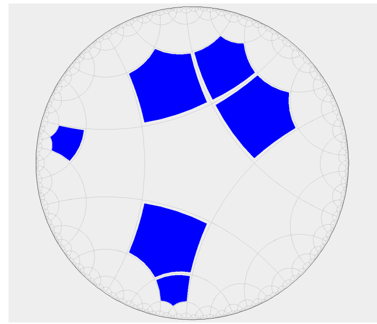
Definition 10. *We start with a set of cells A and after k generations those cells go back to set A. Set A is a* cycle *of k.*

See Figure 14 for an example.

Definition 11. *A cycle of 1 is also called a* still life*.*



(a) Connected cells



(b) Not connected cells

Figure 11: Examples of connected and not connected cells (Figure generated by pentagrid-0.2)

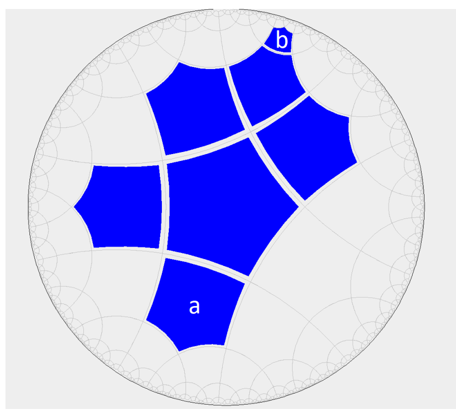


Figure 12: $d(a, b) = 3$ (Figure generated by pentagrid-0.2)

See Figure 15 for an example.
These are all the basic and most commonly used definitions of the Game of Life on the hyperbolic plane. There are still some other definitions and we will mention them when we need to use them. In section 3, we are going to explore the difference between the Euclidean plane and the hyperbolic plane by counting the number of cells.

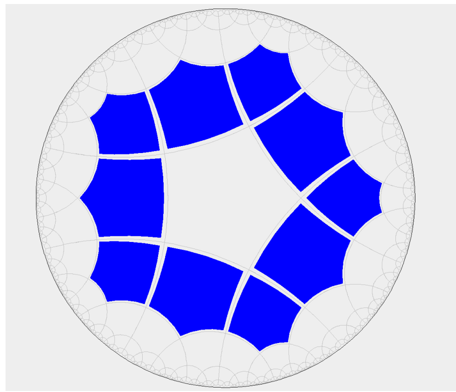
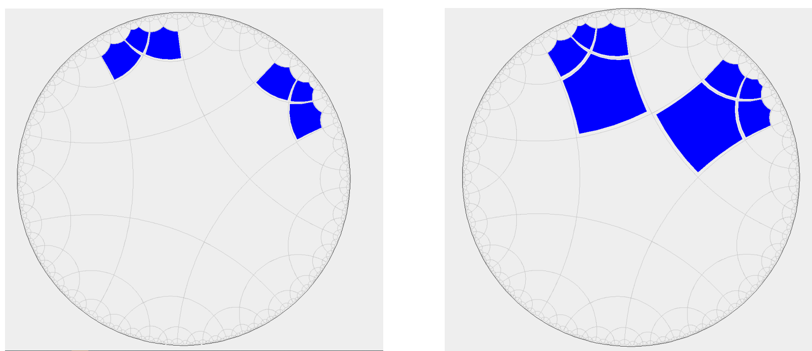


Figure 13: An example of level 1 (Figure generated by pentagrid-0.2)



(a) Generation 1 of a beacon

(b) Generation 2 of a beacon

Figure 14: An example of a Beacon (Figure generated by pentagrid-0.2)

3 Number of Sides and Cells

For the Euclidean plane, if we cover the plane with uniform squares, there are $(2n-1)^2$ total cells located within the n_{th} level. However, now we focus on the hyperbolic plane and we cover the plane with regular polygons. The number of cells within the n_{th} level will increase exponentially. For example, the total number of cells in the 100_{th} level of the Euclidean plane is 39601. There are 165453271441970579549508184324937692830004526480505567 8601 (Shintyakov, 2014b) cells in the hyperbolic plane. In next two subsections, we will count the number of sides and cells for the pentagon tessellation.

3.1 Sides of the n_{th} Level

Definition 12. *On the boundary of the n_{th} level, the edge of a pentagon is named as a* short side *of the level. A line that is made up of two edges is named as a* long side *of the level. The vertex intersected by two sides of a level is named*

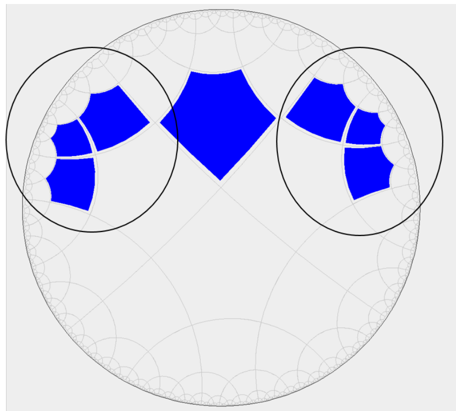


Figure 15: An example of a still life (Figure generated by pentagrid-0.2)

as a corner.

See Figure 16 for an example.

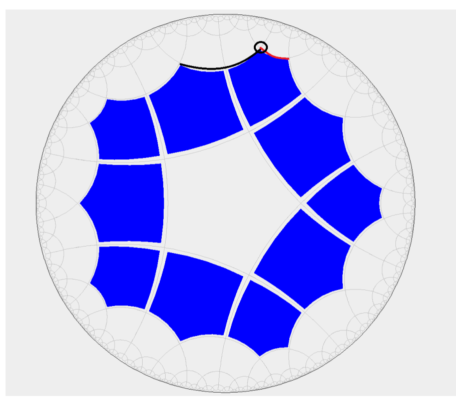


Figure 16: An example of a short side, a long side and a corner. (Figure generated by pentagrid-0.2)

Let a_n be the number of sides of the n_{th} level. First we focus on the $(n-1)_{th}$ level, there are a_{n-1} corners and a_{n-2} short sides. It follows there are $a_{n-1} - a_{n-2}$ long sides in the $(n-1)_{th}$ level. For each corner in the $(n-1)_{th}$ level, it follows one side is in the n_{th} level. For each short side in the $(n-1)_{th}$ level, it follows two sides are in the n_{th} level. For each long side in the $(n-1)_{th}$ level, it follows three sides are in the n_{th} level. Thus, we have a recursive relation, $a_n = a_{n-1} + 2a_{n-2} + 3(a_{n-1} - a_{n-2})$, i.e. $a_n = 4a_{n-1} - a_{n-2}$.
So we have the second-order recursive relation.

$$a_n = 4a_{n-1} - a_{n-2}, n \geq 3, a_1 = 5, a_2 = 15$$

The characteristic equation $x^2 - 4x = 1 = 0$ has roots $2 + \sqrt{3}$ and $2 - \sqrt{3}$. Consequently, $a_n = c_1(2+\sqrt{3})^n + c_2(2-\sqrt{3})^n$, $n \geq 1$. By the initial conditions, $5 = c_1(2 + \sqrt{3}) + c_2(2 - \sqrt{3})$ and $15 = c_1(2 + \sqrt{3})^2 + c_2(2 - \sqrt{3})^2$, we have $c_1 = \frac{5}{2} - \frac{5}{6}\sqrt{3}$ and $c_2 = \frac{5}{2} + \frac{5}{6}\sqrt{3}$. Consequently we have

$$a_n = \left(\frac{5}{2} - \frac{5}{6}\sqrt{3}\right)(2+\sqrt{3})^n + \frac{5}{2} + \frac{5}{6}\sqrt{3}(2-\sqrt{3})^n, n \geq 1. \tag{1}$$

3.2 Cells of the n_{th} Level

Let b_n be the number of cells on the n_{th} level. We can use the number of sides to help us to represent the number of cells on the n_{th} level. For each corner in the $(n-1)_{th}$ level, it follows one cell is in the n_{th} level. For each short side in the $(n-1)_{th}$ level, it follows one cell is in the n_{th} level. For each long side in the $(n-1)_{th}$ level, it follows two cells are in the n_{th} level. Thus, we have a recursive relation, $b_n = a_{n-1} + a_{n-2} + 2(a_{n-1} - a_{n-2})$, i.e. $b_n = 3a_{n-1} - a_{n-2}$. Then we can plug in the solution of a_n to find b_n and sum b_n to find the total number of cells.

$$b_n = \frac{15 + 10\sqrt{3}}{21 + 12\sqrt{3}}(2+\sqrt{3})^n + \frac{-15 + 10\sqrt{3}}{-21 + 12\sqrt{3}}(2-\sqrt{3})^n \tag{2}$$

Then we find the total number of cells by the summation of two geometric sequences.

$$\sum_{i=1}^{n} b_i = \left(\frac{15 + 10\sqrt{3}}{21 + 12\sqrt{3}}\right)\frac{2+\sqrt{3}-(2+\sqrt{3})^{n+1}}{-1-\sqrt{3}} + \left(\frac{-15 + 10\sqrt{3}}{-21 + 12\sqrt{3}}\right)\frac{2-\sqrt{3}-(2-\sqrt{3})^{n+1}}{-1+\sqrt{3}} \tag{3}$$

The cells on the hyperbolic plane grow exponentially. The Game of Life will have more fun on the hyperbolic plane than on the Euclidean plane. In the next section, we work on the cycles for the pentagon. We try to find some general properties of the cycles.

4 Group of 3

We found that for each cycle, there must be at least two groups of 3. See Figure 15 for an example. There are two groups of 3 inside those two circles. One is on the left and the other is on the right. We find one conjecture and two propositions for groups of 3.

Definition 13. *In the pentagon tessellation,* 4 *pentagons meet in each vertex. If* 3 *of them are alive, we call this configuration the* Group of 3.

Proposition 4.1. *The boundary of any set of cells can never become larger under rule B3/S23.*

Proof. For all cells located on the $(n+1)_{th}$ or greater level of the patterns, they can only have at most two neighbors in the n_{th} level. Thus, no cell will become alive and the boundary can never become larger. □

Proposition 4.2. *The boundary of any cycles can never change under the rule B3/S23.*

Proof. Suppose, for contradiction, we have a cycle of n such that k_i be the level of boundary of the i_{th} generation. If there exists a k_i such that $k_i < k_j$ for $j < i$, then $k_m \leq k_i$, for $m > i$. It follows $k_{j+n} \leq k_i < k_j$. Contradiction!

□

Conjecture 4.1. *If the boundary of set A does not get smaller and it is not a block, there is a group of 3 under the rule B3/S23.*

We think that the group of 3 just likes a fix point of a set of cells. It can keep the boundary of a cycle. However, we still cannot find a proper way to prove it.

5 B2/S3

Under rule B2/S3, cells can be born in the outer levels, i.e. the boundary of the cells can become larger. The cells are more unstable than the cells in B3/S3.

Definition 14. *For a set A with the center $a \in A$ such that the boundary of set A is level n, the* outer levels *are the levels k such that $k \geq n$.*

5.1 Two Consecutive Cells

First, we begin with the two consecutive cells in the hyperbolic plane. In order to make sure the cells will grow along a line, we need to avoid S1 and any combination of S1, such as S12 and S123, since only S1 will keep the cells alive in the previous generation and we can not make sure the cells only live along the line between the beginning two cells.

Proposition 5.1. *We denote two consecutive cells to be* 1 *and a two by three group of dead cells to be* 0*. For any positive integer i, the i_{th} generation of 2 consecutive cells can be represented by the i_{th} level of Pascal's triangle mod 2.*

Proof. In the beginning, there are only two cells just like the first level of Pascal's triangle mod 2. At the end of each pattern, a group of two cells can be born freely. This process is the same as how there is a 1 at the beginning and the end of each level of Pascal's triangle mod 2. For every two consecutive pairs of cells, they will die in the next generation because they get too close. In the next generation, we will have a two by three group of dead cells. For each pair of cells that have at least 3 dead cells away from others, this pair will make cells born just next to it. This process is the same as the i_{th} number in the j_{th} level equals the sum of the i_{th} and the $(i-1)_{th}$ numbers in the $(j-1)_{th}$ level of Pascal's triangle mod 2.

□

Lemma 5.1. *Suppose t_n is the number of odd numbers on the n_{th} row of Pascal's triangle. Then we have $t_{2n} = t_n$ and $t_{2n+1} = 2t_{2n}$.*

Proof (Scholes, 1999). Firstly, we consider the numbers at odd indexes of row $2n$.

$$\binom{2n}{2m+1} = \frac{2n}{2m+1} \cdot \binom{2n-1}{2m} \tag{4}$$

We can conclude that the numbers at odd indexes of row $2n$ are always even. Then we consider the numbers at even indexes of row $2n$.

$$\begin{aligned}\binom{2n}{2m} &= \frac{(2n)!}{(2m)!(2n-2m)!} \\ &= \frac{\prod_{i=1}^{n}(2i-1)}{\prod_{i=1}^{m}(2i-1)\prod_{i=1}^{n-m}(2i-1)} \cdot \frac{n!}{m!(n-m)!} \\ &= \frac{\prod_{i=1}^{n}(2i-1)}{\prod_{i=1}^{m}(2i-1)\prod_{i=1}^{n-m}(2i-1)} \cdot \binom{n}{m}\end{aligned} \tag{5}$$

So the numbers at even indexes of row $2n$ have the same parity as the numbers in row n. By the previous two equations, we can conclude that row $2n$ has the same amount of odd numbers as row n, i.e. $t_{2n} = t_n$. For the next part of the lemma, we consider the numbers at even and odd indexes at row $2n+1$ respectively.

$$\binom{2n+1}{2m} = \binom{2n+1}{2m-1} + \binom{2n}{2m} \tag{6}$$

$$\binom{2n+1}{2m+1} = \binom{2n}{2m+1} + \binom{2n}{2m} \tag{7}$$

Notice that $(2n)C(2m-1)$ and $(2n)C(2m+1)$ are even, then $(2n+1)C(2m)$ and $(2n+1)C(2m+1)$ have the same parity as $(2n)C(2m)$. It means numbers at odd and even indexes of row $2n+1$ have the same party as the number at even indexes of row $2n$, i.e. $t_{2n+1} = 2t_{2n}$. □

Proposition 5.2. *If t_n be the sum of 1's in the n_{th} level of Pascal's triangle mod 2 then $2t_j = t_{2^i+j}$ for $0 \le j < 2^i$.*

Proof. Prove it by induction. Base cases: When $i = 0$, $j = 0$ then $2t_0 = 2 = t_1$. Assumption: Suppose $2t_j = t_{2^i+j}$ for $i = n$, $0 \le j < 2^i$. Then we consider $t_{2^{n+1}+j}$. $j = 2k$ is even. Then $t_{2^{n+1}+j} = t_{2(2^n+k)} = t_{2^n+k} = 2t_k = 2t_{2k} = t_j$ by $t_{2n} = t_n$. $j = 2k+1$ is odd. Then $t_{2^{n+1}+j} = t_{2(2^n+k)+1} = 2t_{2^n+k} = 2t_k = 2t_{2k+1} = t_j$ by $t_{2n+1} = t_n$. □

Proposition 5.3. *If $S(n)$ be the number of 1's in the binary representation of n then $t_n = 2^{S(n)}$.*

Proof. Let $n = \sum_{i=1}^{k} 2^{x_i}$, where x_i represents the indexes of $1's$ of the binary representation. Then we consider $t_n = t_{\sum_{i=1}^{k} 2^{x_i}} = 2t_{\sum_{i=2}^{k} 2^{x_i}} = ... = 2^k$. □

5.2 Three Cells with "L" Shape

This type of cell in B2/S3 is quite similar to the 2 consecutive cells. For the most generations of 3 cells, the configuration of cells is just like two separate Pascal's triangles. However, for some generations, 3 cells have some special configuration of cells. For the $(2^n - 1)_{th}$ level, the population of cells is an odd number. For the $(2^n)_{th}$ level, the population of cells is always 5.

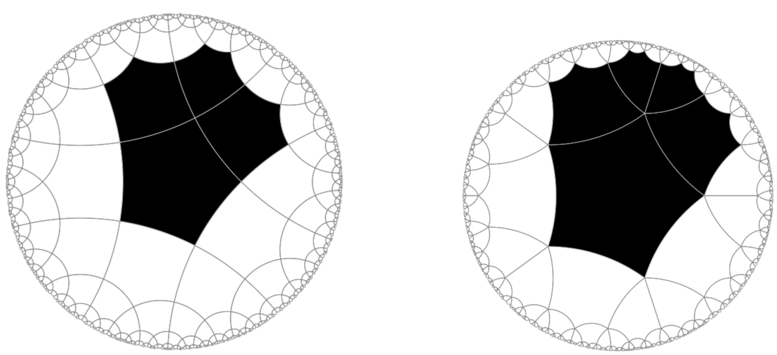
6 Existence of Still Lives in Different Tessellations and Rules

On the Euclidean plane, we only have three types of tessellations: square, triangle, and hexagon. However, on the hyperbolic plane, we have many more types of tessellations. We can also change the rules of the Game of Life. Thus, there is much more interesting work we can do on the hyperbolic plane.

6.1 A Block is a Still Life

Definition 15. *A* block *is a set of cells populated around a vertex.*

See Figure 17 for an example.



(a) (b)

Figure 17: Examples of blocks (Figure generated by hyperbolic-ca-simulator (Shintyakov, n.d.))

Proposition 6.1. *For $m \geq 4$ with rule $Bi/S(m-1)$ where $i > 2$, a block is a still life.*

Proof. For each cell around a vertex, it has exactly $m-1$ neighbors alive. Under the rule $S(m-1)$, all the cells will stay alive in the future. Note the edges between the two cells, one end of the edge is the center of the block, the other end is surrounded by two populated cells and $m-2$ unpopulated cells. Thus, for all unpopulated cells around the block, they can have at most two alive neighbors. The rule Bi with $i > 2$ can make sure no cells will be born. □

6.2 Existence of Still Lives with Rule B1

Proposition 6.2. *In the tessellation $\{n,m\}$, each polygon has $n(m-2)$ neighbors.*

Proof. Suppose in the tessellation {n,m}, cell a has n vertexes and each vertex corresponds to $m-3$ polygons which are connected to a by a vertex. There are n polygons which are connected by a by edges. Therefore, $n(m-3)+n = n(m-2)$ is the total number of neighbors of a. □

Proposition 6.3. *Suppose we have a tessellation $\{n,m\}$. If $mn-2n-3m+5 \geq 1$ then there is no still life for B1.*

Proof. For a set of cells A, we can pick $a \in A$ to be the center of the set A such that the maximum distance between any $b \in A$ and a by k. Then we only focus on the cells on the k_{th} level. There are two types of cells on the k_{th} level. One is connected to the vertex of the $(k-1)_{th}$ level, the other is connected to the edge of it. For the cell c which is connected by the vertex, we will prove that the minimum number of neighbors have a chance to be populated in next generation is $n(m-2)-2(m-1)-(m-3) = mn-2n-3m+5$. We want to consider the minimum number. So we should assume that neighbors of c are in the k_{th} level is populated. By proposition 6.2, $n(m-2)$ is the total number of neighbors. For the two vertexes on the boundary which are connected to the neighbors on the k_{th} level, there are $m-2$ cells on the $(k+1)_{th}$ level that have two neighbors alive. For the vertexes which are connected to the $(k-1)_{th}$ level, all the cells are populated. So we need to eliminate $m-3$ cells. Then for the cells which are connected to the cell in the $(k-1)_{th}$ level by the edge, with the similar reasons, the minimum number of neighbors that have the chance to be populated in next generation is $n(m-2)-2(m-1)-2(m-2)+1 = mn-2n-4m+7$. We can make the difference between these two quantities and it equals to $m-2$. By $\frac{1}{n}+\frac{1}{m} < \frac{1}{2}$, we know that $m \geq 3$, thus $mn-2n-3m+5 > mn-2n-4m+7$. See section 1.3. So if $mn-2n-3m+5 \geq 1$, then there is no still life. □

Proposition 6.4. *For $n = 3$, there exists a still life for rule B1.*

Proof. When $n = 3$ then $mn-2m-3m+5 = 3m-6-3m+5 = -1$. This means for a set of cells populated all the places in the k_{th} level, there is no cell that will be born in the next generation. So we only need to set 'S' to be as sufficient as possible to make sure all the populated cells will not die in the next generation. Then we can have a still life. □

In Figure 18, with rule $B1/S5,6,10$ this is a still life.

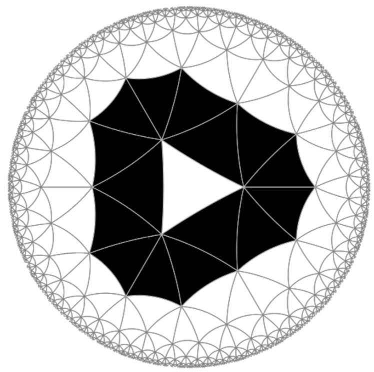
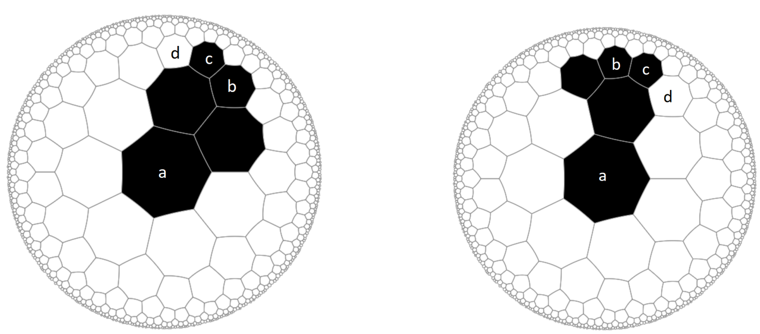


Figure 18: An example of a still life with $n = 3$ and rule $B1$ (Figure generated by hyperbolic-ca-simulator)

6.3 Still Lives with Rule S3

Proposition 6.5. *If $n \geq 7$, $m = 3$ and with rule $S3$, there is no still life.*

Proof. Suppose there is a still life for $n \geq 7$, $m = 3$ and with rule $S3$, called set A. Note, all cells in A are connected and have exactly 3 neighbors. Then we pick a cell $a \in A$ to be the center such that the maximum distance between any $b \in A$ and a by k. Now we consider the cells on the k_{th} level of set A. In Figure 19, a is the center of set A and b is on the outermost level. In order to have 3 neighbors, there are two conditions of the outermost level cell with 3 neighbors. Cell c also needs to satisfy the rule $S3$, so cell d must be alive. Then the cell surrounded by b, c, and d will have 4 neighbors. This contradicts rule $S3$. Thus there is no still life. □



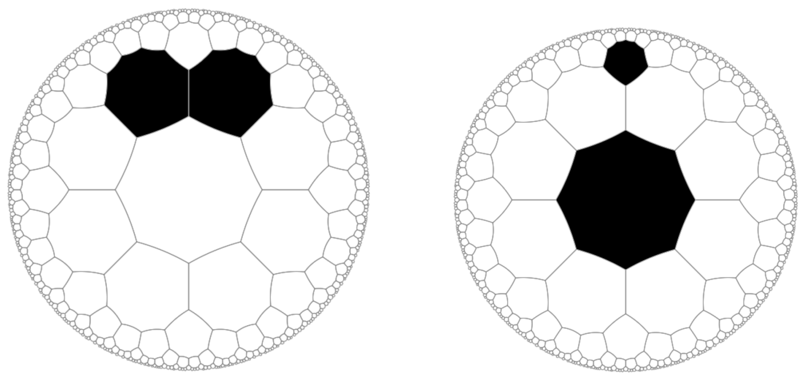
(a) (b)

Figure 19: Two conditions of outermost level cells with 3 neighbors. (Figure generated by hyperbolic-ca-simulator)

6.4 Still Lives with Rule B2

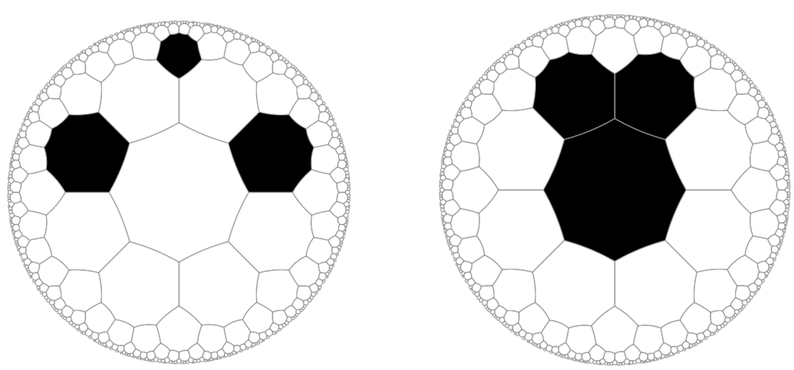
Conjecture 6.1. *Suppose $n = 8, m = 3$, with the rule $B2/S3$, then there are only two different kinds of cycles, which are called blinkers and spinners. (The names are from (Summers, 2013)).*

See Figure 20 and Figure 21 for pictures.



(a) Generation 1 (b) Generation 2

Figure 20: Cycle of 2, blinker. (Figure generated by hyperbolic-ca-simulator)



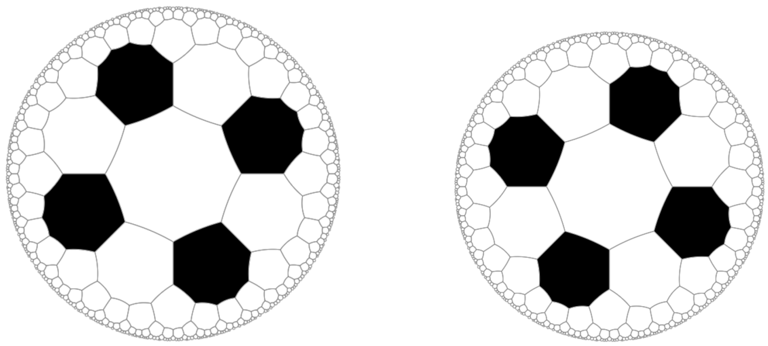
(a) Generation 1 (b) Generation 2

Figure 21: Cycle of 2, spinner. (Figure generated by hyperbolic-ca-simulator)

Definition 16. *When n is even, a* star *is a set of $1 + \frac{n}{2}$ cells with a cell on the center and every second edge connects others.*

Definition 17. *When $m = 4$, a* firework *is a set of $1 + n$ cells with a cell on the center and vertexes connect others.*

Proposition 6.6. *Suppose n is even and $m = 3$ with rule $B2/S1k$ where $k = \frac{n}{2}$, then a star is a still life.*

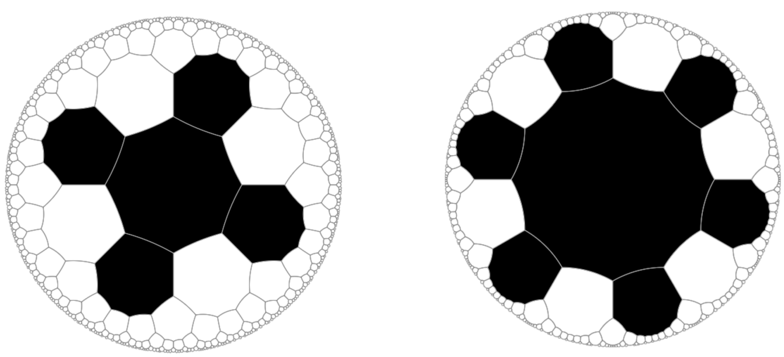


(a) Generation 1 (b) Generation 2

Figure 22: Cycle of 2. (Figure generated by hyperbolic-ca-simulator)

See Figure 23 for an example.

Proof. Suppose $n = 2k, m = 3$, and with rule $B2/S1k$, k is an integer, a star is a still life. Note there are $2k(3-2) = 2k$ neighbors for each cell. For the cell in the center, it has k neighbors alive. For others, they have 1 neighbor alive. The rule $S1k$ can make sure all the cells will stay alive in the future. Then we consider the cells on the second level. They can have at most 1 neighbor alive. Under the rule $B2$, no cell will be born. Thus a star is a still life. □



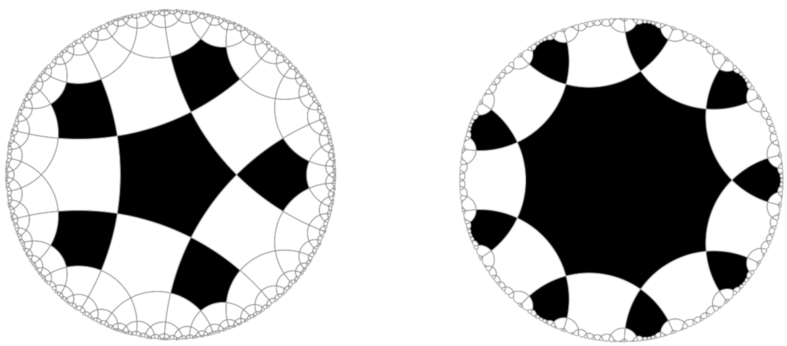
(a) $N = 8$ (b) $N = 12$

Figure 23: Examples of stars. (Figure generated by hyperbolic-ca-simulator)

Proposition 6.7. *For any n, $m = 4$ and with rule $B2/S1n$, then a firework is a still life.*

See Figure 24 for an example.

Proof. Suppose there is a tessellation $n, m = 4$, and with the rule $B2/S1n$. Now we consider the fireworks. Note there are $n(4-2) = 2n$ neighbors for each cell. For the cell in the center, it has n neighbors alive. For others, they have 1 neighbor alive. The rule $S1n$ can make sure all the cells will stay alive in the future. Then we consider the cells on the second level. They can have at most 1 neighbor alive. Under the rule $B2$, no cell will be born. Thus a firework is a still life. □



(a) $n = 5$ (b) $n = 9$

Figure 24: Examples of fireworks. (Figure generated by hyperbolic-ca-simulator)

7 Conclusion

In this paper, we mainly discuss still lifes on the hyperbolic plane with different tessellations and different rules. We find several different still lifes on the hyperbolic plane.
In the pentagon tessellation, we first prove the exponential growth of cells. Then we work on the cycles and make a conjecture of the group of 3. We also connect a special configuration to the Pascal's triangle. Apart from the pentagon tessellation, we also work on the other tessellations in order to generalize the existence of still lifes.
Still lifes are easier to analyze than the other configurations. We only need to focus on one generation. However, cycles and spaceships required us to analyze several generations. In the future, we can work on the conjectures in this paper. For example, there are only two cycles with $n = 8, m = 3$ under the rule $B2/S3$. These cycles have a period of 2 and they are the easiest cycles. So these cycles are good starting points for analyzing cycles. We can also find a tessellation under a special rule such that there exists still lifes, cycles, and spaceships. Then we can construct logic gates on that tessellation by the same method of constructing logic gates on the Euclidean plane.
We can work on the strobing rules, where empty field becomes completely filled in the next generation, and then it completely dies out one generation later. We can try to prove the following conjectures based on D. Shintyakov (Shintyakov, 2014a).

Conjecture 7.1. *There are some other spaceships in these rules.*

Conjecture 7.2. *There exists a pattern that causes very slow but infinite growth in some of the rules.*

Conjecture 7.3. *There are some significantly different rules with spaceships.*

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