A Combinatorial Proof of an Identity of Andrews

Katie Evans
St. Olaf College

Trygve Wastvedt
St. Olaf College Trygve Wastvedt, wastvedt@stolaf.edu

Follow this and additional works at: https://scholar.rose-hulman.edu/rhumj

Recommended Citation
Available at: https://scholar.rose-hulman.edu/rhumj/vol9/iss2/9
A Combinatorial Proof of an Identity of Andrews

Trygve Wastvedt
Department of Mathematics, Statistics, and Computer Science
St. Olaf College, Minnesota, USA
wastvedt@stolaf.edu

Katie A. Evans
Department of Mathematics, Statistics, and Computer Science
St. Olaf College, Minnesota, USA
evansk@stolaf.edu

July 25, 2008

Abstract
We give a combinatorial proof of an identity originally proved by G. E. Andrews in [1]. The identity simplifies a mock theta function first discovered by Rogers.

1 Background and Definitions

First we review the basics of partition theory.

Definition 1.1. A partition \( \lambda \) of a positive integer \( n \) is a finite nonincreasing sequence of positive integers \( \lambda_1, \lambda_2, \ldots, \lambda_r \) such that \( \sum_{i=1}^{r} \lambda_i = n \). The \( \lambda_i \) are called the parts of the partition and the notation \( \lambda \vdash n \) denotes “\( \lambda \) is a partition of \( n \).” We call \( n \) the size of \( \lambda \) and \( \lambda_i \) the size of the \( i^{th} \) part.

Here we will expand this definition so that \( \lambda \) may include parts of size 0.

Graphically, a partition \( \lambda \) can be represented as a left-justified array of boxes called a Ferrers shape where the \( k^{th} \) row contains \( \lambda_k \) boxes. For example, the partition \( \lambda = (6, 3, 2, 1, 1) \) of 13 can be represented by the Ferrers shape shown in Figure 1.
We use standard generating functions as well as the standard notation for basic hypergeometric series as defined by Andrews in [2]:

**Definition 1.2.**

\[
(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),
\]

\[(a)_0 = 1.\]

2 Introduction

We begin with the identity

\[
\sum_{m=0}^{\infty} \frac{q^{n^2}x^m}{(yq^2)^{m+1}} = \sum_{m=0}^{\infty} (-xq/y; q^2)_m y^m
\]

which was introduced and proven algebraically by Andrews in [1]. When \(x = 1\), the left-hand side of (1) becomes the mock theta function

\[
\phi(a; q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(-aq; q^2)_m}
\]

where \(y = -a/q\). Here we will give a combinatorial proof of (1) using integer partitions.

3 Proof

We will now give a combinatorial proof of (1). Both sides of the identity are generating functions for certain sets of partitions. We will construct a bijection between these sets of partitions that will establish (1).

For any partition \(\lambda\) let \(a = \) the largest part of \(\lambda\), \(b = \) the number of non-empty parts, and \(c = \) the number of empty parts (parts of size 0).
3.1 Left-Hand Side of (1)

Let $A$ be the set of partitions with $m$ distinct odd parts, no even parts, and any number of empty parts. We will show that the left-hand side of (1) is the generating function for the partitions in set $A$.

Let $\lambda \in A, \lambda \trianglerighteq n$. Break $\lambda$ into the three pieces $\lambda_1, \lambda_2,$ and $\lambda_3$ in the following way (see Figure 3):

1. The piece $\lambda_1$ consists of the partition $(1, 3, 5, \ldots, 2m - 1) \trianglerighteq m^2$, a staircase Ferrers shape with exactly $m$ odd parts. If we let $x$ count the number of parts and $q$ the size of the partition, $\lambda_1$ contributes $q^{m^2}x^m$ to the generating function for $A$.

2. To construct $\lambda_2$ we remove $\lambda_1$ and any empty parts from $\lambda$, and left-justify the remaining Ferrers shape. Now we separate $\lambda_2$ into pairs of columns. Since each part of $\lambda$ is odd and $\lambda_1$ took away an odd number from each part, $\lambda_2$ consists of even-sized parts. When we let $q$ count the
size of $\lambda_2$, and $y$ count the number of pairs of columns, $\lambda_2$ contributes

$$
(1 + yq^2 + (yq^2)^2 + \cdots)(1 + yq^4 + (yq^4)^2 + \cdots) \cdots (1 + yq^{2m} + (yq^{2m})^2 + \cdots) =
\frac{1}{(1 - yq^2)} \cdots \frac{1}{(1 - yq^{2(m-1)})} \frac{1}{(1 - yq^{2m})} = \frac{1}{(yq^2; q^2)_m}
$$

to the generating function for $A$.

3. The final piece $\lambda_3$, then, consists of all of the empty parts of $\lambda$ which are left after $\lambda_1$ and $\lambda_2$ have been removed. There can be any number of empty parts, and so $\lambda_3$ contributes

$$
(1 + y + y^2 + y^3 + \cdots) = \frac{1}{(1 - y)}
$$

to the generating function for $A$, where $y$ counts the number of empty parts.

From the above it is clear that the left-hand side of (1) is the generating function

$$
\sum_{\lambda \in A} q^{|\lambda|} x^b y^c + \frac{a - (2b - 1)}{2}
$$

(\text{where } |\lambda| = \text{size of } \lambda)

for $A$, the set of partitions with distinct odd parts and any number of empty parts.

Example 3.1. Let $\lambda \in A = (15, 13, 9, 5, 0, 0, 0)$. Then $\lambda$ contributes the term $q^{42}x^4y^{3+1}$ to the generating function for $A$.

![Ferrers shape for $\lambda$, split into $\lambda_1$, $\lambda_2$, and $\lambda_3$](image)

Figure 4: Ferrers shape for $\lambda$, split into $\lambda_1$, $\lambda_2$, and $\lambda_3$
3.2 Right-Hand Side

Let $B$ be the set of partitions with $m$ parts which are either distinct odd parts or empty parts where the largest part is smaller than $2m$. We will show that the right-hand side of (1) is the generating function for these partitions. Let $\lambda \vdash n$ be a partition in $B$ and let the generating function for $B$ have the form

$$\sum_{\lambda \in A} q^{\lambda} x^a y^c.$$

Note that $q$ and $x$ count the same thing as in the left-hand side while $y$ now counts only the empty parts.

To construct $\lambda$ we start with a partition with no more than $m$ distinct odd parts, which contributes $(1+qx)(1+qx^3)\cdots(1+xq^{2m-1}) = (−xq; q^2)_m$ to the generating function for $B$. We then add $m$ empty parts, which contributes $y^m$. In order to keep a total of $m$ parts we must take off an empty part for each distinct odd part added. To do this, we include a $1/y$ in the generating function for each of the distinct odd parts. Thus, the generating function for partitions in the set $B$ is

$$F(q) = \sum_{m=0}^{\infty} (-xq/y; q^2)_m y^m$$

which is the right-hand side of (1).

3.3 Bijection

We will give a bijection between sets $A$ and $B$ that establishes (1).

Define a mapping $\phi : B \rightarrow A$ and let $\lambda \in B$. Construct $\phi(\lambda)$ by removing $(a - (2b - 1))/2$ empty parts from $\lambda$.

Since $a \leq 2m - 1$,

$$a - (2b - 1) \leq 2m - 1 - (2b - 1)$$
$$a - (2b - 1) \leq 2(m - b)$$
$$\frac{a - (2b - 1)}{2} \leq c$$

and so there will always be enough empty parts to strip off.

In order for $\phi$ to be a bijection we must show that it is weight-preserving and that its inverse is well-defined. The mapping $\phi$ is weight-preserving if
for all $\lambda \in B$ the term that $\lambda$ contributes to the right-hand side of (1) is the same as the term that $\phi(\lambda)$ contributes to the left-hand side of (1). As shown in Example 3.2, we are moving the $y$ for each part that is stripped off to a pair of columns at the end of the partition. Since $y$ in the left-hand side of (1) counts these columns as well as the empty parts, the exponent of $y$ in the term in the right-hand side of (1) that counts $\lambda$ will be the same as that in the left-hand side of (1) that counts $\phi(\lambda)$. Likewise, since the partition of non-empty parts in $\lambda$ does not change, the exponents of $x$ and $q$ will remain constant under $\phi$. Therefore, the terms contributed to the generating functions by $\lambda$ and $\phi(\lambda)$ are identical and $\phi$ is weight-preserving.

The inverse of $\phi$ is very simple. Let $\lambda \in A$ and construct $\phi^{-1}(\lambda)$ by adding as many empty parts as there are pairs of columns in $\lambda_2$.

**Example 3.2.** Let $\lambda \in B = (15, 13, 11, 3, 0, 0, 0)$. Then $\lambda$ contributes the term $q^{12}x^4y^4$ to the generating function for $B$. Also, $\phi(\lambda) \in A = (15, 13, 11, 3)$ and $\phi(\lambda)$ contributes the same term to the generating function for $A$.

From the above it is clear that $\lambda$ is a bijection and so

$$
\sum_{\lambda \in A} q^{\lambda_1} x^{\lambda_2} y^{\lambda_3} = \sum_{\lambda \in B} q^{\lambda_1} x^{\lambda_2} y^{\lambda_3} = \sum_{\lambda \in B} q^{\lambda_1} x^{\lambda_2} y^{\lambda_3},
$$

establishing (1).

4 Acknowledgments

We would like to thank Kristina Garrett for her invaluable suggestions, advice, discussions, and overall support. We would also like to thank the
Howard Hughes Medical Institute for funding the grant that supported this research. Finally, a special thanks to our editors Kay Smith and Ross Wastvedt for their comments.

References
