

Order Dimension of Subgroups

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Order Dimension of Subgroups

Iordan Ganev

1 Introduction

Along with the statement and proof of the Theorem of Lagrange, undergraduate Abstract Algebra textbooks generally include several exercises that explore applications of the theorem [1, 2, 3]. One such exercise motivated the ideas of this paper. A version of the exercise reads: In a group G of order 65, let a and b be nonidentity elements of different orders and prove that no proper subgroup of G contains a and b [1]. A solution is: Suppose H is a subgroup of G that contains a and b . If either a or b has order 65, then $H = G$. Otherwise, if the two elements have orders 5 and 13, then 5 and 13 must divide the order of the H (by Lagrange). The least common multiple of 5 and 13 is 65 and $H \subseteq G$, so $|H| = 65$ and $H = G$. Thus, no proper subgroup of G contains a and b . More generally, if $|G| = pq$, where p and q are distinct primes, then the same argument shows that the only subgroup that contains two nonidentity elements of different orders is G itself.

Now let r be another prime and let $|G| = pqr$. In this case, it is possible that G has a proper subgroup of order pq that contains three nonidentity elements of orders p , q , and pq . Can a proper subgroup H contain four nonidentity elements of different orders? No, because a proper subgroup H must have order p , q , r , pq , pr , or qr . Each of these cases allows for less than four nonidentity elements of different orders in H . Hence, three is the “bound” for the number of nonidentity elements of different orders that can be contained in a proper subgroup of G .

2 Order Dimension for Groups and Subgroups

A natural generalization of these observations is to determine the maximum number of nonidentity elements of different orders that can be contained in a subgroup of a group of any order. We define $\text{odim}(G)$, the *order dimension* of the group G , as the number of different orders of its nonidentity elements. For example, the $\text{odim}(D_3) = 2$ since nonidentity elements are either rotations with order 3, or reflections with order 2. In \mathbb{Z}_{12} , there are nonidentity elements with orders 2, 3, 4, 6, and 12, so $\text{odim}(\mathbb{Z}_{12}) = 5$.

Let G be a group with order $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$. It is well-known from number theory that the number of divisors of n is $(n_1 + 1)(n_2 + 1) \dots (n_k + 1)$. This is because each divisor has the form $p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}$, where each l_i can take on the $n_i + 1$ possible values $0, 1, \dots, n_i$. Since the order of each nonidentity element is a divisor other than 1, the order dimension of G cannot exceed the number of divisors of n minus one; that is, $\text{odim}(G) \leq (n_1 + 1)(n_2 + 1) \dots (n_k + 1) - 1$. For a group G of order $65 = 5 \cdot 13$, $\text{odim}(G) \leq (1 + 1)(1 + 1) - 1 = 3$. Indeed, nonidentity elements may have order 5, 13, or 65. Similarly, when $|G| = pq$, $\text{odim}(G) \leq 3$, and when $|G| = pqr$, $\text{odim}(G) \leq (1 + 1)(1 + 1)(1 + 1) - 1 = 7$.

We seek to establish a more specific maximum for the order dimension of any proper subgroup of G . For example, if $|G| = 65$ or any integer of the form pq [a product of two distinct primes], we know from above that although $\text{odim}(G) \leq 3$, the order dimension of any proper subgroup is less than or equal to 1. Also, if $|G| = pqr$ and H is a proper subgroup, then $\text{odim}(H) \leq 3$, more specific than the maximum of 7 for $\text{odim}(G)$. In order to generalize, we renumber the prime factors of $|G| = n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ so that n_1 is the greatest of all the powers [$n_1 \geq n_i$ for $i = 2, 3, \dots, k$], and we define $b_n = n_1(n_2 + 1) \dots (n_k + 1) - 1$.

Theorem 1. If G is a group of order n and H is a proper subgroup of G , then $\text{odim}(H) \leq b_n$.

Proof. Since the order of H divides the order of G and H is proper, $|H|$ has the form $p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}$ where $0 \leq l_i \leq n_i$ for $i = 1, 2, \dots, k$ and $l_i < n_i$ for some i . The number of divisors of $|H|$, excluding the divisor one, is an upper bound to the order dimension of H , so $\text{odim}(H) \leq (l_1 + 1)(l_2 + 1) \dots (l_k + 1) - 1$.

In the case that $l_1 < n_1$, we note that $l_1 + 1 \leq n_1$. Then,

$$\begin{aligned} \text{odim}(H) &\leq (l_1 + 1)(l_2 + 1)\dots(l_k + 1) - 1 \\ &\leq (l_1 + 1)(n_2 + 1)\dots(n_k + 1) - 1 \\ &\leq n_1(n_2 + 1)\dots(n_k + 1) - 1 = b_n. \end{aligned}$$

Now we may assume that $l_1 = n_1$. Since H is proper, $l_i < n_i$ and $l_i + 1 \leq n_i$ for some $i \neq 1$ and $\text{odim}(H) \leq (l_1 + 1)(l_2 + 1)\dots(l_i + 1)\dots(l_k + 1) - 1 \leq (n_1 + 1)(n_2 + 1)\dots(l_i + 1)\dots(n_k + 1) - 1$. Suppose this quantity exceeds b_n :

$$b_n < (n_1 + 1)(n_2 + 1)\dots(l_i + 1)\dots(n_k + 1) - 1.$$

Then,

$$\begin{aligned} n_1(n_2 + 1)\dots(n_i + 1)\dots(n_k + 1) - 1 &< (n_1 + 1)(n_2 + 1)\dots(l_i + 1)\dots(n_k + 1) - 1 \\ \Rightarrow n_1(n_i + 1) &< (n_1 + 1)(l_i + 1) \leq (n_1 + 1)n_i \\ \Rightarrow n_1n_i + n_1 &< n_1n_i + n_i \\ \Rightarrow n_1 &< n_i \text{ — a contradiction.} \end{aligned}$$

Thus, $b_n \geq (n_1 + 1)(n_2 + 1)\dots(l_i + 1)\dots(n_k + 1) - 1 \geq (l_1 + 1)(l_2 + 1)\dots(l_i + 1)\dots(l_k + 1) - 1 \geq \text{odim}(H)$. \diamond

A group G may (or may not) have a subgroup H with $\text{odim}(H) = b_n$ where n is the order G . In such cases, we say that the group G has *property #*.

Example 1. \mathbb{Z}_n always has property # because cyclic groups have an element of every possible order. For instance, take \mathbb{Z}_{56} . Since $56 = 2^3 \cdot 7$ and $n_1 = 3$, we have $b_{56} = 3(1 + 1) - 1 = 5$. Indeed, $\text{odim}(\langle 2 \rangle) = 5$ since $\langle 2 \rangle$ contains the elements 2, 4, 8, 14, and 28, which have orders 28, 14, 7, 4, and 2, respectively. In general, if $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, where n_1 is the greatest power, then $\text{odim}(\langle p_1 \rangle) = b_n$.

Example 2. The maximal power (n_1) in the prime-power expansion of $|G| = n$ may not be unique to a single prime, and there may be subgroups of different orders whose order dimension is b_n . In \mathbb{Z}_{216} , for instance, the order of the group is $216 = 2^3 \cdot 3^3$. The subgroups $\langle 2 \rangle$ and $\langle 3 \rangle$ have different orders, but $\text{odim}(\langle 2 \rangle) = \text{odim}(\langle 3 \rangle) = b_{216} = 3(3 + 1) - 1 = 11$.

Example 3. The alternating group A_4 of order $12 = 2^2 \cdot 3$ does not have property #. In this case, property # would require there to be a proper subgroup with order dimension equal to $b_{12} = 2(1 + 1) - 1 = 3$. This can only occur if the subgroup has order 6, since groups of orders 2, 3, or 4 have order dimension less than

3. A_4 has no subgroup of order 6, hence it does not have property #.

Example 4. The group $\mathbb{Z}_3 \oplus \mathbb{Z}_{90}$ has order $270 = 3^3 \cdot 2 \cdot 5$ and $b_{270} = 3(1+1)(1+1) - 1 = 11$. Now, $\langle(0, 1)\rangle$ is isomorphic to \mathbb{Z}_{90} , which has order dimension $(2+1)(1+1)(1+1) - 1 = 11$ since $90 = 3^2 \cdot 2 \cdot 5$. But this is precisely b_{270} , so $\mathbb{Z}_3 \oplus \mathbb{Z}_{90}$ has property #. Also, the elements $(1, 1)$ and $(2, 1)$ have order 90, so $\langle(1, 1)\rangle$ and $\langle(2, 1)\rangle$ are subgroups isomorphic to \mathbb{Z}_{90} . We can conclude that $\text{odim}(\langle(0, 1)\rangle) = \text{odim}(\langle(1, 1)\rangle) = \text{odim}(\langle(2, 1)\rangle) = b_{270}$.

Example 5. Consider D_8 , the group of symmetries of a regular octagon. Here, $|D_8| = 16 = 2^4$ and $b_{16} = 4 - 1 = 3$. D_8 has property # and a subgroup whose order dimension is b_{16} is $\langle R_{45} \rangle$. Indeed, R_{45} , R_{135} , R_{225} , and R_{315} are elements of $\langle R_{45} \rangle$ with order 8; R_{90} , and R_{270} have order 4; R_{180} has order 2; and R_0 is the identity.

Example 6. The group of symmetries of a regular nonagon, D_9 , has order $18 = 3^2 \cdot 2$ and $b_{18} = 2(2+1) - 1 = 5$. Reflections in D_9 all have order 2 and rotations have order 3 or 9, hence the order dimension of D_9 is 3. It is impossible for a proper subgroup of D_9 to have higher order dimension than the group itself, so D_9 does not have property #.

Continue to assume $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, where p_1 is the prime or one of the primes with the highest power; that is, $n_1 \geq n_i$ for $i = 1, 2, \dots, k$. The examples suggest the following theorem.

Theorem 2. Let G be a group of order n , and let H be a proper subgroup of G . Then $\text{odim}(H) = b_n$ if and only if $H \approx \mathbb{Z}_{n/p_1}$.

Proof. Suppose $\text{odim}(H) = b_n$. Since H is a proper subgroup of G , $|H|$ has the form $p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}$ where $0 \leq l_i \leq n_i$ for $i = 1, 2, \dots, k$ and $l_i < n_i$ for some i .

If $l_1 < n_1$, then the proof of Theorem 1 shows that

$$\text{odim}(H) \leq (l_1 + 1)(l_2 + 1) \dots (l_k + 1) - 1 \leq (l_1 + 1)(n_2 + 1) \dots (n_k + 1) - 1 \leq n_1(n_2 + 1) \dots (n_k + 1) - 1 = b_n.$$

Since $\text{odim}(H) = b_n$, the inequalities become equalities. Therefore, $l_1 + 1 = n_1$, $l_i = n_i$ for $i = 2, 3, \dots, k$,

and $|H| = p_1^{n_1-1} p_2^{n_2} \dots p_k^{n_k} = \frac{n}{p_1}$. This subgroup must have an element with order corresponding to every divisor of $|H| = p_1^{n_1-1} p_2^{n_2} \dots p_k^{n_k}$, and in particular H contains an element with order $p_1^{n_1-1} p_2^{n_2} \dots p_k^{n_k} = |H|$.

That element generates the whole subgroup, so H is cyclic. Every cyclic group of order $\frac{n}{p_1}$ is isomorphic to

\mathbb{Z}_{n/p_1} , so $H \approx \mathbb{Z}_{n/p_1}$.

Otherwise, $l_1 = n_1$. Since H is proper, we know $l_i < n_i$ for some i with $2 \leq i \leq k$. The proof of Theorem 1 demonstrates that

$$b_n \geq (n_1 + 1)(n_2 + 1)\dots(l_i + 1)\dots(n_k + 1) - 1 \geq (l_1 + 1)(l_2 + 1)\dots(l_i + 1)\dots(l_k + 1) - 1 \geq \text{odim}(H).$$

Now, $\text{odim}(H) = b_n$ implies

$$b_n = (n_1 + 1)(n_2 + 1)\dots(l_i + 1)\dots(n_k + 1) - 1 = (l_1 + 1)(l_2 + 1)\dots(l_i + 1)\dots(l_k + 1) - 1 = \text{odim}(H),$$

so $l_j = n_j$ for all j from 1 to k except i . To find l_i , we substitute the definition of b_n and reduce:

$$\begin{aligned} n_1(n_2 + 1)\dots(n_i + 1)\dots(n_k + 1) - 1 &= (n_1 + 1)(n_2 + 1)\dots(l_i + 1)\dots(n_k + 1) - 1 \\ \Rightarrow n_1(n_i + 1) &= (n_1 + 1)(l_i + 1) \\ \Rightarrow n_1n_i + n_1 &= n_1l_i + l_i + n_1 + 1 \\ \Rightarrow n_1n_i &= n_1l_i + l_i + 1 \\ \Rightarrow n_1(n_i - l_i) &= l_i + 1. \end{aligned}$$

Now, $n_1 \leq n_1(n_i - l_i) = l_i + 1 \leq n_i \leq n_1$, so we get $n_i = n_1$. This means p_i could have been chosen as the prime with the greatest power instead of p_1 . We refer to the first case of this proof to see that

$H \approx \mathbb{Z}_{n/p_i}$.

For the converse, suppose $\mathbb{Z}_{n/p_1} \approx H$. Then $|H| = p_1^{n_1-1}p_2^{n_2}\dots p_k^{n_k}$ and, as a cyclic group, H has at least one element for every divisor of $|H|$. Therefore,

$$\text{odim}(H) = n_1(n_2 + 1)\dots(n_k + 1) - 1 = b_n. \diamond$$

Corollary. A group G of order n has property $\#$ if and only if it has a subgroup isomorphic to \mathbb{Z}_{n/p_1} .

3 Finitely Generated Abelian Groups

Example 7. The group $\mathbb{Z}_2 \oplus \mathbb{Z}_{54}$ has order $108 = 3^3 \cdot 2^2$, so property $\#$ would require cyclic subgroup isomorphic to $\mathbb{Z}_{108/3} = \mathbb{Z}_{36}$, or, equivalently, an element of order 36. Elements in \mathbb{Z}_2 have order 1 or 2, while elements in \mathbb{Z}_{54} have order 1, 2, 3, 6, 9, 18, or 54. The order of an element $(x, y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_{54}$ is the least common multiple of $|x|$ and $|y|$, hence $|(x, y)| = 1, 2, 3, 6, 9, 18, \text{ or } 54$. Since 36 does not appear on this list, $\mathbb{Z}_2 \oplus \mathbb{Z}_{54}$ does not have property $\#$.

We established in Example 4 that $\mathbb{Z}_3 \oplus \mathbb{Z}_{90}$ has property #, but Example 7 shows that not all external direct products of finite cyclic groups — that is, not all finitely generated abelian groups — have property #. Theorem 3 explores these groups further and continues to assume $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, where p_1 is the prime or one of the primes with the highest power; that is, $n_1 \geq n_i$ for $i = 1, 2, \dots, k$.

Theorem 3. Let G be a finitely generated abelian group of order n . G has property # if and only if G is isomorphic to \mathbb{Z}_n or $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{n/p_1}$

Proof. Suppose G has property #. The corollary to Theorem 2 asserts that there must be an element of order n/p_1 . By the Fundamental Theorem of Finitely-Generated Abelian Groups, G is isomorphic to $\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \dots \oplus \mathbb{Z}_{m_r}$ where m_1 divides m_2 , m_2 divides m_3 , and so on. We examine $\mathbb{Z}_{m_r} = \mathbb{Z}_{n/d}$ for some divisor d of n . Let (x_1, x_2, \dots, x_r) be the element of order n/p_1 . Now, $|x_1|$ divides m_1 , $|x_2|$ divides m_2 , ..., and $|x_r|$ divides m_r (by Lagrange). Since all the m 's divide m_r , $|x_i|$ divides m_1 for all $i = 1, 2, \dots, r$.

Consequently,

$$\begin{aligned} \frac{n}{p_1} &= \text{lcm}(|x_1|, |x_2|, \dots, |x_r|) \text{ divides } m_r = \frac{n}{d} \\ &\Rightarrow \left(\frac{n}{p_1}\right)q = \frac{n}{d} \\ &\Rightarrow p_1 = dq \\ &\Rightarrow d = 1, q = p_1; \text{ or } d = p_1, q = 1 \\ &\Rightarrow G \approx \mathbb{Z}_n \text{ or } G \approx \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{n/p_1}. \end{aligned}$$

For the converse, suppose $G \approx \mathbb{Z}_n$ or $G \approx \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{n/p_1}$. Then examples of cyclic groups of order n/p_1 are $\langle p_1 \rangle$ in the first case and $\langle (0, 1) \rangle$ in the second, so in both cases G has property #. \diamond

4 Dihedral Groups

Recall from Examples 5 and 6 that D_8 has property # and D_9 does not. Why do dihedral groups have property # and others do not? The general result is given in the next theorem.

Theorem 4. Let $n = 2m$ and let D_m be the dihedral group of order n — the set of symmetries of a regular m -gon. D_m has property # if and only if 2 is the prime (or one of the primes) with greatest power in the prime power expansion of n .

Proof. Continue to assume $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, with $n_1 \geq n_i$ for $i = 1, 2, 3, \dots, k$. If $p_1 = 2$, then D_m has property # because the subgroup of all rotations $\langle R_{\frac{360}{m}} \rangle$ is isomorphic to $\mathbb{Z}_m = \mathbb{Z}_{n/2} = \mathbb{Z}_{n/p_1}$. Otherwise, $p_1 \neq 2$ and n_1 is greater than the power of 2 in the prime power expansion of n . Then D_m would have property # provided that there is a copy of \mathbb{Z}_{n/p_1} contained in D_m , or, equivalently, that there is an element of order n/p_1 in D_m . If this element is a reflection, it has order 2 and $n = 2p_1$ — a contradiction since we can choose the power of p_1 is not greater than the power of 2. If the element is a rotation, then it is contained in $\langle R_{\frac{360}{m}} \rangle$, a cyclic subgroup of order m and $n/p_1 = 2m/p_1$ divides m . So, for some q ,

$$m = (2m/p_1)q$$

$$\Rightarrow mp_1 = 2qm$$

$$\Rightarrow p_1 = 2q.$$

This contradicts the assumption that p_1 is a prime and not equal to 2, so D_m does not have property # when another prime's power exceeds that of 2 in the prime power expansion of $n = 2m$. \diamond

5 Symmetric Groups

In the next three examples, we consider symmetric groups, which lead us to another general result.

Example 8. The symmetric group S_3 has order $6 = 2 \cdot 3$, so, using the convention above, $p_1 = 2$ or $p_1 = 3$. Any two-cycle has order $2 = 6/3 = n/p_1$ and any three-cycle has order $3 = 6/2 = n/p_1$. By the corollary to Theorem 2, S_3 has property #.

Example 9. The order of the symmetric group S_4 is $24 = 2^3 \cdot 3$. Property # would require an element of order $24/2 = 12$, but elements in S_4 are either the identity, two-cycles, a product of disjoint two-cycles, three-cycles, or four-cycles. Neither of these has order 12, so S_4 does not have property #.

Example 10. S_5 has order $5! = 120 = 2^3 \cdot 3 \cdot 5$, so $n/p_1 = 120/2 = 60$. Elements in S_5 have order at most 6 (the product of a two-cycle and a disjoint three-cycle), so S_5 does not have property #.

Theorem 5. S_m does not have property # when $m \geq 4$.

Proof. Let $n = m! = |S_m|$ where $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, with $n_1 \geq n_i$ for $i = 1, 2, 3, \dots, k$.

First, we show that p_1 must equal 2. Intuitively, more factors of 2 appear in $m!$ than of other primes since 2 is a divisor of every other integer from 1 to m , while other primes divide fewer of those integers. To verify the conclusion more technically, suppose for a given prime p that $p^s \leq m < p^{s+1}$. We proceed to find the exponent of p , call it $n(p)$, in the prime power expansion of $n = m!$. There are $\lfloor \frac{m}{p} \rfloor$ multiples of p less than m , so $n(p) \geq \lfloor \frac{m}{p} \rfloor$. Next, there are $\lfloor \frac{m}{p^2} \rfloor$ multiples of p^2 less than or equal to m , each of which adds another factor of p to $n = m!$. So far,

$$n(p) \geq \lfloor \frac{m}{p} \rfloor + \lfloor \frac{m}{p^2} \rfloor$$

Likewise, we may continue to count the factors of powers of p in $n = m!$ until we reach p^s . This gives

$$n(p) = \lfloor \frac{m}{p} \rfloor + \lfloor \frac{m}{p^2} \rfloor + \dots + \lfloor \frac{m}{p^s} \rfloor.$$

Since $p \geq 2$ for all primes p , $n(p) \leq n(2)$. Therefore, 2 is the prime with the greatest power in the prime-power expansion of n , that is $p_1 = 2$.

Next, suppose that there is an element in S_m with order $n/p_1 = n/2 = m!/2$. This element can be written as the product of r disjoint cycles of orders m_1, m_2, \dots, m_r so that

$$m_1 + m_2 + \dots + m_r \leq m \quad \text{and} \quad \text{lcm}(m_1, m_2, \dots, m_r) = \frac{m!}{2}$$

Since 2^{n_1} divides $m!$, 2^{n_1-1} divides $m!/2$. In order for the factor 2^{n_1-1} to appear as the least common multiple of the m_i 's, it must divide at least one of the m_i 's, and so $2^{n_1-1} \leq m$.

Now, we demonstrate that this is impossible for $m \geq 6$ with a proof by induction that $2^{n_1-1} > m$ for $m \geq 6$.

Base step: For $m = 6$, $n = 6! = 720 = 2^4 \cdot 3^2 \cdot 5$ and $n_1 = 4$. Indeed, $2^{4-1} = 2^3 = 8 > 6$.

Induction hypothesis: Suppose $2^{n_1-1} > m$ for some $m \geq 6$.

Induction step: Let 2^s be the greatest power of 2 that divides $m + 1$. Multiplying $m!$ by $m + 1$ adds a factor of 2^s to 2^{n_1} , and we must show that $2^{n_1-1+s} > m + 1$.

Well, if $m + 1$ is even, then $s \geq 1$. This means that

$$2^{n_1-1+s} \geq 2^{n_1} = 2^{n_1-1} + 2^{n_1-1} > m + 2^{n_1-1} \geq m + 1$$

Otherwise, $m + 1$ is odd and $s = 0$. By the induction hypothesis, $2^{n_1-1} > m$, which implies $2^{n_1-1} \geq m + 1$.

But $m + 1$ is odd, so $2^{n_1-1} \neq m + 1$. Hence, $2^{n_1-1+s} = 2^{n_1-1} > m + 1$.

Therefore, it is impossible for S_m to contain an element of order $m!/2$ when $m \geq 6$, $m = 4$ (Example 9), or $m = 5$ (Example 10); consequently, S_m does not have property $\#$ when $m \geq 4$. \diamond

6 Conclusion

Recall the original problem: In a group G of order 65, let a and b be nonidentity elements of different orders and prove that no proper subgroup of G contains a and b . A “new” solution that applies the ideas of this paper may be: $65 = 5 \cdot 13$ and $b_{65} = 1(1 + 1) - 1 = 1$. For a proper subgroup H of G , $\text{odim}(H) \leq b_{65} = 1$ (Theorem 1) so no proper subgroup of G can contain two nonidentity elements of different orders. However, G has property $\#$. By Cauchy’s Theorem, G contains at least one element of order 13 and at least one of order 5 that each generate cyclic subgroups isomorphic to $\mathbb{Z}_{13} = \mathbb{Z}_{65/5}$ and $\mathbb{Z}_5 = \mathbb{Z}_{65/13}$, respectively. Both 5 and 13 have the greatest powers in the prime power expansion of $|G| = 65$; therefore, by the corollary to Theorem 2, any group of order 65 has property $\#$.

Although it is not likely to be found in an Abstract Algebra text, a more general version of the original problem is: Let G be a group of order $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, where $n_1 \geq n_i$ for $i = 1, 2, 3, \dots, k$. Show that the only subgroup of G that contains a collection S of $n_1(n_2 + 1) \dots (n_k + 1)$ nonidentity elements of different orders is G itself. Solution: Suppose a subgroup H of G contains S . Then $\text{odim}(H) \geq n_1(n_2 + 1) \dots (n_k + 1) = b_n + 1 > b_n$ and by Theorem 1, H cannot be proper. It follows that $H = G$.

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