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Branching matrices for the automorphism group lattice of a Riemann surface

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Abstract

Let S be a Riemann surface and $G \subseteq \mathrm{Aut}(S)$ a large subgroup, ($\mathrm{Aut}(S)$ may be unknown). We are particularly interested in regular n-gonal surfaces, i.e., the quotient surface S/G (and hence $S/\mathrm{Aut}(S)$) has genus zero. For various $H \subset K \subseteq G$ the ramification information of the branched coverings $S/K \to S/H$ may be captured in a matrix. The ramification information, in particular strong branching, may be then be used in analyzing the structure of $\mathrm{Aut}(S)$. The ramification information is conjugation invariant so the matrix's rows and columns may be indexed by conjugacy classes of subgroups. The only required information is a generating vector for the action of G, and the subgroup structure. The latter may be computed using Magma or GAP. The signatures and generating vectors of the subgroups are not required.

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1 Introduction

In [1], Accola discussed the concept of strong branching to show that, for large genus, prime cyclic n-gonal action groups are normal in the automorphism group of S. This condition then allows us to determine the full automorphism group of the surface. In this note we examine ramification and strong branching of branched coverings $S/K \to S/H$ for $K \subset H$ in the space of subgroup pairs of a large subgroup G of $\mathrm{Aut}(S)$. We use G as an approximation of $\mathrm{Aut}(S)$ since the full automorphism group may be unknown. The ramification information can be encoded in a matrix whose rows and columns are indexed by conjugacy classes of subgroups of G. Strong branching can be identified by positive entries in the matrix. The entire matrix can be determined by a generating vector of the group G, and the permutation representations determined by subgroups.

In sections 2 and 3 we recall some information on branched coverings, ramification of branched coverings, and conformal group actions. In section 4 we introduce our ramification matrices, using a particular example with $G = A_5$ to illustrate the ideas. The ramification information can be determined without computing signatures and generating vectors of the subgroups. Finally, in Section 5 we show additional examples computed by means of Magma

[6]. Magma is primarily used for finding the subgroup lattice and computing permutation representations. See section 5.2.

Let us sketch why we might be interested in such normality results. Suppose the surface S has a known group H of automorphisms. Suppose further that S/H has genus zero, a common occurrence for highly symmetric surfaces. If we may conclude that H is normal in $\mathrm{Aut}(S)$, then we may identify $\mathrm{Aut}(S)/H$ as an automorphism group of the sphere that permutes the branch points. We may then determine $\mathrm{Aut}(S)$.

The computations we show for specific groups are not really needed to determine group structure since we know the group structure beforehand. Rather, the matrices may be used to study small examples and look for additional constraints on ramification that may be conjectured to impose constraints on an unknown automorphism group. For instance, in Accola's result a simple group structure and a large genus are used to conclude normality. It should be noted that if the prime is large then the genus has to be quite high for the normality result to hold, and so there are many cases for which there are no strong branching constraints on the prime n-gonal actions. The Magma code has been posted along with the paper [3] so that readers may look at their own examples.

2 Ramification of branched covers

2.1 Branched covers

Let S_1, S_2 be two Riemann surfaces of genus σ_1 and σ_2, respectively, and $p : S_1 \to S_2$ a branched covering (holomorphic map) of degree n. Related to the map p we have several interesting objects.

1. There an inclusion of meromorphic function fields:

$$p^* : \mathbb{C}(S_2) \hookrightarrow \mathbb{C}(S_1),$$

   defined by

$$p^*(f) = f \circ p.$$

2. Conversely, given an inclusion of function fields $\iota : \mathbb{C}(S_2) \hookrightarrow \mathbb{C}(S_1)$, there is a map $p : S_1 \to S_2$ such that $\iota = p^*$.

3. There is a pullback map of meromorphic differential 1-forms

$$dp^* : \Omega^1(S_2) \to \Omega^1(S_1),$$

defined locally by

$$dp^*(df) = d(f \circ p).$$

4. There is a divisor dp^* defined on S_1 by

$$(dp^*) = \sum_{P \in S_1} ord_P(dp)P.$$

The value $ord_P(dp)$ is computed by first writing, in local coordinates centered at 0 in the domain and target,

$$p(z) = z^{e(P)} f(z),\ \ f(z) \neq 0.$$

Then

$$dp = z^{e(P)-1}(e(P)f(z)dz + zdf(z)).$$

Since

$$e(P)f(z)dz + zdf(z) = e(P)f(0)dz$$

at $z = 0$ then $ord_P(dp) = e(P) - 1$. Now $e(P) \geq 1$ for all P, is independent of the coordinatization chosen, and $e(P) > 1$ for at most finitely many points. Thus

$$(dp^*) = \sum_{P \in S_1} (e(P) - 1)\, P. \tag{1}$$

2.2 Ramification and the Riemann-Hurwitz equation

Definition 1 *The total ramification of a branched covering of p is the degree of the divisor in equation 1:*

$$ram(p) = \sum_{P \in S_1} (e(P) - 1)\,.$$

If ω is a differential form on S_2 then the degree of the divisor $(dp^*(\omega))$ may be computed in two ways: first as a differential form on S_1 with degree $2(\sigma_1 - 1)$ and secondly as the degree of the pull back $dp^*(\omega)$ to get $2n(\sigma_2 -$

$1) + \sum_{P \in S_1} (e(P) - 1)$. The first term comes from pulling back the zeros and poles of ω and the second term comes from the ramification of the branched cover. The Riemann-Hurwitz equation may then be written:

$$2(\sigma_1 - 1) = 2n(\sigma_2 - 1) + \sum_{P \in S_1} (e(P) - 1) \tag{2}$$

or

$$2(\sigma_1 - 1) - 2n(\sigma_2 - 1) = \sum_{P \in S_1} (e(P) - 1)\,. \tag{3}$$

We see that either side of equation 3 equals the total ramification.

Next we look at some different ways of rewriting the Riemann-Hurwitz equation. Let $Q_1, \ldots, Q_t$ be the points in S_2 over which p is ramified. Then

$$\sum_{P \in S_1} (e(P) - 1) = \sum_{j=1}^{t} \sum_{p(P)=Q_j} (e(P) - 1)\,.$$

Now

$$\sum_{p(P)=Q_j} (e(P) - 1) = \sum_{p(P)=Q_j} e(P) - \sum_{p(P)=Q_j} 1 = n - \left|p^{-1}(Q_j)\right|,$$

so we get another version of the Riemann-Hurwitz theorem:

$$2(\sigma_1 - 1) = 2n(\sigma_2 - 1) + \sum_{j=1}^{t} \left(n - \left|p^{-1}(Q_j)\right|\right),$$

$$\frac{2(\sigma_1 - 1)}{n} = 2(\sigma_2 - 1) + \sum_{j=1}^{t} \left(1 - \frac{|p^{-1}(Q_j)|}{n}\right). \tag{4}$$

It follows then, that if we can count singular preimages, then the total ramification is easily calculated.

Our next Proposition tells us about the behaviour of ramification under composition of branched covers.

Proposition 2 *Let $p : S_1 \to S_2$ and $q : S_2 \to S_3$ be branched coverings of degree n and m respectively then*

$$ram(q \circ p) = ram\,(p) + n \times ram(q).$$

Proof. We know the $deg(q \circ p) = mn$ and $ram(p) = 2(\sigma_1 - 1) - 2n(\sigma_2 - 1)$, and $ram(q) = 2(\sigma_2 - 1) - 2m(\sigma_3 - 1)$. So,

$$\begin{aligned} ram(q \circ p) &= 2(\sigma_1 - 1) - 2nm(\sigma_3 - 1) \\ &= 2(\sigma_1 - 1) - 2n(\sigma_2 - 1) + 2n(\sigma_2 - 1) - 2nm(\sigma_3 - 1) \\ &= 2(\sigma_1 - 1) - 2n(\sigma_2 - 1) + n(2(\sigma_2 - 1) - 2m(\sigma_3 - 1)) \\ &= ram\,(p) + n \times ram(q) \end{aligned}$$

■

2.3 Strong branching

In [1], Accola introduced strong branching of branched covers.

Definition 3 *The branched covering $p : S_1 \to S_2$ of degree n is* strongly branched *if*

$$ram(p) > 2n(n-1)(\sigma_2 + 1) \tag{5}$$

or

$$\begin{aligned} 2(\sigma_1 - 1) - 2n(\sigma_2 - 1) &> 2n(n-1)(\sigma_2 + 1), \\ \sigma_1 &> n^2\sigma_2 + (n-1)^2. \end{aligned} \tag{6}$$

Let us say that $p : S_1 \to S_2$ is weakly branched *if*

$$\begin{aligned} ram(p) &\leq 2n(n-1)(\sigma_2 + 1), \\ 2(\sigma_1 - 1) - 2n(\sigma_2 - 1) &\leq 2n(n-1)(\sigma_2 + 1), \\ \sigma_1 &\leq n^2\sigma_2 + (n-1)^2. \end{aligned}$$

We shall call the number $sb(p)$ defined by:

$$\begin{aligned} sb(p) &= ram(p) - 2n(n-1)(\sigma_2 + 1) \\ &= 2\sigma_1 - 2n^2\sigma_2 - 2 + 4n - 2n^2 \\ &= 2\sigma_1 - 2n^2\sigma_2 - 2(n-1)^2 \end{aligned} \tag{7}$$

the strong branching indicator, *so that p is strongly branched if and only if $sb(p) > 0$. Finally the inclusion of fields $\mathbb{C}(S_2) \subseteq \mathbb{C}(S_1)$ is strongly branched if and only if the corresponding map $p : S_1 \to S_2$ is strongly branched.*

The following proposition needs only some simple algebra to be proven.

Proposition 4 *If $p : S_1 \to S_2$ and $q : S_2 \to S_3$ of degrees n and m, respectively, are weakly branched, then $q \circ p : S_1 \to S_3$ is weakly branched.*

Proof. We know that

$$sb(p) = 2\sigma_1 - 2n^2\sigma_2 - 2(n-1)^2$$

$$sb(q) = 2\sigma_2 - 2m^2\sigma_3 - 2(m-1)^2,$$

and

$$sb(q \circ p) = 2\sigma_1 - 2m^2n^2\sigma_3 - 2(mn-1)^2.$$

From these equations we see that

$$\begin{aligned} sb(p) + n^2 sb(q) &= 2\sigma_1 - 2n^2\sigma_2 - 2(n-1)^2 + n^2(2\sigma_2 - 2m^2\sigma_3 - 2(m-1)^2) \\ &= 2\sigma_1 - 2m^2n^2\sigma_3 - 2(n-1)^2 - 2n^2(m-1)^2 \\ &= 2\sigma_1 - 2m^2n^2\sigma_3 - 2(mn-1)^2 \\ &\quad + 2(mn-1)^2 - 2(n-1)^2 - 2n^2(m-1)^2 \\ &= sb(q \circ p) + 4n\,(m-1)\,(n-1)\,. \end{aligned}$$

Hence

$$sb(q \circ p) = sb(p) + n^2 sb(q) - 4n\,(m-1)\,(n-1)\,.$$

By hypothesis, both $sb(p)$, $sb(q) \leq 0$ so that $sb(q \circ p) \leq 0$ also. In fact, $sb(q \circ p) \leq -4n\,(m-1)\,(n-1) \leq -8$, since $m, n \geq 2$. ■

3 Actions, generating vectors, and ramification

3.1 Conformal actions and generating vectors

We briefly recall some facts about conformal actions of groups and establish some notation (see [2] for instance). We say that the group G acts *conformally* on the Riemann surface S of genus σ if there is a monomorphism $\epsilon : G \to \mathrm{Aut}(S)$, where $\mathrm{Aut}(S)$ is the group of biholomorphic transformations of S.

If there will be no confusion, we will simply consider G as a subgroup of $\mathrm{Aut}(S)$.

The universal cover of S is the hyperbolic plane $\mathbb{H}$ with covering map $\pi_S : \mathbb{H} \to S$. We denote the group of covering transformations of π_S by $\Pi \backsimeq \pi_1(S)$. The conformal group action of G on S has a covering action by a Fuchsian group Γ defined by an exact sequence

$$\Pi \hookrightarrow \Gamma \overset{\eta}{\twoheadrightarrow} G. \tag{8}$$

The induced isomorphism $\overline{\eta} : \Gamma/\Pi \leftrightarrow G$ defines the action $\epsilon = \overline{\eta}^{-1}$ of G on S through the natural action of Γ/Π on $S = \mathbb{H}/\Pi$.

All our work depends on the following presentation of Γ, see [9]:

$$\Gamma = \left\langle \alpha_1, \ldots, a_\tau, \beta_1, \ldots, \beta_\tau, \gamma_1, \ldots, \gamma_t : \prod_{i=1}^{\tau} [\alpha_i, \beta_i] \prod_{j=1}^{t} \gamma_j = \gamma_1^{n_1} = \ldots \gamma_t^{n_t} = 1 \right\rangle. \tag{9}$$

The integers τ and $n_j, j = 1, \ldots, t$, have topological interpretations, in fact the presentation may be established by topological means.

The quotient map $q : \mathbb{H} \to T = \mathbb{H}/\Gamma$, $z \to \Gamma z$ is branched (ramified) over t points $Q_1, \ldots, Q_t \in T$. The points $Q_1, \ldots, Q_t$ may be ordered so that, for each $z \in q^{-1}(Q_j)$, the isotropy subgroup $\Gamma_z = \{\gamma \in \Gamma : \gamma z = z\}$ is conjugate to the cyclic subgroup of Γ generated by γ_j. Also,

$$o(\gamma_j) = n_j. \tag{10}$$

The integer n_j is called the *branching order* at Q_j. Let $B = \{Q_1, \ldots, Q_t\}$, denote the *branching set* with branching orders $(n_1, \ldots, n_t)$. We call the $(t+1)$–tuple $(\tau \colon n_1, \ldots, n_t)$ the *signature* of Γ or the *signature* or *branching data* of G acting on S.

The quotient space $T = \mathbb{H}/\Gamma \simeq S/G$ may be defined abstractly as the set of orbits $\{Gx : x \in S\}$. The field of functions $\mathbb{C}(S/G)$ may be identified with $\mathbb{C}(S)^G$, the G-invariant rational functions on S. The quotient surface S/G is a Riemann surface with genus $\tau \leq \sigma$, the quotient map $\pi_G : S \to S/G$ is branched precisely over $\{Q_1, \ldots, Q_t\}$, and for $P \in \pi_G^{-1}(Q_j)$ the branching order $e_G(P) = n_j$.

Define the elements a_i, b_i, c_j of G by:

$$a_i = \eta(\alpha_i), 1 \leq i \leq \tau, \quad b_i = \eta(\beta_i), 1 \leq i \leq \tau, \quad c_j = \eta(\gamma_j), 1 \leq j \leq t.$$

These elements generate G:

$$G = \langle a_i, b_i, c_j : i = 1, \ldots, \tau,\ j = 1, \ldots, t \rangle \tag{11}$$

and satisfy these relations

$$\prod_{i=1}^{\tau} [a_i, b_i] \prod_{j=1}^{t} c_j = 1 \tag{12}$$

and

$$o(c_j) = n_j, \tag{13}$$

because of equations 9, 10 and since $\Pi = \ker(\eta)$ is torsion free.

Definition 5 *We call a $(2\tau + t)$-tuple $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t)$ of elements of G, satisfying 12 - 13 , a $(\tau\colon n_1, \ldots, n_t)$*-vector*. Such a vector is called a* generating $(\tau\colon n_1, \ldots, n_t)$-vector *if equation 11 is also satisfied.*

From the above discussion and the Riemann Hurwitz theorem below, we obtain Riemann's existence theorem, see [5], [8].

Theorem 6 *The group G acts conformally on a genus σ surface S, with branching data $(\tau\colon n_1, \ldots, n_t)$ if and only if $2\sigma - 2 = |G|\mu(\tau\colon n_1, \ldots, n_t)$ and G has a generating $(\tau\colon n_1, \ldots, n_t)$-vector.*

3.2 Ramification and the Riemann Hurwitz formula

The signature and the order of G are related by the Riemann–Hurwitz equation, see [4], [9]:

$$\frac{2\sigma - 2}{|G|} = 2\tau - 2 + \sum_{j=1}^{t} \left(1 - \frac{1}{n_j}\right) \tag{14}$$

or in total ramification form:

$$2\sigma - 2 - |G|\,(2\tau - 2) = \sum_{j=1}^{t} \left(|G| - \frac{|G|}{n_j}\right) = |G| \sum_{j=1}^{t} \left(1 - \frac{1}{n_j}\right).$$

This easily follows from the variant equation 4.

Define $\mu(\Gamma)$, $\mu(G,S)$, and $\mu(\tau\colon n_1,\ldots,n_t)$ by:

$$\mu(\Gamma)=\mu(G,S)=\mu(\tau\colon n_1,\ldots,n_t)=2\tau-2+\sum_{j=1}^{t}\left(1-\frac{1}{n_j}\right)=\frac{2\sigma-2}{|G|}.$$

The quantity $2\pi\mu(\Gamma)$ is the orbifold hyperbolic area of $T=S/G$, and depends only on the signature of the action. From the Riemann-Hurwitz equation we get a multiplicative relation for $H\subset G$, namely:

$$\mu(H,S)=[G\colon H]\mu(G,S).$$

3.3 Ramification and permutation representations

Now let $H\subseteq G$ be groups of conformal automorphisms. There is a well defined map $\pi_{G/H}:S/H\rightarrow S/G$ defined by $Hx\rightarrow Gx$. We are interested in the ramification structure of $\pi_{G/H}$. We can recover the ramification information from a generating vector for the G-action and the permutation representation of G on right H-cosets. We denote this representation by M with definition $M(c)\cdot Hg=Hgc^{-1}$, and call it the *monodromy representation.* The monodromy representation is equivalent to the monodromy representation of $\pi_1(T\backslash B)$ on the regular fibres of $\pi_{G/H}:S/H\rightarrow S/G$, hence the name. If S/G has genus zero then we call $(M(c_1),\ldots,M(c_t))$ the monodromy vector of $\pi_{G/H}$. The next proposition follows from the discussion in [7], but we give a detailed proof here.

Proposition 7 *Let $H\subseteq G$ act conformally on the surface S. Assume that the action of G has signature $(\tau\colon n_1,\ldots,n_t)$ and generating vector $(a_1,\ldots,a_\tau,$ $b_1,\ldots,b_\tau,c_1,\ldots,c_t)$, and that the branch points on $T=S/G$ are $\{Q_1,\ldots,Q_t\}$. Then:*

1. *The map $\pi_G:S\rightarrow S/G$ factors as $\pi_G=\pi_{G/H}\circ\pi_H$. For $P\in S$ the branching orders factor, $e_G(P)=e_{G/H}(\pi_{G/H}(P))e_H(P)$, where the subscripts have obvious meanings.*

2. *The $\pi_{G/H}$ image of the branch points of $\pi_H:S\rightarrow S/H$ lie in $\{Q_1,\ldots,Q_t\}$.*

3. *The branch points of $\pi_{G/H}:S/H\rightarrow S/G$ lie in the set $\{Q_1,\ldots,Q_t\}$.*

4. *The points of the inverse image $\pi_{G/H}^{-1}(Q_j)$ are in 1-1 correspondence with the cycles of monodromy permutation $M(c_j)$.*

5. *For each cycle of $M(c_j)$ the branching order $e_{G/H}(x)$ of $\pi_{G/H}$ at the point x, corresponding to the cycle, is the size of the cycle.*

6. *The ramification $ram(\pi_{G/H})$ can be easily computed from the cycle types of the permutations $M(c_k)$, $k = 1, \ldots, t$.*

Proof. The first part of statement 1 is trivial. The rest of the statement may be easily proven by looking at the action of automorphisms in local coordinates. Statements 2 and 3 follow from statement 1.

To prove statement 4, let P be a point in S. The point $Q = \pi_G(P)$ is identified with the set GP. The points in S/H lying over GP are the orbits $Hg_1P, \ldots, Hg_nP$, where

$$G = Hg_1 \cup \cdots \cup Hg_n$$

is a decomposition of G into right cosets, and $n = |G\backslash H|$. Now, two points Hg_iP and Hg_jP determine the same point of S/H if and only if $Hg_iP = Hg_jP$, so that $g_jP = hg_iP$ for some $h \in h$. This means that $g_j^{-1}hg_iP = P$. If P does not lie over some Q_k then $g_j^{-1}hg_i = 1$ or $g_j = hg_i$ and $Hg_j = Hhg_i = Hg_i$ are the same coset. Thus, there are n distinct orbits Hg_iP and $\pi_{G/H}^{-1}(Q)$ is a regular fibre, as expected. Now suppose the $Q = Q_k$, one of the branch points. We may assume that $P = P_k$ is chosen so that the stabilizer $\mathrm{Stab}_G(P_k) = \langle c_k \rangle$. Then, the previous discussion yields

$$g_j^{-1}hg_i = c_k^s$$

for some s. It follows that

$$Hg_i = Hhg_i = Hg_jc_k^s$$

so that Hg_i and Hg_j are in the same $\langle c_k \rangle$ orbit in the coset space $G\backslash H$. From the other direction, if $Hg_i = Hg_jc_k^s$ then

$$Hg_iP = Hg_jc_k^sP = Hg_jP.$$

Hence, $Hg_iP = Hg_jP$ if and only if Hg_i and Hg_j are in the same $\langle c_k \rangle$-orbit in $G\backslash H$. This proves statement 4.

For statement 5, the branching order of $e_H(g_jP_k)$ is the order of $\mathrm{Stab}_H(g_jP_k) = \left\langle g_jc_kg_j^{-1} \right\rangle \cap H$ which is the stabilizer of $\langle c_k \rangle$ at the coset Hg_j. The size of the cycle $Hg_j \langle c_k \rangle$ is given by the orbit stabilizer theorem

$$
\begin{aligned}
|Hg_j \cdot \langle c_k \rangle| &= \frac{|\langle c_k \rangle|}{\mathrm{Stab}_{\langle c_k \rangle}(Hg_j)} \\
&= \frac{|\langle c_k \rangle|}{\mathrm{Stab}_H(g_j P_k)} \\
&= \frac{e_G(P)}{e_H(P)} \\
&= e_{G/H}(\pi_{G/H}(P)).
\end{aligned}
$$

For statement 6 we have

$$
\begin{aligned}
ram(\pi_{G/H}) &= \sum_{j=1}^{t} \left(n - \left| \pi_{G/H}^{-1}(Q_j) \right| \right) \\
&= \sum_{j=1}^{t} \left(n - \#cycles(M(c_j)) \right). \qquad (15)
\end{aligned}
$$

■

3.4 n-gonal actions

We call a surface n-gonal if there is a map $p : S \to P^1(\mathbb{C})$ of degree n and p is called an *n-gonal morphism* or *n-gonal map.* Any rational function is an n-gonal map, though we are typically interested in maps of low degree, or those with a high degree of symmetry. An n-gonal map is a called a *regular n-gonal morphism* if p is induced by a conformal group action of G or, alternatively, if the extension $\mathbb{C}(S)/p^*(\mathbb{C}(z))$ is Galois. In this case we say that G has a *n-gonal action* on S. For an n-gonal action the signature is $(0\colon n_1, \ldots, n_t)$ which we more conveniently write $(n_1, n_2, \ldots, n_t)$. The following is easily proved.

Proposition 8 *Suppose that the groups $H \subseteq G$ act on S, with the H-action induced by restriction from G. Then, if the action of H is n-gonal, then so is the action of G. In particular, if a surface admits an n-gonal action, then the action of the automorphism group of S is an n-gonal action.*

4 Ramification matrices

In this section we will illustrate the development of the ideas with the group A_5 and a specific generating vector. All the computations were done using Magma.

4.1 The space of pairs of subgroups

Suppose ρ is some conjugation invariant function on pairs of subgroups $K \subseteq H$ of G, namely $\rho(K,H) = \rho(K^g, H^g)$ for $g \in G$. Our primary example will be $ram(\pi_{H/K})$, also denoted $ram(S/K \to S/H)$. The function ρ is well defined on the space of conjugacy classes of pairs $K \subseteq H$ of subgroups of G. We wish set up a discrete object that allows us to efficiently construct the orbit space of pairs and present the values of the function ρ.

To this end, let $[H]$ denote the conjugacy class of H in G. We write $[K] \leq [H]$ if a subgroup in $[K]$ is contained in a subgroup of $[H]$. We let $[K,H]$ denote the conjugacy class of the pair (K,H), where we do not assume any inclusion relation between K and H. Set $\mathcal{S}(G) = \{[H_1], \ldots, [H_s]\}$, where the $[H_i]$ are ordered so that if $[H_i] \leq [H_j]$ if then $i \leq j$. This will be achieved if $|H_1|, \ldots, |H_s|$ is an increasing sequence. Also, let $\mathcal{P}(G)$ be the orbit space of conjugacy classes of pairs. We may consider $\mathcal{P}(G)$ as a set lying over $\mathcal{S}(G) \times \mathcal{S}(G)$ via the map $\Theta : [K,H] \to ([K],[H])$.

Remark 9 *For $K \subseteq H$, it turns out that the ramification function $ram(\pi_{H/K})$ depends only on $[K]$ and $[H]$. So the determination of the map Θ is somewhat irrelevant, except to determine when $K \subset H$. Even so, we will conduct a detailed analysis of Θ, since it is interesting and is useful for computing functions that do not just depend on $[K]$ and $[H]$.*

We examine the fibres of the map Θ. Let (K,H) be a specific pair lying over $([K],[H])$ and suppose that (K',H') is another such pair. By definition there are $g_1, g_2 \in G$ such that $K' = g_1 K g_1^{-1}$ and $H' = g_2 H g_2^{-1}$. Conjugating by g_2^{-1} and letting $g = g_2^{-1} g_1$ we see that (K',H') is conjugation equivalent to (gKg^{-1}, H). Thus the conjugacy classes lying over $([K],[H])$ all have representatives of the form (gKg^{-1}, H). Now impose the condition $gKg^{-1} \subseteq H$, and let $D_{K,H}$ be the set

$$D_{K,H} = \left\{ g \in G : gKg^{-1} \subseteq H \right\}.$$

We can try also try an alternative approach in which we fix K and get normal forms (K, gHg^{-1}) and then consider these sets:

$$U_{K,H} = \left\{g \in G : K \subseteq gHg^{-1}\right\}.$$

The sets $D_{K,H}$ and $U_{K,H}$ are dual in the sense that:

$$\begin{aligned} U_{K,H} &= \left\{g^{-1} : g \in D_{K,H}\right\}, \\ D_{K,H} &= \left\{g^{-1} : g \in U_{K,H}\right\}. \end{aligned}$$

We will use the sets $D_{K,H}$ (or alternatively $U_{K,H}$) to construct the fiber of Θ lying over $([K],[H])$. However, the sets contain redundancies, we will eliminate the redundancy using double coset decompositions. The normalizers $\mathrm{Nor}_G(H)$ and $\mathrm{Nor}_G(K)$ act on $D_{K,H}$ on the left and right, respectively. Indeed, if $h \in \mathrm{Nor}_G(H), k \in \mathrm{Nor}_G(K)$, then

$$\begin{aligned} hgkK(hgk)^{-1} &= hg(kKk^{-1})g^{-1}h^{-1} \\ &= hgKg^{-1}h^{-1} \\ &\subseteq hHh^{-1} = H. \end{aligned}$$

It follows then that $D_{K,H}$ is a union of double cosets $\mathrm{Nor}_G(H)g_l\mathrm{Nor}_G(K)$. Since the pair $(hgkK(hgk)^{-1}, H)$ is conjugate to the pair (gKg^{-1}, H) then every conjugacy class of pairs mapping to $([K],[H])$ corresponds to an element of $\mathrm{Nor}_G(H)\backslash D_{K,H}/\mathrm{Nor}_G(K)$. Thus we can compute unique representatives of the pairs lying over $([K],[H])$ by writing

$$G = \bigcup_l \mathrm{Nor}_G(H)g_l\mathrm{Nor}_G(K) \tag{16}$$

and testing to see which g_l satisfy

$$g_lKg_l^{-1} \subseteq H. \tag{17}$$

In the case of normal forms (K, gHg^{-1}) and the sets $U_{K,H}$ write

$$G = \bigcup_l \mathrm{Nor}_G(K)g_m\mathrm{Nor}_G(H) \tag{18}$$

and test

$$K \subseteq g_mHg_m^{-1}. \tag{19}$$

For solutions to equations 18 and 19 we may take the inverses of the solutions to 16 and 17 since

$$\left\{g^{-1} : g \in \mathrm{Nor}_G(H) g_l \mathrm{Nor}_G(K)\right\} = \mathrm{Nor}_G(K) g_l^{-1} \mathrm{Nor}_G(H).$$

It is time for an example and a look at the Burnside matrix in Magma.

Example 10 *As promised, consider the example $G = A_5$. Then, the sequence of subgroups and normalizers, produced by Magma, have these orders:*

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9
$\lvert H_j\rvert$	1	2	3	5	4	6	10	12	60
$\lvert \mathrm{Nor}_G(H)\rvert$	60	5	6	10	12	6	10	12	60

Note that the groups are not in increasing order of size but inclusions are respected by the ordering. The ordering does take into account the number of factors in the size. For A_5, each subgroup class is determined by the subgroup order.

In Magma, the `BurnsideMatrix` command gives us a first pass at the enumerating $\mathcal{P}(G)$. The rows and columns of the Burnside matrix $BM(G) = (b_{i,j})$ are indexed by the subgroup classes $[H_i]$. Above the diagonal, i.e., $i < j$, $b_{i,j}$ is the number of elements of $[H_i]$ contained in H_j and below the diagonal, i.e., $i > j$, $b_{i,j}$ is the number of elements of $[H_i]$ containing H_j. The diagonal obviously consists of 1's. The Burnside matrix does some over-counting as the actions of $\mathrm{Nor}_G(H)$ and $\mathrm{Nor}_G(K)$ are not taken into account.

Example 11 *The Burnside matrix BM below, for $G = A_5$ and the subgroup ordering above, has rows and columns labelled by H_i. The dots indicate no inclusions.*

$$BM = \begin{bmatrix} & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 \\ H_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ H_2 & 15 & 1 & . & . & 3 & 3 & 5 & 3 & 15 \\ H_3 & 10 & . & 1 & . & . & 1 & . & 4 & 10 \\ H_4 & 6 & . & . & 1 & . & . & 1 & . & 6 \\ H_5 & 5 & 1 & . & . & 1 & . & . & 1 & 5 \\ H_6 & 10 & 2 & 1 & . & . & 1 & . & . & 10 \\ H_7 & 10 & 6 & 2 & 1 & 1 & . & 1 & . & 6 \\ H_8 & 5 & 1 & 2 & . & 1 & . & . & 1 & 5 \\ H_9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The first and last columns are identical as we are simply counting the number of different conjugates of H_i. The top and bottom row are 1's since the trivial subgroup and the total group are normal.

Example 12 *We will define the reduced Burnside matrix $rBM(G)$ to be the Burnside matrix with entries reduced to take into account the actions of* $\mathrm{Nor}_G(H)$ *and* $\mathrm{Nor}_G(K)$*. The entries of $rBM(G)$ are the cardinalities of the fibres of Θ. Here is the reduced Burnside matrix for $G = A_5$ and the subgroup ordering above:*

$$rBM = \begin{bmatrix} & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 \\ H_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ H_2 & 1 & 1 & . & . & 1 & 1 & 1 & 1 & 1 \\ H_3 & 1 & . & 1 & . & . & 1 & . & 1 & 1 \\ H_4 & 1 & . & . & 1 & . & . & 1 & . & 1 \\ H_5 & 1 & 1 & . & . & 1 & . & . & 1 & 1 \\ H_6 & 1 & 1 & 1 & . & . & 1 & . & . & 1 \\ H_7 & 1 & 1 & . & 1 & 1 & . & 1 & . & 1 \\ H_8 & 1 & 1 & 1 & . & 1 & . & . & 1 & 1 \\ H_9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Example 13 *Here are some results of experiments with some groups.*

G	$\lvert\mathcal{S}(G)\rvert$	$\lvert\mathcal{P}(G)\rvert$	$rBM(G)$ *values* $\neq 0$
A_5	9	32	1
A_6	22	132	$1, 2, 3$
A_7	40	349	$1, 2, 3$
$PSL_2(7)$	15	79	$1, 2, 3$
$PSL_2(8)$	12	51	1
$PSL_2(11)$	16	81	$1, 3$

4.2 Ramification matrices

Now assume that we have a pair $K \subseteq H$ and suppose that we want to compute $ram(\pi_{H/K})$ where $\pi_{H/K} : S/K \to S/H$ is the natural projection. Set σ_K, σ_H to be the genus of S/K and S/H, respectively, and set $n = \frac{|H|}{|K|} =$ degree $\pi_{H/K}$. Then,

$$ram(\pi_{H/K}) = 2(\sigma_K - 1) - 2n(\sigma_H - 1), \qquad (20)$$

by the Riemann-Hurwitz formula 3. In turn, considering $S/K \to S/G$ and $S/H \to S/G$ we get

$$2(\sigma_K - 1) = 2\frac{|G|}{|K|}(\sigma_G - 1) + ram(\pi_{G/K}), \tag{21}$$

$$2(\sigma_H - 1) = 2\frac{|G|}{|H|}(\sigma_G - 1) + ram(\pi_{G/H}). \tag{22}$$

We may compute the $ram(\pi_{G/K})$ and $ram(\pi_{G/H})$ by means of the permutation representations on $G\backslash K$ and $G\backslash H$, and then $ram(\pi_{H/K})$ is easily computed by formula 20. The genus of S/H can be computed by

$$\sigma_H = 1 + \frac{|G|}{|H|}(\sigma_G - 1) + \frac{ram(\pi_{G/H})}{2}.$$

Since G is n-gonal, $\sigma_G = 0$ and we get the simplification

$$\sigma_H = \frac{ram(\pi_{G/H})}{2} - \frac{|G|}{|H|}. \tag{23}$$

Before proceeding with computing the matrices, let us first prove that $ram(\pi_{G/K})$ and $ram(\pi_{G/H})$ are indeed conjugation invariants. To this end, we shall take a detour in the next subsection and prove Proposition 15 on the action of $\mathrm{Aut}(G)$ on the permutation representations of G. We shall then return to the computation of ramification matrices.

4.3 Permutation representations and $\mathrm{Aut}(G)$

The discussion in this section is a little pedantic, but we want a precise statement and proof.

Given a subgroup H of G we get permutation representations of G on left and right cosets

$$L_g : hH \to ghH,$$
$$R_g : Hh \to Hhg^{-1}.$$

With these definitions, we have $L_{gh} = L_g \circ L_h$, $R_{gh} = R_g \circ R_h$. By selecting a specific ordering of the cosets

$$G = g_1 H \cup \cdots \cup g_n H \tag{24}$$

or

$$G = Hh_1 \cup \cdots \cup Hh_n, \tag{25}$$

we get homomorphisms $\rho_L : G \to \Sigma_n$, $\rho_R : G \to \Sigma_n$, where Σ_n is regarded as a permutation group on indices of the g_i or the h_i. Specifically we have:

$$\rho_L(g)i = j \text{ iff } gg_iH = g_jH,$$
$$\rho_R(g)i = j \text{ iff } Hh_ig^{-1} = Hh_j.$$

The permutations in Σ_n are multiplied by composition.

Remark 14 *Notice that we may pick $h_i = g_i^{-1}$ and in this case $\rho_L = \rho_R$.*

If θ is an automorphism of G, then setting $H^\theta = \theta(H)$, we get:

$$G = \theta(g_1)H^\theta \cup \cdots \cup \theta(g_n)H^\theta \tag{26}$$

or

$$G = H^\theta\theta(h_1) \cup \cdots \cup H^\theta\theta(h_n). \tag{27}$$

Let ρ_L^θ and ρ_R^θ denote the representations defined by the coset decompositions 26 and 27. They are permutation representations defined on the coset spaces defined by H^θ. Now writing,

$$gg_iH = g_jH,$$
$$Hh_ig^{-1} = Hh_j,$$

we obtain

$$\theta(g)\theta(g_i)H^\theta = \theta(gg_iH) = \theta(g_j)H^\theta,$$
$$H^\theta\theta(h_i)\theta(g^{-1}) = \theta(Hh_ig^{-1}) = H^\theta\theta(h_j).$$

It follows that

$$\rho_L^\theta(\theta(g)) = \rho_L(g),\ \ \rho_L^\theta = \rho_L \circ \theta^{-1},$$
$$\rho_R^\theta(\theta(g)) = \rho_R(g),\ \ \rho_R^\theta = \rho_R \circ \theta^{-1}.$$

We now easily obtain the following proposition.

Proposition 15 *For any subgroup H and $g \in G$, and any coset decomposition of G/H, $G/H^g, G\backslash H$, or $G\backslash H^g$ all of the permutation representations of $c \in G$ are conjugate Σ_n, and hence have the same cycle structure.*

Proof. First, note that any coset decomposition is the same as one of the decompositions in 24 or 25 with a relabelling. Any two relabellings of the cosets G/H or $G\backslash H$ yield

$$\rho'_L(c) = \pi_1 \rho_L(c) \pi_1^{-1},$$
$$\rho'_R(c) = \pi_2 \rho_R(c) \pi_2^{-1},$$

where π_1 and π_2 are labelling permutations. Also, by Remark 14 the choice $h_i = g_i^{-1}$ yields $\rho_L(c) = \rho_R(c)$. Thus all the permutation representations defined G/H, or $G\backslash H$ are conjugate. Next let θ be the automorphism $h \to ghg^{-1}$, then

$$\rho_L^{\theta}(c) = \rho_L(\theta^{-1}(c)) = \rho_L(g^{-1}cg) = \rho_L(g)^{-1}\rho_L(c)\rho_L(g)$$

and there is a similar equation for $\rho_R(c)$. Thus, no matter what conjugate H^g is chosen, what coset decomposition is chosen, or what labelling is chosen, $\rho_L(c)$ and $\rho_R(c)$ always belong to the same conjugacy class in Σ_n. ■

4.4 Ramification matrices continued

To compute the ramification and the strong branching matrix we follow these steps:

1. Set up $s \times s$ matrices RM and SM over $\mathbb{Z}[\epsilon]$ and initialize all entries with ϵ. The value ϵ in the location (i, j) indicates that $[H_i] \leq [H_j]$ is false. Since positive and negative integer entries are possible in the strong branching matrix, it is convenient for Magma to use a ring-based subterfuge to indicate non-inclusion.

2. For each conjugacy class of subgroups $[H]$ compute $ram(S/H \to S/G)$ using equation 15. Place the values in the last column of RM.

3. Using the reduced Burnside matrix, for each pair (i, j) satisfying $i \leq j$ and $[H_i] \leq [H_j]$ compute $ram(S/H_i \to S/H_j)$ using formulas 20, 21, and 22. The only numerical entries will be in the upper triangle of the matrix RM. The diagonal entries are 0, as the identity map has no ramification. In the lower triangle, below the diagonal, $[H_i] \nleq [H_j]$ so all entries equal ϵ.

4. Using formula 7 ,we fill in the entries of the strong branching matrix SM.

We now illustrate with an example.

Example 16 *Let $G = A_5$ and let (c_1, c_2, c_3) be the generating vector defined by: $c_1 = (3,4,5), c_2 = (1,2,3), c_3 = (1,5,4,3,2)$. Note that $c_1c_2c_3 = 1$, in the multiplication method used in Magma. The genera of the various quotient surfaces, computed using equation 23 are:*

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9
$\lvert H_j \rvert$	1	2	3	5	4	6	10	12	60
genus S/H_j	5	3	1	1	2	1	1	0	0

The ramification matrix is

$$RM = \left[\begin{array}{cccccccccc} & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 \\ H_1 & 0 & 0 & 8 & 8 & 0 & 8 & 8 & 32 & 128 \\ H_2 & . & 0 & . & . & 0 & 4 & 4 & 16 & 64 \\ H_3 & . & . & 0 & . & . & 0 & . & 8 & 40 \\ H_4 & . & . & . & 0 & . & . & 0 & . & 24 \\ H_5 & . & . & . & . & 0 & . & . & 8 & 32 \\ H_6 & . & . & . & . & . & 0 & . & . & 20 \\ H_7 & . & . & . & . & . & . & 0 & . & 12 \\ H_8 & . & . & . & . & . & . & . & 0 & 8 \\ H_9 & . & . & . & . & . & . & . & . & 0 \end{array}\right]$$

As sample calculations, let us compute the ramification of $S/H_8 \to S/G$ and $S/H_6 \to S/G$. For H_8 there are five cosets and the cycle structures of c_1, c_2, c_3 remain unchanged. It follows that

$$ram(S/H_8 \to S/G) = (3-1+1-1+1-1)+(3-1+1-1+1-1)+(5-1) = 8.$$

For H_7 we determine the monodromy vector, using Magma, to be $((1,2,3)(5,4,6),$ $(1,3,5)(2,4,6),\ (1,6,5,2,4))$, and

$$ram(S/H_6 \to S/G) = (3-1+3-1)+(3-1+3-1)+(5-1+1-1) = 12.$$

The entries of the ramification matrix are then easily computed. For instance the $(3,8)$ entry equals

$$2 \times (1-1) - 2 \times \frac{12}{3}(0-1) = 8.$$

Also, the last column of RM may be used to compute the genera. Finally, the strong branching matrix is

$$SM = \left[\begin{array}{cccccccccc}
 & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 \\
H_1 & 0 & -16 & -16 & -72 & -72 & -112 & -352 & -232 & -6952 \\
H_2 & . & 0 & . & . & -12 & -20 & -76 & -44 & -1676 \\
H_3 & . & . & 0 & . & . & -8 & . & -16 & -720 \\
H_4 & . & . & . & 0 & . & . & -4 & . & -240 \\
H_5 & . & . & . & . & 0 & . & . & -4 & -388 \\
H_6 & . & . & . & . & . & 0 & . & . & -160 \\
H_7 & . & . & . & . & . & . & 0 & . & -48 \\
H_8 & . & . & . & . & . & . & . & 0 & -32 \\
H_9 & . & . & . & . & . & . & . & . & 0
\end{array}\right]$$

There are no strongly branched pairs as we would expect in a simple group.

5 Examples and Magma code

5.1 Examples

Example 17 *Let G be `SmallGroup(24,1)` which is isomorphic to $Z_8 \ltimes Z_3$ $= \langle x, y : x^8 = y^3 = 1, xyx^{-1} = y^2 \rangle$, and pick the generating vector (x, x^2, x^4, y, y^2x). The signature is $(8,4,2,3,8)$ and the genus of S is 21. By fiddling with the branch points we can ensure that $\mathrm{Aut}(S) = G$. We get for subgroups and quotient surfaces:*

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$\lvert H_j \rvert$	1	2	3	4	6	8	12	24
genus S/H_j	21	5	5	0	1	0	0	0

The ramification matrix is

$$RM = \begin{array}{cc} & \begin{array}{cccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 \end{array} \\ \begin{array}{c} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ H_6 \\ H_7 \\ H_8 \end{array} & \begin{bmatrix} 0 & 24 & 16 & 48 & 40 & 56 & 64 & 88 \\ . & 0 & . & 12 & 8 & 16 & 20 & 32 \\ . & . & 0 & . & 8 & . & 16 & 24 \\ . & . & . & 0 & . & 2 & 4 & 10 \\ . & . & . & . & 0 & . & 4 & 8 \\ . & . & . & . & . & 0 & . & 4 \\ . & . & . & . & . & . & 0 & 2 \\ . & . & . & . & . & . & . & 0 \end{bmatrix} \end{array}$$

and the strong branching matrix is

$$SM = \begin{array}{cc} & \begin{array}{cccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 \end{array} \\ \begin{array}{c} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ H_6 \\ H_7 \\ H_8 \end{array} & \begin{bmatrix} 0 & 0 & -56 & 24 & -80 & -56 & -200 & -1016 \\ . & 0 & . & 8 & -16 & -8 & -40 & -232 \\ . & . & 0 & . & 0 & . & -8 & -88 \\ . & . & . & 0 & . & -2 & -8 & -50 \\ . & . & . & . & 0 & . & 0 & -16 \\ . & . & . & . & . & 0 & . & -8 \\ . & . & . & . & . & . & 0 & -2 \\ . & . & . & . & . & . & . & 0 \end{bmatrix} \end{array}.$$

The map $S \to S/H_4$ is strongly branched so, according to Accola's theory, H_4 should contain a proper normal subgroup in $\mathrm{Aut}(S)$*, namely H_4 itself. The fixed field of H_4 is the only strongly branched subfield of the Galois field/subgroup lattice. We determine this by looking at the first row. Only the first row of the matrix is useful to prove normality of subgroups. Also note the H_2, H_4 entries show that the fixed field of H_4 is strongly branched in the fixed filed of H_2.*

Example 18 *Without presenting the matrices we list some results:*

G	*signature*	*genus* S	$\lvert\mathcal{S}(G)\rvert$	$\lvert\mathcal{P}(G)\rvert$	*strongly branched* $\pi_{H/K}$
$\mathbb{Z}_5 \times \mathbb{Z}_5$	$(5,5,5)$	6	8	21	*none*
$\mathbb{Z}_3^3$	$(3,3,3,3)$	10	28	133	*none*
$\mathbb{Z}_4 \ltimes \mathbb{Z}_5$	$(4,2,5,4)$	8	6	18	$S \to S/H_2$

5.2 Magma code

The Magma code consists of the two scripts `pairs.mgm` and `rammatrix.mgm` posted as supplementary files to this paper [3]. The `pairs.mgm` script computes the orbit space of pairs. The script `rammatrix.mgm` computes genera, the ramification matrix, and the strong branching matrix. The `pairs.mgm` code is built into the script `rammatrix.mgm`. All the objects discussed in this paper can be computed with the script `rammatrix.mgm`.

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