A special case of the Yelton-Gaines Conjecture on Isomorphic Dessins

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by

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Abstract

Let \((\rho_0, \rho_1)\) and \((\rho'_0, \rho'_1)\) be two ordered pairs of permutations in \(S_n\) and let \(t\) be a divisor of \(n\). The Yelton-Gaines conjecture states that if at least one of these four permutations is a product of \(n/t\) disjoint \(t\)-cycles, and if there is a strong isomorphism (definition below) \(\phi: \langle \rho_0, \rho_1 \rangle \rightarrow \langle \rho'_0, \rho'_1 \rangle\) between the two subgroups of \(S_n\) generated by the elements in each ordered pair, then there is a fixed permutation \(\tau \in S_n\) that simultaneously conjugates \(\rho_i\) to \(\rho'_i\) for \(i = 0, 1\). The conclusion of this conjecture can be restated to say that the two dessins d'enfants corresponding to the two ordered pairs are isomorphic.

In this paper a proof of this conjecture is given in the case in which all of the initial four permutations are fixed-point-free involutions.

1 Introduction

The term \(dessin\ d'enfant\) was coined by Grothendieck to refer to a bipartite graph that is embedded in a compact, oriented Riemann surface, that is, into a torus with \(g \geq 0\) holes. Every dessin is determined up to isomorphism by an ordered pair of permutations from a symmetric group. We emphasize that this pair of permutations is ordered and that the dessin \(D(\rho_0, \rho_1)\) determined by the pair \((\rho_0, \rho_1)\) is not usually isomorphic to the dessin \(D(\rho_1, \rho_0)\) obtained by reversing the order. Two ordered pairs of permutations are considered to be the same when there is a \(\tau \in S_n\) that simultaneously conjugates each component of the first ordered pair into the corresponding component of the second ordered pair. Every dessin gives rise to a graph obtained by ignoring the embedding into the surface of a torus. It frequently happens that two non-isomorphic dessins have isomorphic underlying graphs.

A Gassmann triple consists of a group \(G\) and two locally conjugate subgroups \(H\) and \(H'\), meaning that there exists a bijection \(\psi: H \rightarrow H'\) such that \(\psi(h)\) is conjugate to \(h\) in \(G\) for every \(h \in H\). Let \((G, H, H')\) be a Gassmann triple and note that \(H\) and \(H'\) must have the same index \(n\) in \(G\). Choose an ordered pair \((g_0, g_1)\) of elements of \(G\). These elements act on the set of cosets \(G/H\) by left multiplication, giving an ordered pair of permutations \((\rho_0, \rho_1)\) in \(S_n\). The same elements \(g_0, g_1\) act on the cosets \(G/H'\) giving a second ordered pair of permutations \((\rho'_0, \rho'_1)\). In turn, these two ordered pairs give rise to two dessins \(D(\rho_0, \rho_1)\) and \(D(\rho'_0, \rho'_1)\). These two dessins are called Gassmann equivalent.

We now describe the construction by which a pair of permutations yields a dessin. Let \(\rho_0, \rho_1 \in S_n\). Then each cycle in the permutations corresponds to a vertex in our dessin. The cycles of \(\rho_0\) correspond to black vertices and the cycles of \(\rho_1\) correspond white vertices. If a vertex is induced by an \(n\)-cycle it is then endowed with \(n\) branches. The branches of a vertex are labeled in counter-clockwise order with the elements permuted by the corresponding cycle. After this process is complete, for all \(n \in \{1, ..., n\}\) there is a branch, labeled \(n\), attached to a white vertex and another attached to a black vertex. These two branches are then connected. This is done for each \(n \in \{1, ..., n\}\) and we, thereby, produce the underlying graph of our dessin.
**Example 1:** Let $\rho_0 = (12)(34)(56)$ and $\rho_1 = (13)(456)(2)$. These permutations correspond to the following graph:

![Graph Image](image)

Figure 1: This is the dessin corresponding to $\rho_0$ and $\rho_1$.

The monodromy group of a dessin is the subgroup of $S_n$ generated by the defining permutations. When two dessins are Gassmann equivalent, there exists an isomorphism $\phi$ between the monodromy groups that is a local conjugation in $S_n$ and that maps $\rho_0$ to $\rho'_0$ and maps $\rho_1$ to $\rho'_1$; see [M-P] for details. We refer to such an isomorphism between subgroups of a symmetric group as a *strong* isomorphism.

In 2007 Jeff Yelton examined many examples of Gassmann equivalent dessins. (See [Y]). He was interested in knowing when the underlying graphs were isomorphic. He conjectured that if $D(\rho_0, \rho_1)$ and $D(\rho'_0, \rho'_1)$ are Gassmann equivalent dessins and if at least one of $\rho_0, \rho_1, \rho'_0, \rho'_1$ is composed entirely of $t$-cycles for some fixed $t$, then the underlying graphs are isomorphic. In 2008, Ben Gaines expressed the opinion that Yelton’s hypotheses should imply the stronger conclusion that the dessins are isomorphic. In fact, Gaines not only strengthened the conclusion, but also slightly weakened the hypothesis. He did so by removing the reference to Gassmann equivalent dessins, but still requiring there to be a strong isomorphism between monodromy groups. His work is found in [G]. We refer to Gaines’ formulation as the *Yelton-Gaines* conjecture. Here is the precise statement.

**Yelton-Gaines Conjecture** Let $\rho_0, \rho_1, \rho'_0, \rho'_1$ be permutations in $S_n$ for which there is a group isomorphism $\phi : \langle \rho_0, \rho_1 \rangle \longrightarrow \langle \rho'_0, \rho'_1 \rangle$ which is a local conjugation in $S_n$ and which maps $\rho_i$ to $\rho'_i$ for $i = 0, 1$. If at least one of the four permutations is composed entirely of $t$-cycles for some fixed divisor $t$ of $n$, then there is a permutation $\tau \in S_n$ that conjugates $\rho_i$ into $\rho'_i$ for $i = 0, 1$.

## 2 Proof in the Special Case of Fixed-Point-Free Involutions

In this paper we prove the conjecture in the case when each of the four permutations $\rho_0, \rho_1, \rho'_0, \rho'_1$ is a product of disjoint 2-cycles (and contains no 1-cycles). This forces $n$ to be even. Write $n = 2m$. 
We begin with a lemma.

**Zigzag Lemma** For $i = 0, 1$ let $\rho_i$ be a product of $m$ disjoint transpositions in $S_{2m}$. Let $\rho_2$ denote the product $\rho_1 \rho_0$, with $\rho_0$ acting first. Fix an element $a_1$ in $\{1, \ldots, 2m\}$, let $c$ denote the $\rho_2$-orbit of $a_1$, and let $s = |c|$ be the length of the cycle $c$. Then there exist a set of pairwise distinct elements $\{a_2, \ldots, a_s, b_1, \ldots, b_s\}$ in $\{1, \ldots, 2m\}$ each different from $a_1$, for which

\[
\rho_0 = (a_1, b_1)(a_2, b_2) \cdots (a_{s-1}, b_{s-1})(a_s, b_s) * * *
\]

\[
\rho_1 = (b_1, a_2)(b_2, a_3) \cdots (b_{s-1}, a_s)(b_s, a_1) * * *
\]

are the cycle decompositions, with $* * *$ denoting products of 2-cycles disjoint from the displayed cycles. Moreover, the $\rho_2$-orbit $c$ is $c = (a_1, \ldots, a_s)$ and another orbit in $\rho_2$ is $d = (b_s, \ldots, b_1)$, disjoint from $c$.

**Proof.** Define $b_1 = \rho_0(a_1)$. Then $b_1 \neq a_1$ since $\rho_0$ has no fixed-points. Let $x_1 = \rho_1(b_1)$. Then $x_1 \neq b_1$ since $\rho_1$ has no fixed points. We distinguish two cases: either $x_1 = a_1$ or $x_1 \neq a_1$. If $x_1 = a_1$, then

\[
\rho_0 = (a_1, b_1) * * *
\]

\[
\rho_1 = (b_1, a_1) * * *
\]

In this case, $\rho_2(a_1) = a_1$ and $c = (a_1)$ is the $\rho_2$-orbit of $a_1$ so the length of $c$ is $s = 1$. Moreover, $d = (b_1)$ is also cycle of length $s = 1$ in $\rho_2$ disjoint from $c$, as claimed by the lemma. Now consider the case $x_1 \neq a_1$, and rename $x_1$ to $a_2$. Then $a_1, b_1, a_2$ are pairwise distinct. Define $b_2 = \rho_0(a_2)$. Then $b_2$ is distinct from each of $a_1, b_1, a_2$. Let $x_2 = \rho_1(b_2)$. Then $x_2$ is different from $b_1, a_2, b_2$. We again distinguish two cases: Either $x_2 = a_1$ or $x_2 \neq a_1$. If $x_2 = a_1$, then

\[
\rho_0 = (a_1, b_1)(a_2, b_2) * * *
\]

\[
\rho_1 = (b_1, a_2)(b_2, a_1) * * *
\]

In this case, the $\rho_2$-orbit of $a_1$ is $c = (a_1, a_2)$ of length $s = 2$, and $d = (b_2, b_1)$ is another length 2 cycle in $\rho_2$ that is disjoint from $c$, as the lemma claims. In the other case when $x_2 \neq a_1$ then rename $x_2$ to $a_3$. Continue this process. If after $m - 1$ iterations we have not yet produced an element $x_i = a_1$ then necessarily in the $m$th iteration we will have $x_m = a_1$ since $a_1$ must occur somewhere in the cycle decomposition of $\rho_1$. Let $s \leq m$ be the first (and in fact only) integer $i$ for which this process produces $x_i = a_1$. Then

\[
\rho_0 = (a_1, b_1)(a_2, b_2) \cdots (a_{s-1}, b_{s-1})(a_s, b_s) * * *
\]

\[
\rho_1 = (b_1, a_2)(b_2, a_3) \cdots (b_{s-1}, a_s)(b_s, a_1) * * *
\]

Then $c = (a_1, a_2, \ldots, a_s)$ and $d = (b_s, b_{s-1}, \ldots, b_1)$ are disjoint cycles of $\rho_2$ of length $s$. This proves the lemma. $\square$

We note a corollary.

**Corollary 1:** Let $\rho_0, \rho_1$, and $\rho_2$ be as in the zigzag lemma, and let $M = \langle \rho_0, \rho_1 \rangle$ be the monodromy group. Then the number of cycles of $\rho_2$ of a given length is even and the total number of cycles in
\( \rho_2 \) is even. The cycles in \( \rho_2 \) of a given length can be paired as \( c_i \) and \( d_i \) so that the \( M \)-orbits in \( \{1, \ldots, 2m\} \) are precisely the union of the elements permuted by \( c_i \) and \( d_i \).

**Proof**: Fix an element \( a_1 \) and let the \( \rho_2 \)-orbit of \( a_1 \) be \( c = (a_1, a_2, \ldots, a_s) \). According to the zigzag lemma, there are elements \( b_1, b_2, \ldots, b_s \) for which \( d = (b_s, \ldots, b_1) \) is another orbit of \( \rho_2 \) of length \( s \) and disjoint from \( c \). Moreover, \( \rho_0 \) and \( \rho_1 \) interchange the set of elements of \( c \) and with the set of elements of \( d \). The map \( \rho_0 \) preserves the order while \( \rho_1 \) doesn’t, but the order doesn’t matter here. So the union \( c \cup d \) (with minor notational liberty) is mapped to itself by the monodromy group \( M \). And this union is the \( M \)-orbit of \( a_1 \) as can be seen by letting the generators \( \rho_0, \rho_1 \) alternately act on \( a_1 \).

We are now ready to prove the main result of this paper.

**Theorem**: Let each of \( \rho_0, \rho_1, \rho'_0, \rho'_1 \in S_{2m} \) be products of \( m \) disjoint 2-cycles. If there exists a strong isomorphism of monodromy groups \( \phi : \langle \rho_0, \rho_1 \rangle \to \langle \rho'_0, \rho'_1 \rangle \) then there is an element \( \tau \in S_{2m} \) that simultaneously conjugates \( \rho_i \) to \( \rho_i' \) for \( i = 0, 1 \). In other words, there is an isomorphism of dessins \( D(\rho_0, \rho_1) \cong D(\rho'_0, \rho'_1) \).

**Proof**: Let \( \rho_2 = \rho_1 \rho_0 \) and let \( \rho'_2 = \rho'_1 \rho'_0 \). The strong isomorphism \( \phi \) maps \( \rho_2 \) to \( \rho'_2 \). Since \( \phi \) is a local conjugation, then \( \rho_2 \) and \( \rho'_2 \) have the same cycle structure. By Corollary 1, there are an even number, say \( 2r \), of cycles of \( \rho_2 \), and the set of these cycles can be written as \( \{c_1, d_1, \ldots, c_r, d_r\} \) where each pair \( c_i, d_i \) is as described in the zigzag lemma. That is, we can write \( c_i = (a_{i,1}, \ldots, a_{i,s_i}) \) and \( d_i = (b_{i,s_i}, \ldots, b_{i,1}) \) so that \( \rho_0, \rho_1, \rho_2 \) are given by

\[
\begin{align*}
\rho_0 &= \prod_{i=1}^{r} (a_{i,1}, b_{i,1}) \ldots (a_{i,s_i}, b_{i,s_i}) \\
\rho_1 &= \prod_{i=1}^{r} (b_{i,1}, a_{i,2}) \ldots (b_{i,s_i}, a_{i,1}) \\
\rho_2 &= \prod_{i=1}^{r} (a_{i,1}, a_{i,2}, \ldots, a_{i,s_i})(b_{i,1}, b_{i,2}, \ldots, b_{i,s_i}).
\end{align*}
\]

Similarly, there are \( 2r \) cycles in \( \rho'_2 \) and the set of these cycles can be written as \( \{c'_1, d'_1, \ldots, c'_r, d'_r\} \) with \( c_i \) and \( c'_i \) having the same length for \( i = 1, \ldots, r \). Moreover, we can write \( c'_i = (a'_{i,1}, \ldots, a'_{i,s_i}) \) and \( d'_i = (b'_{i,s_i}, \ldots, b'_{i,1}) \) so that \( \rho'_0, \rho'_1, \rho'_2 \) are given by

\[
\begin{align*}
\rho'_0 &= \prod_{i=1}^{r} (a'_{i,1}, b'_{i,1}) \ldots (a'_{i,s_i}, b'_{i,s_i}) \\
\rho'_1 &= \prod_{i=1}^{r} (b'_{i,1}, a'_{i,2}) \ldots (b'_{i,s_i}, a'_{i,1}) \\
\rho'_2 &= \prod_{i=1}^{r} (a'_{i,1}, a'_{i,2}, \ldots, a'_{i,s_i})(b'_{i,1}, b'_{i,2}, \ldots, b'_{i,s_i}).
\end{align*}
\]

With the notation just established, define the permutation \( \tau \in S_{2m} \) by declaring that for each \( i = 1, \ldots, r \) and for each \( j = 1, \ldots, s_i \) that \( \tau(a_{i,j}) = a'_{i,j} \) and \( \tau(b_{i,j}) = b'_{i,j} \). This \( \tau \) gives the desired conjugation. We note in passing that the given strong isomorphism \( \phi \) agrees with conjugation by \( \tau \) since both maps are group isomorphisms taking the same values on generators of the monodromy group \( M \).

\[ \square \]

With the discussion above, we conclude:

**Corollary 2**: If the fixed-point-free involutions \( \rho_0, \rho_1, \rho'_0, \rho'_1 \in S_n \) arise from a pair of elements \( g_0, g_1 \) in a Gassmann triple \((G, H, H')\) of finite groups, then the corresponding dessins are isomor-
phic: $\mathbb{D}(\rho_0, \rho_1) \cong \mathbb{D}(\rho'_0, \rho'_1)$.

References


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