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Topological and \mathcal{H}^q Equivalence of Prime Cyclic p -gonal Actions on Riemann Surfaces

S. Allen Broughton

July 25, 2016

Abstract

Two Riemann surfaces S_1 and S_2 with conformal G -actions have topologically equivalent actions if there is a homeomorphism $h : S_1 \rightarrow S_2$ which intertwines the actions. A weaker equivalence may be defined by comparing the representations of G on the spaces of holomorphic q -differentials $\mathcal{H}^q(S_1)$ and $\mathcal{H}^q(S_2)$. In this note we study the differences between topological equivalence and \mathcal{H}^q equivalence of prime cyclic actions, where S_1/G and S_2/G have genus zero.

Contents

1	Introduction	2
1.1	Problem statement	3
1.2	Previous and related work	3
1.3	Outline of the paper	5
2	Actions of groups on surfaces	5
2.1	Generating vectors and equivalence of actions	5
2.2	Actions of \mathbb{Z}_p	8
2.3	Rotation numbers	9
3	Representations of G on $\mathcal{H}^q(S)$	10
3.1	Character Patterns	10
3.2	Multiplicity matrices for prime cyclic groups	12
3.3	Alternate perspective	17

4	Examples and conjectures	21
4.1	Examples	21
4.2	Conjectures and Remarks	22

1 Introduction

The finite group G acts orientably on the Riemann surface S if there is a monomorphism:

$$\epsilon : G \rightarrow \text{Homeo}^+(S),$$

the group of orientation preserving homeomorphisms of S . If the image $\epsilon(G)$ consists of conformal automorphisms of S , i.e.,

$$\epsilon : G \rightarrow \text{Aut}(S),$$

we say that G acts conformally on S . Two actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are topologically equivalent if there is an intertwining homeomorphism $h : S_1 \rightarrow S_2$ and an automorphism $\omega \in \text{Aut}(G)$ such that

$$\epsilon_2(g) = h\epsilon_1(\omega(g))h^{-1}, \forall g \in G. \tag{1}$$

For conformal actions, we have in diagram form:

$$\begin{array}{ccc} G & \xrightarrow{\epsilon_1} & \text{Aut}(S_1) \\ \downarrow \omega & & \downarrow Ad_h \\ G & \xrightarrow{\epsilon_2} & \text{Aut}(S_2) \end{array} \tag{2}$$

If h is a conformal map then we say that the actions are conformally equivalent.

The classification of topological equivalence classes of finite group actions is equivalent to the classification of conjugacy classes of finite subgroups of the mapping class group Mod_σ . Therefore, topological classification is a natural classification. In our problem statement below we consider a weaker equivalence through the representation on q -differentials. The equivalence relation defined by Abelian differentials, $q = 1$, has been used to classify actions in low genus, by some authors. We shall show in this paper that this equivalence relation is strictly weaker than topological equivalence and the discrepancy gets quite dramatic for even moderate genus.

1.1 Problem statement

The group G always acts on the homology $H_1(S)$ or cohomology $H^1(S)$ of S . Any topologically equivalent action yields equivalent representations. In the conformal action case, the group G also acts linearly on the various spaces of holomorphic q -differentials $\mathcal{H}^q(S)$. The space $\mathcal{H}^q(S)$ is the \mathbb{C} -vector space of holomorphic sections of $T^*(S) \otimes \cdots \otimes T^*(S)$ (q times), where $T^*(S)$ is the cotangent bundle, a holomorphic line bundle. Given actions of G on two surfaces S_1, S_2 of the same genus we say that the actions of G are $\mathcal{H}^q(S)$ -equivalent if the two different representations on $\mathcal{H}^q(S_1)$ and $\mathcal{H}^q(S_2)$ are equivalent over \mathbb{C} . It turns out that topologically equivalent actions are $\mathcal{H}^q(S)$ -equivalent for every q . However, two $\mathcal{H}^q(S)$ -equivalent actions may be topologically inequivalent. In this case we call the actions *holomorphically q -conflated*. The goal of this paper is to explore the relationship between topological equivalence and $\mathcal{H}^q(S)$ -equivalence for cyclic prime p -gonal actions. This means that S/G has genus zero and G is a cyclic group of prime order p .

1.2 Previous and related work

A review of various methods of classification of group actions is given in [3], a paper which is devoted to the complete topological classification of actions on surfaces in genus 2 and 3. The methods frequently make use of the G -action on the homology or cohomology of the surface. For prime cyclic p -gonal actions, the methods are ineffective as prime cyclic p -gonal actions yield the same complex representation on $H_1(S)$ and $H^1(S)$. On the analytic side, another classification scheme, has been considered by A. Kuribayashi, I. Kuribayashi and H. Kimura in various papers [5], [6], [7], [8], [9] by considering the representations on Abelian differentials $\mathcal{H}^1(S)$. These authors have worked out the classification of all subgroups of $GL(\sigma, \mathbb{C})$ that occur in this way for $2 \leq \sigma \leq 5$. In these papers the authors sought to classify the Riemann surfaces of low genus, by relating the characteristics of their Weierstrass points, their equations and their automorphism groups. None of these works consider the problem of topologically inequivalent actions of groups. In fact, the classification of actions by topological equivalence is strictly finer than the classification scheme considered by these authors. The smallest genus example where the abelian differentials fail to distinguish topologically distinct actions occurs in genus 4 with $G = \mathbb{Z}_5$ and 4 branch

points. For an interesting example of conflation, see Example 3.1 of [10], where it is shown that there are 7 inequivalent actions of $J(1)$, Janko's first group, on a surface of genus 2091. From the results of [2] none of these actions can be distinguished by the action of $J(1)$ on the space of Abelian differentials.

In the paper [11], Streit and Wolfart propose to use the representations on q -differentials as Galois invariants to distinguish various quasi-platonic surfaces under the absolute Galois group action. Though not strictly relevant to our discussion on q -conflation, the cited paper gives an application of characters of G obtained from q -differentials.

A cyclic p -gonal Riemann surface with t branch points is "superelliptic" and has an equation of the form

$$y^p = \prod_{j=1}^t (x - a_j)^{c_j}, \quad (3)$$

where a_1, \dots, a_t are distinct complex numbers and the exponents satisfy:

$$\begin{aligned} 1 &\leq c_j \leq p-1, \\ c_1 + \dots + c_t &= 0 \pmod{p}. \end{aligned}$$

These surfaces have been extensively studied, and we focus our attention on these surfaces as our main example, though we shall not exploit their explicit equations.

We noted previously that topological classes of G actions correspond to conjugacy classes of the mapping class group. In [2] and [3] a detailed description of the relationship among the following three concepts is given:

- the conjugacy classification of finite subgroups of the mapping class group Mod_σ ,
- topological equivalence of finite groups acting conformally on surfaces, and
- the branch locus \mathcal{B}_σ of \mathcal{M}_σ , the moduli space of surfaces of genus $\sigma \geq 2$ which consists of surfaces with a non-trivial automorphism group (at least for $\sigma > 2$).

The branch locus is a union of smooth subvarieties (strata) each of which corresponds to a topological equivalence class of actions, determined by the common automorphism group of the surfaces in the stratum. Because q -conflation the characters of the $\mathcal{H}^q(S)$ representations may fail to separate the strata for \mathbb{Z}_p actions.

Finally, our discussion will follow the results of [1] closely. In that work a decomposition of the various $\mathcal{H}^q(S)$ into irreducibles is given.

1.3 Outline of the paper

In Section 2 we recall the basics of conformal actions on surfaces and reduce topological equivalence to relations on generating vectors of G . We focus specifically on \mathbb{Z}_p actions and give a canonical form for generating vectors corresponding to a specific topological action class. In Section 3 we recall the basics of the representations $\mathcal{H}^q(S)$, specifically the Eichler trace formula. We then derive explicit formulas for the characters of \mathbb{Z}_p actions which can be determined explicitly in terms of arithmetic formulas dependent on the prime p , the number of branch points t , the degree q of the differential and the canonical generating vector of the \mathbb{Z}_p action. Our main results are Propositions 7 and 11. In Section 4 we give some examples and make some conjectures.

Acknowledgement 1 *I would like to thank Victor Gonzalez for conversations and questions that inspired this note.*

2 Actions of groups on surfaces

2.1 Generating vectors and equivalence of actions

The quotient surface $S/G = T$ of a conformal action is a closed Riemann surface of genus τ with a unique conformal structure so that

$$\pi_G : S \rightarrow S/G = T \tag{4}$$

is holomorphic. The quotient map $\pi_G : S \rightarrow T$ is branched uniformly over a finite set $B_G = \{Q_1, \dots, Q_t\}$ such that π_G is an unramified covering exactly over $T^\circ = T - B_G$. Let $S^\circ = \pi_G^{-1}(T^\circ)$ so that $\pi_G : S^\circ \rightarrow T^\circ$ is an unramified covering space whose group of deck transformation equals $\epsilon(G)$, restricted to S° . This covering determines a normal subgroup $\Pi_G = \pi_1(S^\circ) \triangleleft \pi_1(T^\circ)$

and an exact sequence $\Pi_G \hookrightarrow \pi_1(T^\circ) \twoheadrightarrow \epsilon(G)$ by mapping loops to deck transformations. Combine the last map with $\epsilon(G) \xrightarrow{\epsilon^{-1}} G$ to get an exact sequence

$$\Pi_G \hookrightarrow \pi_1(T^\circ) \xrightarrow{\xi} G. \quad (5)$$

We have left out base points to simplify the exposition, and so ξ is ambiguous up to inner automorphisms. However this is no matter since we are only concerned about actions up to automorphisms of G .

In the opposite direction we start with a sequence as in equation 5. We can construct a unramified, holomorphic, regular covering space which we still denote $\pi_G : S^\circ \rightarrow T^\circ$. The deck transformations are automatically holomorphic. We can fill in the punctures to get a closed surface, and the action of $\epsilon : G \rightarrow \text{Aut}(S^\circ)$ extends to a conformal action $\epsilon : G \rightarrow \text{Aut}(S)$ at the filled in punctures, using the removable singularity theorem.

The fundamental group $\pi_1(T^\circ)$ has the following presentation:

$$\text{generators} : \{\alpha_i, \beta_i, \gamma_j, 1 \leq i \leq \tau, 1 \leq j \leq t\}, \quad (6)$$

$$\text{relations} : \prod_{i=1}^{\sigma} [\alpha_i, \beta_i] \prod_{j=1}^s \gamma_j = 1. \quad (7)$$

Define

$$a_i = \xi(\alpha_i), b_i = \xi(\beta_i), c_j = \xi(\gamma_j).$$

Then the $2\tau + t$ tuple $(a_1, \dots, a_\tau, b_1, \dots, b_\tau, c_1, \dots, c_t)$ is called a *generating vector* for the action. We observe that

$$G = \langle a_1, \dots, a_\tau, b_1, \dots, b_\tau, c_1, \dots, c_t \rangle, \quad (8)$$

$$o(c_j) = n_j, \quad (9)$$

and

$$\prod_{i=1}^{\tau} [a_i, b_i] \prod_{j=1}^t c_j = c_1^{n_1} = \dots = c_t^{n_t} = 1, \quad (10)$$

for some integers $n_j \geq 2$. We call $(\tau : n_1, \dots, n_t)$ the signature of the action and we write (n_1, \dots, n_t) if $\tau = 0$. Finally we define

$$\mu(\tau : n_1, \dots, n_t) = 2\tau - 2 + t - \left(\frac{1}{n_1} + \dots + \frac{1}{n_t} \right).$$

The genus of S is given by the Riemann- Hurwitz equation

$$\frac{2\sigma - 2}{|G|} = \mu(\tau : n_1, \dots, n_t). \quad (11)$$

From this equation we see that the quantity $\mu(\tau : n_1, \dots, n_t)$ may be interpreted as an orbifold Euler characteristic of S/G or the area of a fundamental region for the G action on S , divided by 2π .

Remark 2 *When S/G has genus zero we say that the action is n -gonal since the projection $S \rightarrow S/G$ is an $|G|$ -gonal projection. In this case the signature is (n_1, \dots, n_t) , the generating vector is (c_1, \dots, c_t) , and the Riemann-Hurwitz equation becomes*

$$\frac{2\sigma - 2}{|G|} = t - 2 - \left(\frac{1}{n_1} + \dots + \frac{1}{n_t} \right).$$

The generating vector of a topological equivalence class is not unique. The topological equivalence relation acts on the generating vectors. First the action of $\omega \in \text{Aut}(G)$:

$$\begin{aligned} & (a_1, \dots, a_\tau, b_1, \dots, b_\tau, c_1, \dots, c_t) \\ \rightarrow & (\omega(a_1), \dots, \omega(a_\tau), \omega(b_1), \dots, \omega(b_\tau), \omega(c_1), \dots, \omega(c_t)) \end{aligned} \quad (12)$$

For an intertwining homeomorphism $h : S_1 \rightarrow S_2$, as in equation 1, we see from the diagram 2, that there is an induced diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{h} & S_2 \\ \downarrow \pi_{\epsilon_1(G)} & & \downarrow \pi_{\epsilon_2(G)} \\ T_1 & \xrightarrow{\bar{h}} & T_2 \end{array} \quad (13)$$

and hence an induced isomorphism

$$h_* : \pi_1(T_1^\circ) \rightarrow \pi_1(T_2^\circ).$$

The isomorphism h_* is base point dependent, but a different choice of base point results in composing h_* an inner automorphism. For $\alpha_i, \beta_i, \gamma_j \in \pi_1(T_1^\circ)$ each loop $h_*(\alpha_i), h_*(\beta_i), h_*(\gamma_j)$ can be written as words in the corresponding $\alpha'_i, \beta'_i, \gamma'_j \in \pi_1(T_2^\circ)$. The action on $(a_1, \dots, a_\tau, b_1, \dots, b_\tau, c_1, \dots, c_t)$ is to rewrite each group element in the vector as a word in a'_i, b'_i, c'_j using the words

defined by h_* . The general types of h_* can be found in many places including [2] and [3] but we will only need the following “braid automorphisms” for cyclic p -gonal actions. We may choose a homeomorphism h , preserving the branch sets, satisfying $h(Q_i) = Q'_{i+1}$, $h(Q_{i+1}) = Q'_i$, $Q_i, Q_{i+1} \in T_1$, $Q'_i, Q'_{i+1} \in T_2$, and preserving the indices of all other branch points. The homeomorphism h may be further restricted so that we have

$$\begin{aligned} h_*(\gamma_i) &= \gamma'_{i+1}, \\ h_*(\gamma_i) &= (\gamma'_{i+1})^{-1} \gamma'_i \gamma'_{i+1}. \end{aligned}$$

Then, the action on generating vectors is:

$$\begin{aligned} &(a_1, \dots, a_\tau, b_1, \dots, b_\tau, c_1, \dots, c_t) \\ \rightarrow &\left(a'_1, \dots, a'_\tau, b'_1, \dots, b'_\tau, c'_1, \dots, c'_{i+1}, (c'_{i+1})^{-1} c'_i c'_{i+1}, \dots, c'_t \right) \end{aligned} \quad (14)$$

When $T_1 = T_2$, the group generated by these transformations induces an action of the symmetric group on the indices (n_1, \dots, n_t) . We choose the subgroup which preserves the orders of the (n_1, \dots, n_t) . Now assume that G is abelian and that $\tau = 0$. Then the action generated by the braid transformations is

$$(c_1, \dots, c_t) \rightarrow (c_{\theta(1)}, \dots, c_{\theta(t)}) \quad (15)$$

where θ is a transformation preserving the orders n_i , namely $n_{\theta(i)} = n_i$. If $G = \mathbb{Z}_p$ then θ can be any permutation.

2.2 Actions of \mathbb{Z}_p

For a cyclic p -gonal action we take $G = \mathbb{Z}_p$, and the c_i may be taken as integers in the range $1 \leq c_i \leq p - 1$ such that $c_1 + \dots + c_t = 0 \pmod{p}$. The c_i may be interpreted as the exponents in the superelliptic equation 3. The genus of such a surface is $\sigma = (t - 2) \frac{p-1}{2}$, according to the Riemann-Hurwitz equation 11. From equation 15 we see that we may select the c_i in any order we shall assume that $c_1 \leq \dots \leq c_t$. This will give us a normal form under the braid transformations. The automorphism action allows us to multiply the c_i by any e satisfying $1 \leq e \leq p - 1$, namely

$$(c_1, \dots, c_t) \rightarrow (ec_1, \dots, ec_t) \pmod{p}. \quad (16)$$

It follows that each topological equivalence class of actions we may assume that one of the c_i equals 1. We use the following algorithm to find the topological equivalence classes of p -gonal actions of \mathbb{Z}_p with t branch points.

Algorithm 3 Step 1: Find all sorted tuples (c_1, \dots, c_t) of integers satisfying

$$1 \leq c_i \leq p - 1$$

$$1 = c_1 \leq \dots \leq c_t$$

$$c_1 + \dots + c_t = 0 \pmod{p}$$

Step 2: For each (c_1, \dots, c_t) in Step 1 lexicographically sort the set

$$\{(ec_1, \dots, ec_t)\} : e \in \mathbb{Z}_p\}$$

and select the first tuple.

Step 3: Lexicographically sort all the tuples obtained in Step 2 into a single list. This is a set of representative generating vectors for the topological equivalence classes of actions.

Remark 4 A process for enumeration of topological equivalence classes of actions of elementary abelian groups is described in [4]. In Example 9, of Section 3 the topological equivalence classes for \mathbb{Z}_5 with signature, $(5, 5, 5, 5)$ are discussed, the smallest case in which holomorphic conflation occurs. For this case, the algorithm produces three class representatives: $(c_1, c_2, c_3, c_4) = (1, 1, 1, 2), (1, 1, 4, 4),$ and $(1, 2, 3, 4)$. Even for modest genus the number of topological equivalence classes can get quite large. See Example 16 in Section 4.

2.3 Rotation numbers

For $g \in \text{Aut}(S)$ we denote $S^g = \{P \in S : gP = P\}$, the set of fixed points of g on S . If $P \in S^g$, then the induced map of tangent spaces $dg^{-1} : T_P(S) \rightarrow T_P(S)$ is multiplication by an $o(g)$ -th root of unity, denoted by $\varepsilon(g, P)$. It is easy to show that we may pick the c_i , and $P_i \in \pi_G^{-1}(Q_i)$ such that $G_{P_i} = \{g \in G : gP_i = P_i\} = \langle c_i \rangle$ and

$$\varepsilon(c_i, P_i) = \exp\left(\frac{2\pi\sqrt{-1}}{n_i}\right).$$

If we let (c_1, \dots, c_t) be a generating vector for \mathbb{Z}_p . Each c_i has a unique fixed point $P_i \in \pi_G^{-1}(Q_i)$ since the stabilizer at P_i is the entire group. In this case

$$\varepsilon(c_i, P_i) = \exp\left(\frac{2\pi i}{p}\right) = \zeta.$$

To calculate rotation numbers we define d_i so that

$$c_i d_i = 1 \pmod{p}. \quad (17)$$

Taking the additive structure of \mathbb{Z}_p into account, we compute:

$$\varepsilon(g, P_i) = \varepsilon(g c_i d_i, P_i) = \varepsilon(c_i, P_i)^{g d_i} = \zeta^{g d_i}. \quad (18)$$

3 Representations of G on $\mathcal{H}^q(S)$

In this section we first fix an action for general G and discuss the character decomposition as q varies. Then we focus on the prime cyclic, p -gonal case getting very specific results. Finally, we switch perspectives, fixing t and q and consider the variation of the character decomposition over all topological types. In particular we describe a process to determine which actions are conflated.

3.1 Character Patterns

The characters of the representations of G on the spaces $\mathcal{H}^q(S)$ exhibit a pattern composed of a linear term plus a periodic term. This pattern was exhibited via Poincaré series in [1]. Here we reformulate the results of [1] by separating the characters into a regular character with multiplicity linear in q and a projective character which is periodic in q . Using the notation of [1], let $\text{ch}_{\mathcal{H}^q(S)}$ denote the character of the representation of G on $\mathcal{H}^q(S)$, χ_0, \dots, χ_t denote the irreducible characters of G , and χ_{reg} denote the character of the regular representation of G . Then, we have the following pattern formulas.

Proposition 5 *Let notation be as above. For $q > |G|$, write $q = a|G| + b$ with $1 \leq b \leq |G|$. Then,*

$$\begin{aligned} \text{ch}_{\mathcal{H}^q(S)} &= 2a(\sigma - 1)\chi_{reg} + \text{ch}_{\mathcal{H}^b(S)} - \chi_0, \quad b = 1, \\ \text{ch}_{\mathcal{H}^q(S)} &= 2a(\sigma - 1)\chi_{reg} + \text{ch}_{\mathcal{H}^b(S)}, \quad b \neq 1. \end{aligned} \quad (19)$$

Remark 6 *Using Proposition 5 we can completely determine \mathcal{H}^q -equivalence of actions by looking at q in the range $1 \leq q \leq |G|$. If G is not cyclic then the modulus $|G|$ can be replaced by a smaller number divisible by the exponent of G . However, we are mostly interested in prime cyclic actions. In the cyclic*

case the characters $\text{ch}_{\mathcal{H}^q(S)}$ can be completely encoded in a $|G| \times |G|$ matrix of integers determined by the multiplicities in the character decomposition of $\text{ch}_{\mathcal{H}^q(S)}$, $1 \leq q \leq |G|$. We develop this matrix for prime cyclic groups in Section 3.2.

To prove Proposition 5 we recall the Eichler trace formula. Define $\lambda_q : G \rightarrow \mathbb{C}$, $q \geq 1$

$$\begin{aligned}\lambda_q(1) &= (\sigma - 1)(2q - 1), \\ \lambda_q(g) &= \sum_{P \in S^g} \frac{(\varepsilon(P, g))^q}{1 - \varepsilon(P, g)}.\end{aligned}$$

where the last sum is zero if the fixed S^g is empty. From the Riemann-Roch theorem and the Eichler trace formula the characters $\text{ch}_{\mathcal{H}^q(S)}$ are given by

$$\begin{aligned}\text{ch}_{\mathcal{H}^1(S)}(g) &= 1 + \lambda_1(g), \\ \text{ch}_{\mathcal{H}^q(S)}(g) &= \lambda_q(g), q > 1,\end{aligned}$$

The first equation may be rewritten

$$\text{ch}_{\mathcal{H}^1(S)}(g) - \chi_0(g) = \lambda_1(g)$$

We are ready to prove Proposition 5.

Proof. Given $q = a|G| + b$, we have for $b = 1$:

$$\text{ch}_{\mathcal{H}^q(S)}(1) = 2a(\sigma - 1)|G| + \text{ch}_{\mathcal{H}^1(S)}(1) - \chi_0(1)$$

and

$$\begin{aligned}\text{ch}_{\mathcal{H}^q(S)}(g) &= \sum_{P \in S^g} \frac{(\varepsilon(P, g))^q}{1 - \varepsilon(P, g)} \\ &= 1 + \sum_{P \in S^g} \frac{(\varepsilon(P, g))^{a|G|+1}}{1 - \varepsilon(P, g)} - \chi_0(1) \\ &= \text{ch}_{\mathcal{H}^1(S)}(g) - \chi_0(g)\end{aligned}$$

For $b > 1$, we get

$$\text{ch}_{\mathcal{H}^q(S)}(1) = 2a(\sigma - 1)|G| + \text{ch}_{\mathcal{H}^b(S)}(1)$$

and

$$\begin{aligned} \text{ch}_{\mathcal{H}^q(S)}(g) &= \sum_{P \in S^q} \frac{(\varepsilon(P, g))^{a|G|+b}}{1 - \varepsilon(P, g)} \\ &= \text{ch}_{\mathcal{H}^b(S)}(g) \end{aligned}$$

Thus, the class function $\text{ch}_{\mathcal{H}^1(S)}(g) - (\text{ch}_{\mathcal{H}^1(S)} - \chi_0)$ for $b = 1$ and, respectively, the class function $\text{ch}_{\mathcal{H}^q(S)} - \text{ch}_{\mathcal{H}^b(S)}$ for $b > 1$ is zero for $g \neq 1$ and equals $2a(\sigma - 1)|G|$ at $g = 1$. But this is just the character $2a(\sigma - 1)\chi_{reg}$. ■

3.2 Multiplicity matrices for prime cyclic groups

Let $1 \leq q \leq |G|$ we decompose the character $\text{ch}_{\mathcal{H}^q(S)}$ into irreducible characters:

$$\text{ch}_{\mathcal{H}^q(S)} = \mu_q^0 \chi_0 + \cdots + \mu_q^l \chi_l, \quad (20)$$

where $\mu_q^j = \langle \text{ch}_{\mathcal{H}^q(S)}, \chi_j \rangle$ is the multiplicity inner product. We define the representation vector

$$R_q = \begin{bmatrix} \mu_q^0 \\ \vdots \\ \mu_q^l \end{bmatrix}$$

and multiplicity matrix

$$M = [\mu_q^j] = [R_1 \quad \cdots \quad R_{|G|}]. \quad (21)$$

If needed, we denote the dependence of M on p and t via the notation $M_{p,t}$. By Proposition 5, the \mathcal{H}^q representations are completely determined once M is known. We may use the Poincaré series in [1] to derive M but in the case of cyclic p -gonal actions it is simpler to just rework the calculation. We shall work with the class functions λ_q and then tack on χ_0 for $\text{ch}_{\mathcal{H}^1(S)}$.

We break up the class λ function into a regular piece and a piece for each

branch point Q_i . Define

$$\begin{aligned}
\lambda_q^0(1) &= (\sigma - 1)(2q - 1), \\
\lambda_q^0(g) &= 0, g \neq id(G), \\
\lambda_q^i(1) &= 0, i > 0, \\
\lambda_q^i(g) &= \sum_{P \in \pi_G^{-1}(Q_i)} \frac{(\varepsilon(P, g))^q}{1 - \varepsilon(P, g)}, g \neq id(G), i > 0, \\
\lambda_q(g) &= \sum_{i=0}^t \lambda_q^i(g).
\end{aligned}$$

With the exception of μ_1^0 , we have

$$\mu_q^j = \langle \text{ch}_{\mathcal{H}^q(S)}, \chi_j \rangle = \sum_{i=0}^t \langle \lambda_q^i, \chi_j \rangle = \langle \lambda_q, \chi_j \rangle$$

For μ_1^0 we have:

$$\mu_1^0 = \langle \lambda_1^0, \chi_0 \rangle + 1.$$

Since $\lambda_q^0 = \frac{(\sigma-1)(2q-1)}{p} \chi_{reg}$ then

$$\langle \lambda_q^0, \chi_j \rangle = \frac{(\sigma - 1)(2q - 1)}{p}$$

Now assume that $G = \mathbb{Z}_p$. The character χ_j on \mathbb{Z}_p is defined by $\chi_j(g) = \zeta^{jg}$. Also, there is a single fixed point P_i lying over Q_i , since $G_{P_i} = \mathbb{Z}_p$. Thus,

$$\lambda_q^i(g) = \sum_{P \in \pi_G^{-1}(Q_i)} \frac{(\varepsilon(P, g))^q}{1 - \varepsilon(P, g)} = \frac{(\varepsilon(P_i, g))^q}{1 - \varepsilon(P_i, g)}$$

We may compute the other inner products as follows. Assuming a generating vector (c_1, \dots, c_t) and d_i defined by equation 17, we obtain, using equation

18:

$$\begin{aligned}
\langle \lambda_q^i, \chi_j \rangle &= \frac{1}{p} \sum_{g=0}^{p-1} \lambda_q^i(g) \overline{\chi(g)} \\
&= \frac{1}{p} \sum_{g=1}^{p-1} \frac{(\varepsilon(P_i, g))^q}{1 - \varepsilon(P_i, g)} \overline{\chi(g)} \\
&= \frac{1}{p} \sum_{g=1}^{p-1} \frac{(\zeta^{gd_i})^q}{1 - \zeta^{gd_i}} \zeta^{jg}
\end{aligned}$$

Setting $h = gd_i$ we get $g = hc_i$, and the change of variable $g \rightarrow h$ yields

$$\langle \lambda_q^i, \chi_j \rangle = \frac{1}{p} \sum_{h=1}^{p-1} \frac{(\zeta^h)^q}{1 - \zeta^h} \zeta^{jhc_i} = \frac{1}{p} \sum_{h=1}^{p-1} \frac{\zeta^{h(q-jc_i)}}{1 - \zeta^h}.$$

To eliminate the denominator, write

$$\begin{aligned}
\langle \lambda_q^i, \chi_j \rangle &= \lim_{x \rightarrow 1^-} \frac{1}{p} \sum_{h=1}^{p-1} \frac{\zeta^{h(q-jc_i)}}{1 - x\zeta^h} \\
&= \lim_{x \rightarrow 1^-} \sum_{k=0}^{\infty} \frac{1}{p} \sum_{h=1}^{p-1} \zeta^{h(q-jc_i)} \zeta^{hk} x^k \\
&= \lim_{x \rightarrow 1^-} \left(\sum_{k=0}^{\infty} \left(\frac{1}{p} \sum_{h=0}^{p-1} \zeta^{h(q-jc_i+k)} \right) x^k - \sum_{k=0}^{\infty} \frac{1}{p} x^k \right)
\end{aligned}$$

The sum $\frac{1}{p} \sum_{h=0}^{p-1} \zeta^{h(q-jc_i+k)}$ is non-zero only if $q - jc_i + k = 0 \pmod{p}$, and when k satisfies this condition the sum is 1. Let

$$k_0(i, j, q) = (jc_i - q) \pmod{p} \tag{22}$$

be the smallest non-negative k for which $q - jc_i + k = 0 \pmod{p}$. Then

$$\begin{aligned}
\langle \lambda_q^i, \chi_j \rangle &= \lim_{x \rightarrow 1^-} \sum_{k'=0}^{\infty} x^{k_0(i, j, q) + pk'} - \sum_{k=0}^{\infty} \frac{1}{p} x^k \\
&= \lim_{x \rightarrow 1^-} \left(\frac{x^{k_0(i, j, q)}}{1 - x^p} - \frac{1}{p(1 - x)} \right).
\end{aligned}$$

Using L'Hôpital's rule (twice) we get

$$\langle \lambda_q^i, \chi_j \rangle = \frac{p-1-2k_0(i, j, q)}{2p}. \quad (23)$$

Putting it all together we get

$$\begin{aligned} \langle \lambda_q, \chi_j \rangle &= \frac{(\sigma-1)(2q-1)}{p} + \sum_{i=1}^t \frac{p-1-2k_0(i, j, q)}{2p} \\ &= \frac{(\sigma-1)(2q-1)}{p} + \frac{t(p-1)}{2p} - \frac{1}{p} \sum_{i=1}^t k_0(i, j, q) \end{aligned}$$

The genus satisfies $\sigma = \frac{(p-1)(t-2)}{2}$ and so

$$\begin{aligned} \langle \lambda_q, \chi_j \rangle &= \frac{ptq-2pq+p-tq}{p} - \frac{1}{p} \sum_{i=1}^t k_0(i, j, q) \\ &= q(t-2) + 1 - \frac{1}{p} \left(tq + \sum_{i=1}^t k_0(i, j, q) \right) \end{aligned} \quad (24)$$

We put the preceding into a proposition

Proposition 7 *Let \mathbb{Z}_p act on a surface S with such that S/G has genus zero, $S \rightarrow S/G$ is branched over t points and that a generating vector for the action is given by (c_1, \dots, c_t) . Then the $p \times p$ multiplicity matrix $M_{p,t}$ defined by equations 20 and 21 is given by*

$$\begin{aligned} M_{pt}(j, q) &= [\mu_q^j] = q(t-2) + 1 - \frac{1}{p} \left(tq + \sum_{i=1}^t (jc_i - q) \bmod p \right) \\ 0 &\leq j \leq p-1, 1 \leq q \leq p, (j, q) \neq (0, 1) \end{aligned} \quad (25)$$

and

$$M_{p,t}(0, 1) = 0 \quad (26)$$

where χ_j , is the j th character of \mathbb{Z}_p in the standard ordering.

Remark 8 We should check that $\langle \lambda_q, \chi_j \rangle$ is an integer. We just need to show that $tq + \sum_{i=1}^t (jc_i - q) \bmod p$ is divisible by p .

$$\begin{aligned}
\left(tq + \sum_{i=1}^t k_0(i, j, q) \right) \bmod p &= \left(tq + \sum_{i=1}^t (jc_i - q) \bmod p \right) \bmod p \\
&= \left(tq + j \left(\sum_{i=1}^t c_i \right) - tq \right) \bmod p \\
&= j \left(\sum_{i=1}^t c_i \right) \bmod p = 0
\end{aligned}$$

Example 9 The smallest possible genus for which $\mathcal{H}^1(S)$ equivalence conflates topological actions is genus 4 and with \mathbb{Z}_5 actions with 4 branch points. There are three topological actions given by the generating vectors $(c_1, c_2, c_3, c_4) = (1, 1, 1, 2)$, $(1, 1, 4, 4)$, and $(1, 2, 3, 4)$. The multiplicity matrices in the three cases are given in Table 1 below. By comparing columns of the matrices we see that the last two classes are conflated by odd degree differentials, but all three classes are separated using quadratic and quartic differentials.

Generating Vector	Multiplicity Matrix
$(c_1, c_2, c_3, c_4) = (1, 1, 1, 2)$	$ \begin{bmatrix} 0 & 1 & 3 & 5 & 7 \\ 2 & 1 & 2 & 4 & 6 \\ 1 & 3 & 2 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \\ 0 & 2 & 4 & 5 & 4 \end{bmatrix} $
$(c_1, c_2, c_3, c_4) = (1, 1, 4, 4)$	$ \begin{bmatrix} 0 & 1 & 3 & 5 & 7 \\ 1 & 1 & 3 & 5 & 5 \\ 1 & 3 & 3 & 3 & 5 \\ 1 & 3 & 3 & 3 & 5 \\ 1 & 1 & 3 & 5 & 5 \end{bmatrix} $
$(c_1, c_2, c_3, c_4) = (1, 2, 3, 4)$	$ \begin{bmatrix} 0 & 1 & 3 & 5 & 7 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} $

Table 1

Remark 10 *All calculations for Table 1, indeed all tables in the paper, were done with Magma. The multiplicity matrices were calculated by direct decomposition of the representations and by using Proposition 7*

3.3 Alternate perspective

Given a prime p , the number of branch points t , and a degree q we can look at the map from generating vectors to representation vectors. To this end, for a generating vector (c_1, \dots, c_t) we define the vector $X = (x_1, \dots, x_{p-1})$ of non-negative integers by

$$x_l = |\{c_i : c_i = l\}| \quad (27)$$

The vector (x_1, \dots, x_{p-1}) must satisfy these equations:

$$\begin{aligned} \sum_{l=1}^{p-1} x_l &= t, \\ \sum_{l=1}^{p-1} lx_l &= 0 \pmod{p}. \end{aligned}$$

Each such vector defines a “braid class” of generating vectors. If we act on (c_1, \dots, c_t) by an element of $\text{Aut}(G)$ then the components of X are permuted. We see that

$$\begin{aligned} \langle \lambda_q^i, \chi_j \rangle &= \frac{p-1-2k_0(i, j, q)}{2p} \\ &= \frac{p-1-2k_1(l, j, q)}{2p} \end{aligned}$$

where k_1 is defined by replacing c_i by l in equation 22 :

$$k_1(l, j, q) = (jl - q) \pmod{p} \quad (28)$$

We writing equation 24

$$\begin{aligned} \langle \lambda_q, \chi_j \rangle &= \frac{ptq - 2pq + p - tq}{p} - \frac{1}{p} \sum_{i=1}^t k_0(i, j, q) \\ &= \frac{ptq - 2pq + p - tq}{p} - \frac{1}{p} \sum_{l=1}^{p-1} k_1(l, j, q)x_l \end{aligned}$$

Define, entry-wise, the $p \times (p - 1)$ matrix N and p vectors R, R_0 by:

$$N(j, l) = k_1(l, j, q) = (jl - q) \bmod p \quad (29)$$

$$\begin{aligned} R(j) &= \langle \lambda_q, \chi_j \rangle \\ R_0(j) &= ptq - 2pq + p - tq \end{aligned} \quad (30)$$

Then the representation decomposition for \mathbb{Z}_p acting on \mathcal{H}^q in vector form

$$R = \frac{1}{p} (R_0 - NX). \quad (31)$$

Let's put this into a proposition.

Proposition 11 *Let \mathbb{Z}_p act on S with t branch points and action vector X (equations 27, 30). Let $1 \leq q \leq p$ and define R, R_0, N as in equation 30. Then, except for the multiplicity μ_1^0 , the character decomposition vector R of \mathbb{Z}_p acting on $\mathcal{H}^q(S)$ is given by equation 31.*

The alternate multiplicity matrix N allows us to consider the all the different topological actions at once, for an arbitrary number of branch points $t \geq 3$ and a fixed q . If needed, we denote the dependence of N on p, q via the notation $N_{p,q}$. The matrices for $p = 5$ are given in Table 2. The column K of that table is matrix whose columns span the kernel of N . For some p, q the kernel can have a fairly large dimension, see Table 5 of Section 4.

q	N	K
1	$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$
2	$\begin{bmatrix} 3 & 3 & 3 & 3 \\ 4 & 0 & 1 & 2 \\ 0 & 2 & 4 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 1 & 0 & 4 \end{bmatrix}$	no kernel
3	$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 4 & 0 & 1 \\ 4 & 1 & 3 & 0 \\ 0 & 3 & 1 & 4 \\ 1 & 0 & 4 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$
4	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 0 \\ 3 & 0 & 2 & 4 \\ 4 & 2 & 0 & 3 \\ 0 & 4 & 3 & 2 \end{bmatrix}$	no kernel
5	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

Table 2

Remark 12 Looking at Tables 1 and 2 we see some patterns.

1. If the kernel is trivial then there is no conflation.
2. If there is conflation the kernel is non trivial.
3. The entries of the top row of N are all equal. The rest of the matrix is symmetric.

4. Upon removing the top row of the matrix N each row and column contains all the numbers $0, \dots, p-1$ except for the number in the top row.

Let us try to explain the bullets in the previous Remark. If two actions X_1 and X_2 are \mathcal{H}^q conflated, then for the degree q matrix

$$\begin{aligned} \frac{1}{p} (R_0 - NX_1) &= \frac{1}{p} (R_0 - NX_2) \\ N(X_1 - X_2) &= 0 \end{aligned}$$

or

$$N(X_1 - X_2) = 0$$

since R_0 depends only on p, t, q . This explains bullets 1 and 2. To explain bullets 3 and 4 we recall the definition of the entries of N

$$N_{j,l} = (jl - q) \bmod p.$$

In the top row $(jl - q) \bmod p = (0 \times l - q) \bmod p = p - q$, so we get a constant row. For the remaining entries $jl \neq 0$ and so $(jl - q) \bmod p \neq p - q$. The lower matrix is formed from the multiplication table of \mathbb{Z}_p^* shifted by $-q$ and the appropriate residue mod p chosen. This explains the format of the lower part of the matrix.

Remark 13 *As noted previously The totality of all p -gonal \mathbb{Z}_p^* actions forms a subset of $\mathbb{Z}^{p-1} = \{X = (x_1, \dots, x_{p-1}) : x_l \in \mathbb{Z}\}$ subject to these constraints only:*

$$\begin{aligned} x_l &\geq 0 \\ \sum_{l=1}^{p-1} x_l &\geq 3 \\ \sum_{l=1}^{p-1} lx_l &= 0 \bmod p \end{aligned}$$

This set is invariant under these operations: addition of vectors and permutation of the indices by $l \rightarrow el \bmod p$ for $e \in \mathbb{Z}_p^$. We have conflation if and only if two inequivalent actions X_1 and X_2 have the same number of branch points and $X_1 - X_2 \in \ker(N)$*

4 Examples and conjectures

4.1 Examples

Example 14 *Here is a sample of low genus actions that are not distinguished by $\mathcal{H}^1(S)$ equivalence and the lowest degree differential that separates them.*

p	t	σ	<i>some conflated actions</i>	<i>smallest degree of separating differential</i>
5	4	4	(1, 1, 4, 4), (1, 2, 3, 4)	2
5	5	6	(1, 1, 1, 3, 4), (1, 1, 2, 2, 4)	2
5	6	8	(1, 1, 1, 4, 4, 4), (1, 1, 2, 3, 4, 4)	2
7	4	6	(1, 1, 6, 6), (1, 2, 5, 6)	2
7	5	9	(1, 1, 1, 5, 6), (1, 1, 2, 5, 5), (1, 1, 3, 4, 5)	2
7	6	12	(1, 1, 1, 6, 6, 6), (1, 1, 2, 5, 6, 6) (1, 1, 3, 4, 6, 6), (1, 2, 3, 4, 5, 6)	3

Table 3

Looking at the last row of the table, we see that quadratic differentials will not always work in separating classes.

Example 15 *In the following tables we record the behaviour of the kernel of $N_{p,q}$. In Table 4 we record the dimensions of the kernels of $N_{p,q}$ in the range $1 \leq q \leq p$ in list form. In Table 5 we give lists of degrees q for which the kernel $N_{p,q}$ is trivial (good degrees) and non-trivial (bad degrees)*

p	<i>Dimensions $\ker(N_{p,q}), 1 \leq q \leq p$</i>
3	0, 0
5	1, 0, 1, 0, 1
7	2, 1, 0, 3, 0, 1, 2
11	4, 1, 0, 1, 0, 4, 0, 1, 0, 1, 4
13	5, 0, 3, 0, 2, 0, 5, 0, 2, 0, 3, 0, 5
17	7, 0, 2, 0, 0, 0, 1, 0, 7, 0, 1, 0, 0, 0, 2, 0, 7
19	8, 1, 0, 0, 0, 3, 0, 0, 0, 8, 0, 0, 0, 3, 0, 0, 0, 1, 8

Table 4

p	<i>good degrees</i>	<i>bad degrees</i>
3	1, 2, 3	
5	2, 4	1, 3, 5
7	3, 5	1, 2, 4, 6, 7
11	3, 5, 7, 9	1, 2, 4, 6, 8, 10, 11
13	2, 4, 6, 8, 10, 12	1, 3, 5, 7, 9, 11, 13
17	2, 4, 5, 6, 8, 10, 12, 13, 14, 16	1, 3, 7, 9, 11, 15, 17
19	3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17	1, 2, 6, 10, 14, 18, 19

Table 5

Example 16 *The number of conflated actions can be quite large. For example, for $p = 11, t = 10, q = 1, \sigma = 40$ there are 854 actions with numerous conflations. In one case 31 different actions all have the same \mathcal{H}^1 character. For $p = 19, t = 10, q = 1, \sigma = 63$ there are 9143 actions with a maximum of 116 conflated actions.*

4.2 Conjectures and Remarks

By reviewing the tables we are led to the following two conjectures.

Conjecture 17 *For every prime $p \geq 3$ there is a $q \leq p$ such that the matrix $N_{p,q}$ has trivial kernel and hence there is no \mathcal{H}^q conflation, for \mathbb{Z}_p actions, for every $t \geq 3$.*

Conjecture 18 *For every prime $p \geq 3$ and $1 \leq q \leq p$ such that $N_{p,q}$ has a non-trivial kernel there is \mathcal{H}^q conflation for infinitely many values of t .*

Conjecture 19 *For every prime $p \geq 3$ and every $1 \leq q \leq p$ kernels of $N_{p,q}$ and $N_{p,q'}$ have the same dimension where $q' = p + 1 - q$.*

Indeed some preliminary calculations show that the kernels of $N_{p,q}$ and $N_{p,q'}$ equal each other for small primes. We cannot claim a conjecture for the kernel of $N_{p,1}$ but we can at least make the following remark

Remark 20 *For all the primes, except $p = 7$ in Table 4 we see that $p = 2 \dim(\ker(N_{p,1})) + 3$.*

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