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Topological and $\mathcal{H}^q$ Equivalence of Prime Cyclic p-gonal Actions on Riemann Surfaces (corrected)

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Abstract

Two Riemann surfaces S_1 and S_2 with conformal G-actions have topologically equivalent actions if there is a homeomorphism $h : S_1 \to S_2$ which intertwines the actions. A weaker equivalence may be defined by comparing the representations of G on the spaces of holomorphic q-differentials $\mathcal{H}^q(S_1)$ and $\mathcal{H}^q(S_2)$. In this note we study the differences between topological equivalence and $\mathcal{H}^q$ equivalence of prime cyclic actions, where S_1/G and S_2/G have genus zero. This paper is a corrected version of the previous paper of the same title.

Contents

1 Introduction

The finite group G *acts orientably* on the Riemann surface S if there is a monomorphism:

$$\epsilon : G \to \text{Homeo}^+(S),$$

the group of orientation preserving homeomorphisms of S. If the image $\epsilon(G)$ consists of conformal automorphisms of S, i.e.,

$$\epsilon : G \to \text{Aut}(S),$$

we say that G *acts conformally* on S. Two actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are *topologically equivalent* if there is an intertwining homeomorphism $h : S_1 \to S_2$ and an automorphism $\omega \in \text{Aut}(G)$ such that

$$\epsilon_2(g) = h\epsilon_1(\omega(g))h^{-1}, \forall g \in G. \tag{1}$$

If h is a conformal map then we say that the actions are conformally equivalent. For conformal actions, we have in diagram form:

$$\begin{tikzcd} G \arrow[r, "\epsilon_1"] \arrow[d, "\omega^{-1}"] & \text{Aut}(S_1) \arrow[d, "Ad_h"] \\ G \arrow[r, "\epsilon_2"] & \text{Aut}(S_2) \end{tikzcd} \tag{2}$$

The classification of topological equivalence classes of finite group actions is equivalent to the classification of conjugacy classes of finite subgroups of the mapping class group Mod_σ. Therefore, topological classification is a natural classification. In our problem statement below we consider a weaker equivalence through the representation of G on q-differentials. The equivalence relation defined by Abelian differentials, $q = 1$, has been used to classify actions in low genus, by some authors. We shall show in this paper that this equivalence relation is strictly weaker than topological equivalence and the discrepancy gets quite dramatic, even for moderate genus.

1.1 Problem statement

The group G always acts on the homology $H_1(S)$ or cohomology $H^1(S)$ of S. Any topologically equivalent action yields equivalent representations. In the conformal action case, the group G also acts linearly on the various spaces of holomorphic q-differentials $\mathcal{H}^q(S)$. The space $\mathcal{H}^q(S)$ is the $\mathbb{C}$-vector space of holomorphic sections of $T^*(S) \bigotimes \cdots \bigotimes T^*(S)$ (q times), where $T^*(S)$ is the cotangent bundle, a holomorphic line bundle. We are going to be "pedantically precise" about the action of G since it will be important later in our discussion of rotation numbers in Section 2.3 (a source of error in the previous paper!). The group G acts on $\mathcal{H}^q(S)$ by the representation

$$H_g^q(\psi(P)) = \left(dg^{-1}\right)^* (\psi(g^{-1}P)), \tag{3}$$

where $\psi \in \mathcal{H}^q(S)$, and we use g to denote the mapping $\epsilon(g)$. The formula makes sense since

$$\begin{aligned} \psi(P) &\in T_P^*(S) \bigotimes \cdots \bigotimes T_P^*(S), \\ \psi(g^{-1}P) &\in T_{g^{-1}P}^*(S) \bigotimes \cdots \bigotimes T_{g^{-1}P}^*(S) \end{aligned}$$

and

$$\left(dg^{-1}\right)^* : T_{g^{-1}P}^*(S) \bigotimes \cdots \bigotimes T_{g^{-1}P}^*(S) \to T_P^*(S) \bigotimes \cdots \bigotimes T_P^*(S).$$

We use g^{-1} as $f \to f^*$ is contravariant.

Given actions of G on two surfaces S_1, S_2 of the same genus we say that the actions of G are $\mathcal{H}^q$-equivalent if the two different representations on $\mathcal{H}^q(S_1)$ and $\mathcal{H}^q(S_2)$ are equivalent over $\mathbb{C}$. It turns out that topologically equivalent actions are $\mathcal{H}^q$-equivalent for every q. However, two $\mathcal{H}^q$-equivalent actions may be topologically inequivalent. In this case we call the actions *holomorphically q-conflated*. The goal of this paper is to explore the relationship between topological equivalence and $\mathcal{H}^q$-equivalence for cyclic prime p-gonal actions. This means that S/G has genus zero and G is a cyclic group of prime order p.

Remark 1 *Using the action ϵ instead of the subgroup $\epsilon(G)$ allows us to consider the group G as the primary object as well giving us a certain preciseness and economy of exposition (e.g., actions $\leftrightarrow$ generating vectors). This comes at the expense of having to take in to account the various actions $\epsilon \circ \omega$ for*

$\omega \in \mathrm{Aut}(G)$, *For topological equivalence we really only care about conjugacy of the subgroup* $\epsilon(G)$, *hence we include* $\omega \in \mathrm{Aut}(G)$ *in the definition of topologival equivalence. Strictly speaking, we should also be considering* $H^q_{\omega g}$, $\omega \in \mathrm{Aut}(G)$, *in the discussion of* $\mathcal{H}^q$*-equivalence, but we may ignore this by specifying a canonically chosen action for each topological equivalence class.*

1.2 Previous and related work

A review of various methods of classification of group actions is given in [3], a paper which is devoted to the complete topological classification of actions on surfaces in genus 2 and 3. The methods frequently make use of the G-action on the homology or cohomology of the surface. For prime cyclic p-gonal actions, the methods are ineffective as prime cyclic p-gonal actions yield the same complex representation on $H_1(S)$ and $H^1(S)$. On the analytic side, another classification scheme, has been considered by A. Kuribayashi, I. Kuribayashi and H. Kimura in various papers [7], [8], [9], [10], [11] by considering the representations on Abelian differentials $\mathcal{H}^1(S)$. These authors have worked out the classification of all subgroups of $GL(\sigma, \mathbb{C})$ that occur in this way for $2 \leq \sigma \leq 5$. In these papers the authors sought to classify the Riemann surfaces of low genus, by relating the characteristics of their Weierstrass points, their equations and their automorphism groups. None of these works consider the problem of topologically inequivalent actions of groups. In fact, the classification of actions by topological equivalence is strictly finer than the classification scheme considered by these authors. The smallest genus example where the Abelian differentials fail to distinguish topologically distinct actions occurs in genus 4 with $G = \mathbb{Z}_5$ and 4 branch points. For an interesting example of conflation, see Example 3.1 of [13], where it is shown that there are 7 inequivalent actions of $J(1)$, Janko's first group, on a surface of genus 2091. From the results of [2] none of these, actions can be distinguished by the action of $J(1)$ on the space of Abelian differentials.

In the paper [14], Streit and Wolfart propose to use the representations on q-differentials as Galois invariants to distinguish various quasi-platonic surfaces under the absolute Galois group action. Though not strictly relevant to our discussion on q-conflation, the cited paper gives an application of characters of G obtained from q-differentials.

We noted previously that topological classes of G actions correspond to conjugacy classes of the mapping class group. In [2] and [3] a detailed description of the relationship among the following three concepts is given:

- the conjugacy classification of finite subgroups of the mapping class group Mod_σ,
- topological equivalence of finite groups acting conformally on surfaces, and
- the branch locus $\mathcal{B}_\sigma$ of $\mathcal{M}_\sigma$, the moduli space of surfaces of genus $\sigma \geq 2$ which consists of surfaces with a non-trivial automorphism group (at least for $\sigma > 2$).

The branch locus is a union of smooth subvarieties (strata) each of which corresponds to a topological equivalence class of actions, determined by the common automorphism group of the surfaces in the stratum. Because of q-conflation the character of the $\mathcal{H}^q(S)$ representation of G may fail to separate the strata for $\mathbb{Z}_p$ actions, for certain values of q. In [6] Guerrero studies the representations of the automorphism group G in holomorphic families determined by branch loci, proving that the characters are constant along a stratum. He also claims that the characters determined by $\mathcal{H}^1(S)$ and $\mathcal{H}^2(S)$ are sufficient to determine the "rotation constant data" for a G action. In the prime cyclic case this is sufficient to determine topological equivalence. We defer the discussion of the results of [6] and q-conflation to a follow-up note the current paper.

A cyclic p-gonal Riemann surface with t branch points is "superelliptic" and has an equation of the form

$$y^p = \prod_{j=1}^{t} (x - a_j)^{c_j}, \tag{4}$$

where $a_1, \ldots, a_t$ are distinct complex numbers and the exponents satisfy:

$$\begin{aligned} 1 &\leq c_j \leq p - 1, \\ c_1 + \cdots + c_t &= 0 \bmod p. \end{aligned}$$

These surfaces have been extensively studied, and we focus our attention on these surfaces as our main example, though we shall not exploit their explicit equations.

Finally, our discussion will follow the results of [1] closely. In that work a decomposition of the various $\mathcal{H}^q(S)$ into irreducibles is given.

1.3 Outline of the paper

In Section 2 we recall the basics of conformal actions on surfaces and reduce topological equivalence to relations on generating vectors of G. We focus specifically on $\mathbb{Z}_p$ actions and give a canonical form for generating vectors corresponding to a specific topological action class. In Section 3 we recall the basics of the representations on $\mathcal{H}^q(S)$, specifically the Eichler trace formula. We then derive explicit formulas for the characters of $\mathbb{Z}_p$ actions which can be determined explicitly in terms of arithmetic formulas dependent on the prime p, the number of branch points t, the degree q of the differential and the canonical generating vector of the $\mathbb{Z}_p$ action. Our main results are Propositions 8 and 12. In Section 4 we give some examples and make some conjectures.

Acknowledgement 2 *I would like to thank Victor Gonzalez for conversations and questions that inspired this note, and Mariela Carvacho for pointing out the work of Ignacio Guerrero.*

2 Actions of groups on surfaces

2.1 Generating vectors and equivalence of actions

The quotient surface $S/G = T$ of a conformal action is a closed Riemann surface of genus τ with a unique conformal structure so that

$$\pi_G : S \to S/G = T \tag{5}$$

is holomorphic. The quotient map $\pi_G : S \to T$ is branched uniformly over a finite set $B_G = \{Q_1, \ldots, Q_t\}$ such that π_G is an unramified covering exactly over $T^\circ = T - B_G$. Let $S^\circ = \pi_G^{-1}(T^\circ)$ so that $\pi_G : S^\circ \to T^\circ$ is an unramified covering space whose group of deck transformation equals $\epsilon(G)$, restricted to S°. This covering determines a normal subgroup $\Pi_G = \pi_1(S^\circ) \lhd \pi_1(T^\circ)$ and an exact sequence $\Pi_G \hookrightarrow \pi_1(T^\circ) \twoheadrightarrow \epsilon(G)$ by mapping loops to deck transformations. Combine the last map with $\epsilon(G) \overset{\epsilon^{-1}}{\to} G$ to get an exact sequence

$$\Pi_G \hookrightarrow \pi_1(T^\circ) \overset{\xi}{\twoheadrightarrow} G. \tag{6}$$

We have left out base points to simplify the exposition, and so ξ is ambiguous up to an inner automorphism. However, this does not matter since we are only concerned about actions up to automorphisms of G.

In the opposite direction we start with a sequence as in equation 6. We can construct a unramified, holomorphic, regular covering space which we still denote $\pi_G : S^\circ \to T^\circ$. The deck transformations are automatically holomorphic. We can fill in the punctures to get a closed surface, and the action of $\epsilon : G \to \mathrm{Aut}(S^\circ)$ extends to a conformal action $\epsilon : G \to \mathrm{Aut}(S)$ at the filled in punctures, using the removable singularity theorem.

The fundamental group $\pi_1(T^\circ)$ has the following presentation:

$$\text{generators} \quad : \quad \{\alpha_i, \beta_i, \gamma_j, 1 \leq i \leq \tau, 1 \leq j \leq t\}, \tag{7}$$

$$\text{relation} \quad : \quad \prod_{i=1}^{\tau} [\alpha_i, \beta_i] \prod_{j=1}^{t} \gamma_j = 1. \tag{8}$$

Define

$$a_i = \xi(\alpha_i), b_i = \xi(\beta_i), c_j = \xi(\gamma_j).$$

Then the $2\tau + t$ tuple $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t)$ is called a *generating vector* for the action. We observe that

$$G = \langle a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t \rangle, \tag{9}$$

$$o(c_j) = n_{j,} \tag{10}$$

and

$$\prod_{i=1}^{\tau} [a_i, b_i] \prod_{j=1}^{t} c_j = c_1^{n_1} = \cdots = c_t^{n_t} = 1, \tag{11}$$

for some integers $n_j \geq 2$. We call $(\tau : n_1, \ldots, n_t)$ the signature of the action and we write the signature as $(n_1, \ldots, n_t)$ if $\tau = 0$. Finally we define

$$\mu(\tau : n_1, \ldots, n_t) = 2\tau - 2 + t - \left(\frac{1}{n_1} + \cdots + \frac{1}{n_t} \right).$$

The genus σ of S is given by the Riemann-Hurwitz equation

$$\frac{2\sigma - 2}{|G|} = \mu(\tau : n_1, , \ldots, n_t). \tag{12}$$

From this equation we see that the quantity $\mu(\tau : n_1, \ldots, n_t)$ may be interpreted as an orbifold Euler characteristic of S/G or the area of a fundamental region for the G action on S, divided by 2π.

Remark 3 *When S/G has genus zero we say that the action is n-gonal since the projection $S \rightarrow S/G$ is an $|G|$-gonal projection. In this case the signature is $(n_1, \ldots, n_t)$, the generating vector is $(c_1, \ldots, c_t)$, and the Riemann-Hurwitz equation becomes*

$$\frac{2\sigma - 2}{|G|} = t - 2 - \left(\frac{1}{n_1} + \cdots + \frac{1}{n_t}\right).$$

The generating vector of a topological equivalence class is not unique. The topological equivalence relation transfers to the generating vectors. First the action of $\omega \in \mathrm{Aut}(G)$:

$$\begin{aligned} & (a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) \\ \rightarrow \ & (\omega(a_1), \ldots, \omega(a_\tau), \omega(b_1), \ldots, \omega(b_\tau), \omega(c_1), \ldots, \omega(c_t)). \end{aligned} \tag{13}$$

For an intertwining homeomorphism $h : S_1 \rightarrow S_2$, as in equation 1, we see from the diagram 2, that there is an induced diagram

$$\begin{tikzcd} S_1 \arrow[r, "h"] \arrow[d, "\pi_{\epsilon_1(G)}"] & S_2 \arrow[d, "\pi_{\epsilon_2(G)}"] \\ T_1 \arrow[r, "\overline{h}"] & T_2 \end{tikzcd} \tag{14}$$

and hence an induced isomorphism

$$h_* : \pi_1(T_1^\circ) \rightarrow \pi_1(T_2^\circ).$$

The isomorphism h_* is base point dependent, but a different choice of base point results in composing with h_*, an inner automorphism. For $\alpha_i, \beta_i, \gamma_j \in \pi_1(T_1^\circ)$ each loop $h_*(\alpha_i), h_*(\beta_i), h_*(\gamma_j)$ can be written as words in the corresponding $\alpha'_i, \beta'_i, \gamma'_j \in \pi_1(T_2^\circ)$. The action on $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t)$ is to rewrite each group element in the vector as a word in a'_i, b'_i, c'_j using the words defined by h_*. The general types of h_* can be found in many places including [2] and [3] but we will only need the following "braid automorphisms" for cyclic p-gonal actions. We may choose a homeomorphism h, preserving the branch sets, satisfying $h(Q_j) = Q'_{j+1}$, $h(Q_{j+1}) = Q'_j$, $Q_j, Q_{j+1} \in T_1$, $Q'_j, Q'_{j+1} \in T_2$, and preserving the indices of all other branch points. The homeomorphism h may be further restricted so that we have

$$\begin{aligned} h_*(\gamma_j) &= \gamma'_{j+1}, \\ h_*(\gamma_j) &= \left(\gamma'_{j+1}\right)^{-1} \gamma'_j \gamma'_{j+1}. \end{aligned}$$

Then, the action on generating vectors is:

$$\begin{aligned} & (a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) \\ \rightarrow\ & \left(a_1', \ldots, a_\tau', b_1', \ldots, b_\tau', c_1', \ldots, c_{j+1}', \left(c_{j+1}'\right)^{-1} c_j' c_{j+1}', \ldots, c_t'\right) \end{aligned} \tag{15}$$

When $T_1 = T_{2,}$ the group generated by these transformations induces an action of the symmetric group on the indices $(n_1, \ldots, n_t)$. Topological equivalence must preserve the orders of the $(n_1, \ldots, n_t)$. Now assume that G is Abelian and that $\tau = 0$. Then the action generated by the braid transformations is

$$(c_1, \ldots, c_t) \rightarrow \left(c_{\theta(1)}, \ldots, c_{\theta(t)}\right) \tag{16}$$

where θ is a transformation preserving the orders n_i, namely $n_{\theta(i)} = n_i$. If $G = \mathbb{Z}_p$ then θ can be any permutation.

2.2 Actions of $\mathbb{Z}_p$

For a cyclic p-gonal action we take $G = \mathbb{Z}_p$, an additive group, and the c_j may be taken as integers in the range $1 \leq c_j \leq p-1$ such that $c_1 + \cdots + c_t = 0 \bmod p$. The c_i may be interpreted as the exponents in the superelliptic equation 4. The genus of such a surface is $\sigma = \frac{1}{2}(t-2)(p-1)$, ($t$ is even if $p = 2$), according to the Riemann-Hurwitz equation 12. From equation 16 we see that we may select the c_j in any order, we shall assume that $c_1 \leq \cdots \leq c_t$. This will give us a normal form under the braid transformations. The automorphism action allows us to multiply the c_i by any e satisfying $1 \leq e \leq p-1$, namely

$$(c_1, \ldots, c_t) \rightarrow (ec_1, \ldots, ec_t) \bmod p. \tag{17}$$

It follows that each topological equivalence class of actions we may assume that one of the c_j equals 1. We use the following algorithm to find the topological equivalence classes of p-gonal actions of $\mathbb{Z}_p$ with t branch points.

Algorithm 4 *Step 1: Find all sorted tuples* $(c_1, \ldots, c_t)$ *of integers satisfying*

$$\begin{aligned} 1 &\leq c_i \leq p-1 \\ 1 &= c_1 \leq \cdots \leq c_t \end{aligned}$$

$$c_1 + \cdots + c_t = 0 \bmod p$$

Step 2: For each $(c_1, \ldots, c_t)$ *in Step 1 lexicographically sort the set*

$$\{(ec_1, \ldots, ec_t)\} : e \in \mathbb{Z}_p\}$$

and select the first tuple.

Step 3: Lexicographically sort all the tuples obtained in Step 2 into a single list. This is a set of representative generating vectors for the topological equivalence classes of actions.

Remark 5 *A process for enumeration of topological equivalence classes of actions of elementary Abelian groups is described in [4]. In Example 10, of Section 3 the topological equivalence classes for* $\mathbb{Z}_5$ *with signature* $(5,5,5,5)$ *are discussed, the smallest case in which holomorphic conflation occurs. For this case, the algorithm produces three class representatives:* $(c_1, c_2, c_3, c_4) = (1,1,1,2)$, $(1,1,4,4)$, *and* $(1,2,3,4)$. *Even for modest genus the number of topological equivalence classes can get quite large. See Example 17 in Section 4.*

2.3 Rotation constants

We now return to the case of general G. For $g \in G$ we identify g with $\epsilon(g) \in \mathrm{Aut}(S)$ and we denote $S^g = \{P \in S : gP = P\}$, the set of fixed points of g on S. If $P \in S^g$, then the induced map of tangent spaces and cotangent spaces $dg^{-1} : T_P(S) \to T_P(S)$ and $(dg^{-1})^* : T_P^*(S) \to T_P^*(S)$ is multiplication by the same $o(g)$-th root of unity, denoted by $\varepsilon(P, g)$. The number $\varepsilon(P, g)$ is called the *rotation constant* of g at P. The rotation constants as defined are naturally related to the action on the spaces of differentials given by equation 3. For covering space constructions, the number $\rho(P, g) = 1/\varepsilon(P, g) = \overline{\varepsilon(P, g)}$, which is called the *multiplier*, gives the local structure of the action of g at P, as it is the map $dg : T_P(S) \to T_P(S)$.

It is easy to show that we may pick the c_j and a $P_j \in \pi_G^{-1}(Q_j)$ such that $G_{P_j} = \{g \in G : gP_j = P_j\} = \langle c_j \rangle$ and

$$\rho(P_j, c_j) = \exp\left(\frac{2\pi i}{n_j}\right) = \zeta_{n_j}$$

and hence

$$\varepsilon(P_j, c_j) = \exp\left(-\frac{2\pi i}{n_j}\right) = \overline{\zeta_{n_j}}.$$

Let $(c_1, \ldots c_t)$ be a generating vector for $\mathbb{Z}_p$. Each c_j has a unique fixed point $P_j \in \pi_G^{-1}(Q_j)$ since the stabilizer at P_j is the entire group. In this case

$$\varepsilon(P_j, c_j) = \exp\left(-\frac{2\pi i}{p}\right) = \overline{\zeta_p} = \overline{\zeta}.$$

To calculate rotation constants we define d_j so that

$$c_j d_j = 1 \bmod p. \tag{18}$$

Taking the additive structure of $\mathbb{Z}_p$ into account, we compute:

$$\varepsilon(P_j, g) = \varepsilon(P_j, g c_j d_j) = \varepsilon(P_j, c_j)^{g d_j} = \overline{\zeta}^{g d_j}. \tag{19}$$

3 Representations of G on $\mathcal{H}^q(S)$

In this section we first fix an action for general G and discuss the character decomposition as q varies. Then we focus on the prime cyclic, p-gonal case getting very specific results. Finally, we switch perspectives, fixing t and q and consider the variation of the character decomposition over all topological types. In particular we describe a process to determine which actions are conflated.

3.1 Character Patterns

The characters of the representations of G on the spaces $\mathcal{H}^q(S)$ exhibit a pattern composed of a linear term plus a periodic term. This pattern was exhibited via Poincaré series in [1]. Here we reformulate the results of [1] by separating the characters into a regular character with multiplicity linear in q and a projective character which is periodic in q. Using the notation of [1], let $\mathrm{ch}_{\mathcal{H}^q(S)}$ denote the character of the representation of G on $\mathcal{H}^q(S)$, $\chi_0, \ldots, \chi_{l-1}$ (l is the number of conjugacy classes) denote the irreducible characters of G, and χ_{reg} denote the character of the regular representation of G. Then, we have the following pattern formulas.

Proposition 6 *Let notation be as above. For $q > |G|$, write $q = a|G| + b$ with $a \geq 1$ and $1 \leq b \leq |G|$. Then,*

$$\begin{aligned} \mathrm{ch}_{\mathcal{H}^q(S)} &= 2a(\sigma - 1)\chi_{reg} + \mathrm{ch}_{\mathcal{H}^b(S)} - \chi_0, b = 1, \\ \mathrm{ch}_{\mathcal{H}^q(S)} &= 2a(\sigma - 1)\chi_{reg} + \mathrm{ch}_{\mathcal{H}^b(S)},\ b \neq 1. \end{aligned} \tag{20}$$

Remark 7 *Using Proposition 6 we can completely determine* $\mathcal{H}^q$*-equivalence of actions by looking at* q *in the range* $1 \leq q \leq |G|$ *. If* G *is not cyclic then the modulus* $|G|$ *can be replaced by a smaller number divisible by the exponent of* G*. However, we are mostly interested in prime cyclic actions. In the cyclic case the characters* $\mathrm{ch}_{\mathcal{H}^q(s)}$ *can be completely encoded in a* $|G| \times |G|$ *matrix of integers determined by the multiplicities in the character decomposition of* $\mathrm{ch}_{\mathcal{H}^q(S)}$*,* $1 \leq q \leq |G|$ *. We develop this matrix for prime cyclic groups in Section 3.2.*

To prove Proposition 6 we recall the Eichler trace formula [5]. Define $\lambda_q : G \to \mathbb{C}$, $q \geq 1$ by

$$\begin{aligned} \lambda_q(1) &= (\sigma - 1)(2q - 1), \\ \lambda_q(g) &= \sum_{P \in S^g} \frac{(\varepsilon(P, g))^q}{1 - \varepsilon(P, g)}, g \neq 1. \end{aligned}$$

where the last sum is zero if the fixed point set S^g is empty. From the Riemann-Roch theorem and the Eichler trace formula [5] the characters $\mathrm{ch}_{\mathcal{H}^q(s)}$ are given by

$$\begin{aligned} \mathrm{ch}_{\mathcal{H}^1(S)}(g) &= 1 + \lambda_1(g), \\ \mathrm{ch}_{\mathcal{H}^q(S)}(g) &= \lambda_q(g), q > 1, \end{aligned}$$

The first equation may be rewritten

$$\mathrm{ch}_{\mathcal{H}^1(S)}(g) - \chi_0(g) = \lambda_1(g)$$

We are ready to prove Proposition 6.

Proof. Given $q = a|G| + b$, we have for $b = 1$:

$$\mathrm{ch}_{\mathcal{H}^q(S)}(1) = 2a(\sigma - 1)|G| + \mathrm{ch}_{\mathcal{H}^1(S)}(1) - \chi_0(1)$$

and

$$\begin{aligned} \mathrm{ch}_{\mathcal{H}^q(S)}(g) &= \sum_{P \in S^g} \frac{(\varepsilon(P, g))^q}{1 - \varepsilon(P, g)} \\ &= 1 + \sum_{P \in S^g} \frac{(\varepsilon(P, g))^{a|G|+1}}{1 - \varepsilon(P, g)} - \chi_0(g) \\ &= \mathrm{ch}_{\mathcal{H}^1(S)}(g) - \chi_0(g) \end{aligned}$$

For $b > 1$, we get

$$\mathrm{ch}_{\mathcal{H}^q(S)}(1) = 2a(\sigma - 1)\,|G| + \mathrm{ch}_{\mathcal{H}^b(S)}(1)$$

and

$$\begin{aligned}\mathrm{ch}_{\mathcal{H}^q(S)}(g) &= \sum_{P \in S^g} \frac{(\varepsilon(P,g))^{a|G|+b}}{1-\varepsilon(P,g)} \\ &= \mathrm{ch}_{\mathcal{H}^b(S)}(g)\end{aligned}$$

Thus, the class function $\mathrm{ch}_{\mathcal{H}^1(S)}(g) - \left(\mathrm{ch}_{\mathcal{H}^1(S)} - \chi_0\right)$ for $b = 1$ and, respectively, the class function $\mathrm{ch}_{\mathcal{H}^q(S)} - \mathrm{ch}_{\mathcal{H}^b(S)}$ for $b > 1$ is zero for $g \neq 1$ and equals $2a(\sigma - 1)\,|G|$ at $g = 1$. But this is just the character $2a(\sigma - 1)\chi_{reg}$. ■

3.2 Multiplicity matrices for prime cyclic groups

For convenience we will use the index i for Q_i, P_i, and c_i and the index j for irreducible characters of G.

Let $1 \leq q \leq |G|$ we decompose the character $\mathrm{ch}_{\mathcal{H}^q(S)}$ into irreducible characters:

$$\mathrm{ch}_{\mathcal{H}^q(S)} = \mu_q^0 \chi_0 + \cdots + \mu_q^{l-1}\chi_{l-1}, \tag{21}$$

where $\mu_q^j = \left\langle \mathrm{ch}_{\mathcal{H}^q(S)}, \chi_j \right\rangle$ is the multiplicity inner product. We define the representation vector

$$R_q = \begin{bmatrix} \mu_q^0 \\ \vdots \\ \mu_q^{l-1} \end{bmatrix}$$

and multiplicity matrix

$$M = \left[\mu_q^j\right] = \begin{bmatrix} R_1 & \cdots & R_{|G|} \end{bmatrix}. \tag{22}$$

In the prime cyclic case, we denote the dependence of M on p and t via the notation $M_{p,t}$. By Proposition 6, the $\mathcal{H}^q$ representations are completely determined once M is known. We may use the Poincaré series in [1] to derive M but in the case of cyclic p-gonal actions it is simpler to just rework the calculation. We shall work with the class functions λ_q and then tack on χ_0 for $\mathrm{ch}_{\mathcal{H}^1(S)}$.

We break up the class function λ into a regular piece and a piece for each branch point Q_i. Define

$$\begin{aligned}
\lambda_q^0(1) &= (\sigma-1)(2q-1), \\
\lambda_q^0(g) &= 0,\ g \neq id(G), \\
\lambda_q^i(1) &= 0, i > 0, \\
\lambda_q^i(g) &= \sum_{P\in\pi_G^{-1}(Q_i)} \frac{(\varepsilon(P,g))^q}{1-\varepsilon(P,g)},\ g \neq id(G), i > 0, \\
\lambda_q(g) &= \sum_{i=0}^{t} \lambda_q^i(g).
\end{aligned}$$

With the exception of μ_1^0, we have

$$\mu_q^j = \left\langle \mathrm{ch}_{\mathcal{H}^q(S)}, \chi_j \right\rangle = \sum_{i=0}^{t} \left\langle \lambda_q^i, \chi_j \right\rangle = \left\langle \lambda_q, \chi_j \right\rangle$$

For μ_1^0 we have:

$$\mu_1^0 = \left\langle \lambda_1^0, \chi_0 \right\rangle + 1.$$

Since $\lambda_q^0 = \frac{(\sigma-1)(2q-1)}{|G|}\chi_{reg}$ then

$$\left\langle \lambda_q^0, \chi_j \right\rangle = \frac{(\sigma-1)(2q-1)}{|G|} \dim\left(\chi_j\right),$$

which satisfies

$$\left\langle \lambda_q^0, \chi_j \right\rangle = \frac{(\sigma-1)(2q-1)}{p}$$

when $G = \mathbb{Z}_p$.

Now assume that $G = \mathbb{Z}_p$. The character χ_j on $\mathbb{Z}_p$ is defined by $\chi_j(g) = \zeta^{jg}$. Also, there is a single fixed point P_i lying over Q_i, since $G_{P_i} = \mathbb{Z}_p$. Thus,

$$\lambda_q^i(g) = \sum_{P\in\pi_G^{-1}(Q_i)} \frac{(\varepsilon(P,g))^q}{1-\varepsilon(P,g)} = \frac{(\varepsilon(P_i,g))^q}{1-\varepsilon(P_i,g)}.$$

We may compute the other inner products as follows. Assuming a generating vector $(c_1,\ldots,c_t)$ and d_i defined by equation 18, we obtain, using equation 19:

$$\begin{aligned}
\left\langle \lambda_q^i, \chi_j \right\rangle &= \frac{1}{p}\sum_{g=0}^{p-1} \lambda_q^i(g)\overline{\chi_j(g)} \\
&= \frac{1}{p}\sum_{g=1}^{p-1} \frac{(\varepsilon(P_i, g))^q}{1-\varepsilon(P_i, g)}\overline{\chi_j(g)} \\
&= \frac{1}{p}\sum_{g=1}^{p-1} \frac{\left(\overline{\zeta}^{gd_i}\right)^q}{1-\overline{\zeta}^{gd_i}}\overline{\zeta^{jg}}.
\end{aligned}$$

Setting $h = gd_i$ we get $g = hc_i$, and the change of variable $g \to h$ yields

$$\left\langle \lambda_q^i, \chi_j \right\rangle = \frac{1}{p}\sum_{h=1}^{p-1} \frac{\left(\overline{\zeta}^{h}\right)^q}{1-\overline{\zeta}^{h}}\overline{\zeta^{jhc_i}} = \frac{1}{p}\sum_{h=1}^{p-1} \frac{\overline{\zeta}^{h(q+jc_i)}}{1-\overline{\zeta}^{h}}.$$

To eliminate the denominator, write

$$\begin{aligned}
\left\langle \lambda_q^i, \chi_j \right\rangle &= \lim_{x\to 1^-} \frac{1}{p}\sum_{h=1}^{p-1} \frac{\overline{\zeta}^{h(q+jc_i)}}{1-x\overline{\zeta}^{h}} \\
&= \lim_{x\to 1^-} \sum_{k=0}^{\infty} \frac{1}{p}\sum_{h=1}^{p-1} \overline{\zeta}^{h(q+jc_i)}\overline{\zeta}^{hk} x^k \\
&= \lim_{x\to 1^-} \left(\sum_{k=0}^{\infty} \left(\frac{1}{p}\sum_{h=0}^{p-1} \overline{\zeta}^{h(q+jc_i+k)} \right) x^k - \sum_{k=0}^{\infty} \frac{1}{p} x^k \right).
\end{aligned}$$

The sum $\frac{1}{p}\sum_{h=0}^{p-1}\overline{\zeta}^{h(q+jc_i+k)}$ is non-zero only if $q + jc_i + k = 0 \bmod p$, and when k satisfies this condition the sum is 1. Let

$$k_0(i,j,q) = (-jc_i - q) \bmod p \tag{23}$$

be the smallest non-negative k for which $q + jc_i + k = 0 \bmod p$. Then,

$$\begin{aligned}
\left\langle \lambda_q^i, \chi_j \right\rangle &= \lim_{x\to 1^-} \sum_{k'=0}^{\infty} x^{k_0(i,j,q)+pk'} - \sum_{k=0}^{\infty} \frac{1}{p} x^k \\
&= \lim_{x\to 1^-} \left(\frac{x^{k_0(i,j,q)}}{1-x^p} - \frac{1}{p(1-x)} \right).
\end{aligned}$$

Using L'Hôpital's rule (twice), we get:

$$\left\langle \lambda_q^i, \chi_j \right\rangle = \frac{p-1-2k_0(i,j,q)}{2p}. \tag{24}$$

Putting it all together, we get:

$$\begin{aligned}\left\langle \lambda_q, \chi_j \right\rangle &= \frac{(\sigma-1)(2q-1)}{p} + \sum_{i=1}^{t} \frac{p-1-2k_0(i,j,q)}{2p} \\ &= \frac{(\sigma-1)(2q-1)}{p} + \frac{t(p-1)}{2p} - \frac{1}{p}\sum_{i=1}^{t} k_0(i,j,q).\end{aligned}$$

The genus satisfies $\sigma = \frac{(p-1)(t-2)}{2}$ and so

$$\begin{aligned}\left\langle \lambda_q, \chi_j \right\rangle &= \frac{ptq-2pq+p-tq}{p} - \frac{1}{p}\sum_{i=1}^{t} k_0(i,j,q) \qquad (25) \\ &= q(t-2)+1-\frac{1}{p}\left(tq+\sum_{i=1}^{t} k_0(i,j,q)\right)\end{aligned}$$

We put the preceding into a proposition.

Proposition 8 *Let $\mathbb{Z}_p$ act on a surface S with such that S/G has genus zero, $S \to S/G$ is branched over t points and that a generating vector for the action is given by $(c_1, \ldots, c_t)$. Then the $p \times p$ multiplicity matrix $M_{p,t}$ defined by equations 21 and 22 is given by*

$$\begin{aligned}M_{p,t}(j,q) &= \left[\mu_q^j\right] = q(t-2)+1-\frac{1}{p}\left(tq+\sum_{i=1}^{t}(-jc_i-q) \bmod p\right) \qquad (26) \\ 0 &\leq j \leq p-1, 1 \leq q \leq p, (j,q) \neq (0,1)\end{aligned}$$

and

$$M_{p,t}(0,1) = 0, \tag{27}$$

where χ_j, is the jth character of $\mathbb{Z}_p$ in the standard ordering.

Remark 9 *We should check that $\left\langle \lambda_q, \chi_j \right\rangle$ is an integer. We just need to show that $tq + \sum_{i=1}^{t}(-jc_i - q) \bmod p$ is divisible by p.*

$$\begin{aligned} \left(tq + \sum_{i=1}^{t} k_0(i,j,q) \right) \bmod p &= \left(tq + \sum_{i=1}^{t} (-jc_i - q) \bmod p \right) \bmod p \\ &= \left(tq - j \left(\sum_{i=1}^{t} c_i \right) - tq \right) \bmod p \\ &= -j \left(\sum_{i=1}^{t} c_i \right) \bmod p = 0 \end{aligned}$$

Example 10 *The smallest possible genus for which $\mathcal{H}^1(S)$ equivalence conflates topological actions is genus 4, with $\mathbb{Z}_5$ actions and 4 branch points. There are three topological actions given by the generating vectors $(c_1, c_2, c_3, c_4) = (1,1,1,2)$, $(1,1,4,4)$, and $(1,2,3,4)$. The multiplicity matrices in the three cases are given in Table 1 below. By comparing columns of the matrices we see that the last two classes are conflated by odd degree differentials, but all three classes are separated using either quadratic and quartic differentials.*

Generating Vector	*Multiplicity Matrix*
$(c_1, c_2, c_3, c_4) = (1,1,1,2)$	$\begin{bmatrix} 0 & 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 5 & 4 \\ 1 & 2 & 4 & 3 & 5 \\ 1 & 3 & 2 & 4 & 5 \\ 2 & 1 & 2 & 4 & 6 \end{bmatrix}$
$(c_1, c_2, c_3, c_4) = (1,1,4,4)$	$\begin{bmatrix} 0 & 1 & 3 & 5 & 7 \\ 1 & 1 & 3 & 5 & 5 \\ 1 & 3 & 3 & 3 & 5 \\ 1 & 3 & 3 & 3 & 5 \\ 1 & 1 & 3 & 5 & 5 \end{bmatrix}$
$(c_1, c_2, c_3, c_4) = (1,2,3,4)$	$\begin{bmatrix} 0 & 1 & 3 & 5 & 7 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

Table 1

Remark 11 *All calculations for Table 1, indeed all tables in the paper, were done with Magma [12]. The multiplicity matrices were calculated both by direct decomposition of the representations and by using Proposition 8.*

3.3 Alternate perspective

Given a prime p, the number of branch points t, and a degree q we can look at the map from generating vectors to representation vectors. To this end, for a generating vector $(c_1, \ldots, c_t)$ we define the vector $X = (x_1, \ldots x_{p-1})$ of non-negative integers by

$$x_l = |\{c_i : c_i = l\}| \tag{28}$$

(Note that l in x_l is not to be confused with the l in χ_{l-1}. For the character χ_{l-1}, $l = p$, the number of conjugacy classes. The vector $(x_1, \ldots x_{p-1})$ must satisfy these equations:

$$\begin{aligned} \sum_{l=1}^{p-1} x_l &= t, \\ \sum_{l=1}^{p-1} l x_l &= 0 \bmod p. \end{aligned}$$

Each such vector defines a "braid class" of generating vectors. If we act on $(c_1, \ldots, c_t)$ by an element of $\mathrm{Aut}(G)$ then the components of X are permuted. We see that

$$\begin{aligned} \left\langle \lambda_q^i, \chi_j \right\rangle &= \frac{p - 1 - 2k_0(i, j, q)}{2p} \\ &= \frac{p - 1 - 2k_1(l, j, q)}{2p} \end{aligned}$$

where k_1 is defined by replacing c_i by l in equation 23 :

$$k_1(l, j, q) = (-jl - q) \bmod p \tag{29}$$

We writing equation 25

$$\begin{aligned} \left\langle \lambda_q, \chi_j \right\rangle &= \frac{ptq - 2pq + p - tq}{p} - \frac{1}{p} \sum_{i=1}^{t} k_0(i, j, q) \\ &= \frac{ptq - 2pq + p - tq}{p} - \frac{1}{p} \sum_{l=1}^{p-1} k_1(l, j, q) x_l \end{aligned}$$

Define, entry-wise, the $p \times (p-1)$ matrix N and vectors R, R_0 of length p by:

$$N(j,l) = k_1(l,j,q) = (-jl-q) \bmod p \tag{30}$$

$$\begin{aligned} R(j) &= \left\langle \lambda_q, \chi_j \right\rangle \\ R_0(j) &= ptq - 2pq + p - tq \end{aligned} \tag{31}$$

Then the representation decomposition for $\mathbb{Z}_p$ acting on $\mathcal{H}^q$ in vector form

$$R = \frac{1}{p}\left(R_0 - NX\right). \tag{32}$$

Let's put this into a proposition.

Proposition 12 *Let $\mathbb{Z}_p$ act on S with t branch points and action vector X (equations 28, 31). Let $1 \leq q \leq p$ and define R, R_0, N as in equation 31. Then, except for the multiplicity μ_1^0, the character decomposition vector R of $\mathbb{Z}_p$ acting on $\mathcal{H}^q(S)$ is given by equation 32.*

The alternate multiplicity matrix N allows us to consider all the different topological actions at once, for an arbitrary number of branch points $t \geq 3$ and a fixed q. If needed, we denote the dependence of N on p, q via the notation $N_{p,q}$. The matrices for $p = 5$ are given in Table 2. The column K of that table is a matrix whose columns span the kernel of N. For some p, q the kernel can have a fairly large dimension, see Table 5 of Section 4.

Remark 13 *Looking at Tables 1 and 2 we see some patterns.*

1. *If the kernel is trivial then there is no q-conflation.*
2. *If there is conflation the kernel is non trivial.*
3. *The entries of the top row of N are all equal. The rest of the matrix is symmetric.*
4. *Upon removing the top row of the matrix N each row and column contains all the numbers $0, \ldots, p-1$ except for the number in the top row.*

Let us try to explain the numbered items in the previous Remark. If two actions X_1 and X_2 are $\mathcal{H}^q$ conflated, then for the degree q matrix

$$\frac{1}{p}\left(R_0 - NX_1\right) = \frac{1}{p}\left(R_0 - NX_2\right)$$

or

$$N(X_1 - X_2) = 0$$

since R_0 depends only on p, t, q. This explains items 1 and 2. To explain items 3 and 4 we recall the definition of the entries of N

$$N_{j,l} = (-jl - q) \bmod p.$$

In the top row $(-jl - q) \bmod p = (-0 \times l - q) \bmod p = p - q$, so we get a constant row. For the remaining entries $jl \neq 0$ and so $(-jl - q) \bmod p \neq p - q \bmod p$. The lower matrix is formed from the negative of the multiplication table of $\mathbb{Z}_p^*$ shifted by $-q$ and the appropriate residue $\bmod\, p$ chosen. This explains the format of the lower part of the matrix.

q	N	K
1	$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 3 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$
2	$\begin{bmatrix} 3 & 3 & 3 & 3 \\ 2 & 1 & 0 & 4 \\ 1 & 4 & 2 & 0 \\ 0 & 2 & 4 & 1 \\ 4 & 0 & 1 & 2 \end{bmatrix}$	no kernel
3	$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 0 & 4 & 3 \\ 0 & 3 & 1 & 4 \\ 4 & 1 & 3 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$
4	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & 3 & 2 \\ 4 & 2 & 0 & 3 \\ 3 & 0 & 2 & 4 \\ 2 & 3 & 4 & 0 \end{bmatrix}$	no kernel
5	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

Table 2

Remark 14 *As noted previously, the totality of all p-gonal* $\mathbb{Z}_p$ *actions forms a subset of* $\mathbb{Z}^{p-1} = \{X = (x_1, \ldots, x_{p-1}) : x_l \in \mathbb{Z}\}$ *subject to these constraints* *only:*

$$\begin{aligned} x_l &\geq 0 \\ \sum_{l=1}^{p-1} x_l &\geq 3 \\ \sum_{l=1}^{p-1} l x_l &= 0 \bmod p \end{aligned}$$

This set is invariant under these operations: addition of vectors and permutation of the indices by $l \to el \bmod p$ for $e \in \mathbb{Z}_p^*$. We have conflation if and only if two inequivalent actions X_1 and X_2 have the same number of branch points and $X_1 - X_2 \in \ker(N)$. We also note that the top row of N is a multiple of the sum of the remaining rows. Therefore, the kernel of N is the kernel of the $(p-1) \times (p-1)$ matrix obtained by deleting the top row.

4 Examples and conjectures

4.1 Examples

Example 15 *Here is a sample of low genus actions that are not distinguished by $\mathcal{H}^1(S)$ equivalence and the lowest degree differential that separates them.*

p	t	σ	*some conflated actions*	*smallest degree of separating differential*
5	4	4	$(1,1,4,4), (1,2,3,4)$	2
5	6	8	$(1,1,1,4,4,4), (1,1,2,3,4,4)$	2
7	4	6	$(1,1,6,6), (1,2,5,6)$	2
7	5	9	$(1,1,1,5,6), (1,1,2,5,5), (1,1,3,4,5)$	2
7	6	12	$(1,1,1,6,6,6), (1,1,2,5,6,6)$ $(1,1,3,4,6,6), (1,2,3,4,5,6)$	3

Table 3

Looking at the last row of the table, we see that quadratic differentials will not always work in separating classes.

Example 16 *In the following tables we record the behaviour of the kernel of $N_{p,q}$. In Table 4 we record the dimensions of the kernels of $N_{p,q}$ in the range*

$1 \leq q \leq p$ in list form. In Table 5 we give lists of degrees q for which the kernel $N_{p,q}$ is trivial (good degrees) and non-trivial (bad degrees)

p	*Dimensions* $\ker(N_{p,q})$, $1 \leq q \leq p$
3	$0, 0$
5	$1, 0, 1, 0, 1$
7	$2, 1, 0, 3, 0, 1, 2$
11	$4, 1, 0, 1, 0, 4, 0, 1, 0, 1, 4$
13	$5, 0, 3, 0, 2, 0, 5, 0, 2, 0, 3, 0, 5$
17	$7, 0, 2, 0, 0, 0, 1, 0, 7, 0, 1, 0, 0, 0, 2, 0, 7$
19	$8, 1, 0, 0, 0, 3, 0, 0, 0, 8, 0, 0, 0, 3, 0, 0, 0, 1, 8$

Table 4

p	*good degrees*	*bad degrees*
3	$1, 2, 3$	
5	$2, 4$	$1, 3, 5$
7	$3, 5$	$1, 2, 4, 6, 7$
11	$3, 5, 7, 9$	$1, 2, 4, 6, 8, 10, 11$
13	$2, 4, 6, 8, 10, 12$	$1, 3, 5, 7, 9, 11, 13$
17	$2, 4, 5, 6, 8, 10, 12, 13, 14, 16$	$1, 3, 7, 9, 11, 15, 17$
19	$3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17$	$1, 2, 6, 10, 14, 18, 19$

Table 5

Example 17 *The number of conflated actions can be quite large. For example, for $p = 11, t = 10, q = 1, \sigma = 40$ there are 854 actions with numerous conflations. In one case 31 different actions all have the same $\mathcal{H}^1$ character. For $p = 19, t = 10, q = 1, \sigma = 63$ there are 9143 actions with a maximum of 116 conflated actions.*

4.2 Conjectures and Remarks

By reviewing the tables we are led to the following two conjectures.

Conjecture 18 *For every prime $p \geq 3$ there is a $q \leq p$ such that the matrix $N_{p,q}$ has trivial kernel and hence there is no $\mathcal{H}^q$ conflation, for $\mathbb{Z}_p$ actions, for every $t \geq 3$.*

Conjecture 19 *For every prime* $p \geq 3$ *and* $1 \leq q \leq p$ *such that* $N_{p,q}$ *has a non-trivial kernel there is* $\mathcal{H}^q$ *conflation for infinitely many values of* t*.*

Conjecture 20 *For every prime* $p \geq 3$ *and every* $1 \leq q \leq p$ *kernels of* $N_{p,q}$ *and* $N_{p,q'}$ *have the same dimension where* $q' = p + 1 - q$*.*

Indeed some preliminary calculations show that the kernels of $N_{p,q}$ and $N_{p,q'}$ equal each other for small primes. We cannot claim a conjecture for the kernel of $N_{p,1}$ but we can at least make the following remark

Remark 21 *For all the primes in Table 4 we see that:*

$$p = 2\dim(\ker(N_{p,1})) + 3.$$

For all the primes in Table 4, except $p = 7$*, we have:*

$$p = 2\dim(\ker(N_{p,(p+1)/2})) + 3.$$

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