

## A Brief History of Elliptic Integral Addition Theorems

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# A Brief History of Elliptic Integral Addition Theorems

Jose Barrios <sup>\*†</sup>

## Abstract

The discovery of elliptic functions emerged from investigations of integral addition theorems. An addition theorem for a function  $f$  is a formula expressing  $f(u+v)$  in terms of  $f(u)$  and  $f(v)$ . For a function defined as a definite integral with a variable upper limit, an addition theorem takes the form of an equation between the sum of two such integrals, with upper limits  $u$  and  $v$ , and an integral whose upper limit is a certain function of  $u$  and  $v$ . In this paper, we briefly sketch the role which the investigation of such addition theorems has played in the development of the theory of elliptic integrals and elliptic functions.

## 1 What is an Addition Theorem?

Consider the following equation involving an integral familiar to calculus students:

$$\int_0^{\sin u} \frac{1}{\sqrt{1-x^2}} dx + \int_0^{\sin v} \frac{1}{\sqrt{1-x^2}} dx = \int_0^{\sin(u+v)} \frac{1}{\sqrt{1-x^2}} dx.$$

If we set  $y = \sin u$  and  $z = \sin v$ , we obtain an equivalent form:

$$\int_0^y \frac{1}{\sqrt{1-x^2}} dx + \int_0^z \frac{1}{\sqrt{1-x^2}} dx = \int_0^{T(y,z)} \frac{1}{\sqrt{1-x^2}} dx, \quad (1)$$

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where  $T$  is an algebraic function of  $y$  and  $z$  as below:

$$T(y, z) = y\sqrt{1 - z^2} + z\sqrt{1 - y^2}.$$

Such an interesting mathematical relationship has gained attention from many famous mathematicians, including Fagnano, Euler, Legendre, Jacobi, and others. Briefly

**Definition 1.1** [7] *An addition theorem for a function  $f$  is a formula expressing  $f(u + v)$  in terms of  $f(u)$  and  $f(v)$ .*

In other words, a function  $f$  has an addition theorem if there exists a two variable function  $T$  such that  $f(u + v) = T(f(u), f(v))$ .

When the function  $T$  is algebraic, we say that  $f$  has an algebraic addition theorem. Familiar functions which have algebraic addition theorems are

$$f(x) = e^x, \quad g(x) = \sin x, \quad h(x) = \tan x, \quad \text{etc.}$$

The corresponding function  $T$  for each of the above functions is

$$T(u, v) = u \cdot v, \quad T(u, v) = u\sqrt{1 - v^2} + v\sqrt{1 - u^2}, \quad T(u, v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

respectively (on appropriate intervals).

In [3], Harris Hancock states that “Such an equation offers a means of determining the value of the function for the sum of two quantities as arguments, when the values of the function for the two arguments taken single are known. It is called an *algebraic addition theorem*.”

Addition theorems may apply in different situations. An integral addition theorem is an equation expressing the sum of two integrals with upper limits  $u$  and  $v$  as a single integral whose upper limit is a function  $T$  of  $u$  and  $v$  as shown in (1). The discovery of elliptic functions emerged from investigations of such integral addition theorems. It was Euler who formulated what is now called the addition theorem for elliptic integrals of the first kind in the middle of the seventeenth century. With Weierstrass [3], *The problem of the theory of elliptic functions is to determine all functions of the complex argument for which there exists an algebraic addition-theorem.*

## 2 Early Encounters with the Elliptic Integrals

In 1702, John Bernoulli conjectured that the integral of any rational function is expressible in terms of other rational functions, trigonometric functions, and logarithmic functions. It turns out that this statement is true [1]. As for the integrals of irrational functions, mathematicians were aware of several which could not be integrated in terms of the elementary functions. They were thus the first integration problems which seemed impossible to properly solve in the sense of Leibniz, who considered a solution to be an  $F(x)$ , composed of elementary functions, such that  $F'(x) = f(x)$ . The rectification of the ellipse and the lemniscate, determination of the period of the simple pendulum, and the shape of an elastica are some of the problems encountered at the time which required integration of irrational functions for their solution. In each of these cases, the particular type of irrational integral involved is the elliptic integral

$$\int R(x, \sqrt{P(x)})dx, \quad (2)$$

where  $R(x, \sqrt{P(x)})$  is a rational function of its arguments and  $P(x)$  is a polynomial of degree 3 or 4. That such integrals cannot be evaluated in terms of the elementary functions was finally proved by Liouville in 1833 [8].

As will be seen, later mathematicians, though unable to evaluate this integral in terms of elementary functions, nevertheless obtained useful related results, such as the aforementioned addition theorem due to Euler, as well as a more general addition theorem due to Abel.

### 2.1 Relation between polynomials of even and odd degree

As stated,  $P(x)$  in (2) is a polynomial of degree 3 or 4. However, as will now be shown [1], there is no loss of generality in restricting one's considerations to  $P(x)$  of degree 4. More generally, if  $P(x)$  is a polynomial of odd degree  $2q - 1$ , it is always possible, by means of a change of variable from  $x$  to  $y$ , to express  $\sqrt{P(x)}$  in a form  $y^{-q}\sqrt{P_1(y)}$ , where  $P_1(y)$  is a polynomial of degree  $2q$ .

Beginning with

$$P(x) = A_0x^{2q-1} + A_1x^{2q-2} + \cdots + A_{2q-1},$$

set  $x = a + 1/y$ , where  $a$  is not a root of  $P(x)$ . This gives

$$P(x) = P(a) + P'(a)\frac{1}{y} + \cdots + \frac{P^{(2q-1)}(a)}{(2q-1)!} \frac{1}{y^{2q-1}} = \frac{P_1(y)}{y^{2q}},$$

$$\sqrt{P(x)} = \frac{\sqrt{P_1(y)}}{y^q}.$$

Thus,

$$\int R(x, \sqrt{P(x)})dx = \int R_1(y, \sqrt{P_1(y)})dy,$$

where  $R(\mu, \omega)$  and  $R_1(\mu, \omega)$  are rational functions of their arguments, the degree of the polynomial having been increased by 1.

### 3 Fagnano

In 1718 [6], Fagnano made a discovery related to the rectification of the arc of the lemniscate. He considered, in particular, the lemniscate whose rectangular equation is  $(x^2 + y^2)^2 = (x^2 - y^2)$ . The graph has the shape of the infinity symbol “ $\infty$ ” and intersects the  $x$ -axis at  $x = 0, \pm 1$  (see Fig. 1). The length of the arc in the first quadrant beginning at the origin and terminating at the point a distance  $r$  from the origin is

$$s(r) = \int_0^r \frac{dx}{\sqrt{1-x^4}}, \quad (3)$$

where  $r$  lies in the closed interval  $[0, 1]$ .

A similar integral is  $\arcsin r = \int_0^r 1/\sqrt{1-x^2}dx$ , where  $r$  lies in the closed interval  $[0, 1]$ . Its integrand can be rationalized by the substitution

$$x = \frac{2w}{1+w^2} \implies \frac{dx}{\sqrt{1-x^2}} = \frac{2dw}{1+w^2}.$$

Perhaps inspired by this analogous case, Fagnano made two similar substitutions in (3):

$$x = \frac{w\sqrt{2}}{\sqrt{1+w^4}}, \text{ then } w = \frac{y\sqrt{2}}{\sqrt{1-y^4}} \implies x = \frac{2y\sqrt{1-y^4}}{1+y^4}.$$

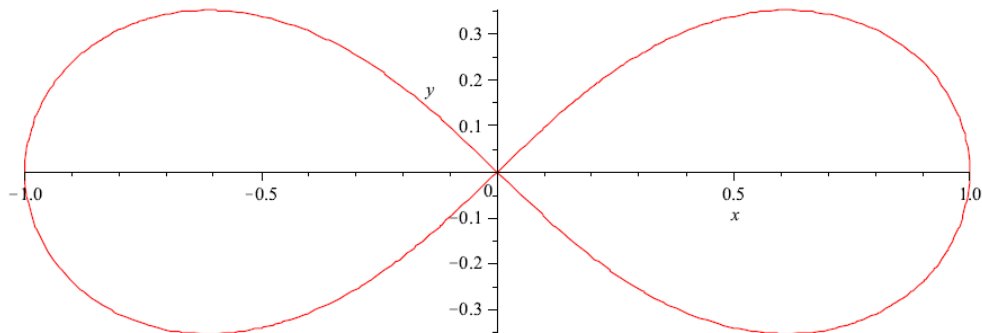


Figure 1: lemniscate

As a result, he obtained

$$\int_0^r \frac{dx}{\sqrt{1-x^4}} = 2 \int_0^u \frac{dy}{\sqrt{1-y^4}},$$

where

$$r = T(u) = \frac{2u\sqrt{1-u^4}}{1+u^4}. \quad (4)$$

This shows that the lemniscatic arc length for a point at distance  $r = T(u)$  from the origin is twice that for a point at distance  $u$  from the origin. Thus, while the lemniscatic integral itself remained unevaluated, Fagnano had in an instance of serendipity acquired an interesting related expression which doubles the arc of the lemniscate.

## 4 Euler

Expanding on Fagnano's work, Euler obtained the addition theorem for lemniscatic integrals. He first found that the general solution of

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}} \quad (5)$$

is given by the algebraic relation

$$x = \frac{y\sqrt{1-c^4} + c\sqrt{1-y^4}}{1+c^2y^2}, \quad (6)$$

where  $c$  is an arbitrary constant. Since the general solution of (5) can also be expressed as

$$\int_0^y \frac{dt}{\sqrt{1-t^4}} + \int_0^c \frac{dt}{\sqrt{1-t^4}} = \int_0^x \frac{dt}{\sqrt{1-t^4}},$$

where the second integral on the left is a convenient expression for the constant of integration, it follows that the upper limits of this equation are related by (6). Written in another form,

$$\int_0^u \frac{dx}{\sqrt{1-x^4}} + \int_0^v \frac{dx}{\sqrt{1-x^4}} = \int_0^{T(u,v)} \frac{dx}{\sqrt{1-x^4}},$$

where,

$$T(u, v) = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1 + u^2v^2}.$$

As can be seen,  $T(u, v)$  is a symmetric algebraic function of two variables representing a plausible generalization of (4) above.

Euler next obtained the addition theorem for elliptic integrals of the first kind. Let  $P(x) = (1-x^2)(1-k^2x^2)$  for a  $k$  such that  $0 < k < 1$ . Then

$$\int_0^u \frac{dx}{\sqrt{P(x)}} + \int_0^v \frac{dx}{\sqrt{P(x)}} = \int_0^{T(u,v)} \frac{dx}{\sqrt{P(x)}},$$

where

$$T(u, v) = \frac{u\sqrt{P(v)} + v\sqrt{P(u)}}{1 - k^2u^2v^2}.$$

Critiquing his own work, Euler pointed out that his methods were restricted to only square roots and polynomials of degree 4. He also noted that he had not found the solution of (5) by a general method. Still, his were the chief results in the theory of elliptic integrals at the time.

## 5 Legendre, Abel, and Jacobi

The next mathematician who investigated elliptic integrals was Legendre. In his *Traite des fonctions elliptiques* [5], he showed that integration of the elliptic integral  $\int R(t)/\sqrt{P(t)}dt$ , where  $R(t)$  is a rational function of  $t$  and

$P(t)$  is a polynomial of fourth degree, can be reduced to the integration of the three integrals

$$\int \frac{dx}{\sqrt{1-x^2}\sqrt{1-l^2x^2}}, \quad \int \frac{x^2 dx}{\sqrt{1-x^2}\sqrt{1-l^2x^2}}, \quad \int \frac{dx}{(x-a)\sqrt{1-x^2}\sqrt{1-l^2x^2}},$$

which he called the elliptic integrals of the first, second, and third kinds, respectively. Applying substitutions, he reduced these integrals to

$$\int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \quad \int_0^\phi \sqrt{1-k^2\sin^2\phi} d\phi, \quad \int_0^\phi \frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-k^2\sin^2\phi}},$$

where  $0 < k < 1$  while  $n$  is an arbitrary constant.

Legendre's work was eclipsed by that of Abel and Jacobi. The crucial idea initiating their discoveries was the inversion of the elliptic integrals.

In analogy with the function

$$\omega(u) = \int_0^u \frac{dx}{\sqrt{1-x^2}},$$

with inverse function  $u = \sin \omega$ , Abel hit on the idea of investigating the inverse function of

$$\omega(\phi) = \int_0^\phi \frac{dt}{\sqrt{1-k^2\sin^2 t}},$$

denoted by Jacobi as  $\phi = \text{am } \omega$ . Gudermann later introduced the notation

$$\text{sn } \omega = \sin \phi, \quad \text{cn } \omega = \cos \phi, \quad \text{dn } \omega = \sqrt{1-k^2\sin^2 \phi}.$$

These functions were observed to satisfy relations similar to those of the trigonometric functions:

$$\text{sn}^2 \omega + \text{cn}^2 \omega = 1, \quad \text{dn}^2 \omega + k^2 \text{sn}^2 \omega = 1,$$

$$\text{sn}(-\omega) = -\text{sn } \omega, \quad \text{cn}(-\omega) = \text{cn } \omega, \quad \text{dn}(-\omega) = \text{dn } \omega.$$

Also, the value  $4K$ , with  $K$  defined by

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2 \phi}},$$



is a period of  $\operatorname{sn} u$  and  $\operatorname{cn} u$ , while  $2K$  is a period of  $\operatorname{dn} u$ .

Just as the trigonometric functions have simple algebraic addition theorems, so too, as Abel found, did the elliptic functions. Thus, for  $\operatorname{sn}(\omega + \eta)$ , he obtained

$$\operatorname{sn}(\omega + \eta) = \frac{\operatorname{sn} \omega \operatorname{cn} \eta \operatorname{dn} \eta + \operatorname{sn} \eta \operatorname{cn} \omega \operatorname{dn} \omega}{1 - k^2 \operatorname{sn}^2 \omega \operatorname{sn}^2 \eta},$$

with similar formulas for  $\operatorname{cn}(\omega + \eta)$  and  $\operatorname{dn}(\omega + \eta)$ .

Having learned about Cauchy's work in complex analysis during a trip to Paris, Abel next defined the elliptic functions for pure imaginary arguments:

$$\operatorname{sn}(iu, k) = i \frac{\operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}, \quad \operatorname{cn}(iu, k) = \frac{1}{\operatorname{cn}(u, k')}, \quad \operatorname{dn}(iu, k) = \frac{\operatorname{dn}(u, k')}{\operatorname{cn}(u, k')},$$

where  $k'$  is defined by  $k^2 + k'^2 = 1$ .

With the elliptic functions defined for real and pure imaginary arguments, the addition theorems provided an obvious definition for general complex arguments  $z = \operatorname{Re}(z) + i\operatorname{Im}(z)$ .

## 6 Weierstrass

At around 1860 [4], Weierstrass, impressed by Abel's and Jacobi's work, began his own investigations into the theory of elliptic functions. In his theory, the fundamental elliptic function was  $p(u)$ , defined as the inverse function of

$$u(p) = \int_0^p \frac{dt}{\sqrt{4t^3 - g_2t - g_3}},$$

where  $g_2$  and  $g_3$  are constants such that  $g_2^3 - 27g_3^2 \neq 0$ . He showed how every elliptic function could be expressed in terms of  $p(u)$  and its derivative.

According to Weierstrass, every function with an algebraic addition theorem is an elliptic function or a limiting case of one (i.e. rational, trigonometric, and exponential functions) [3]. Thus, with Weierstrass's contributions, the intimate connection between algebraic addition theorems and elliptic functions was established.

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