11-1990

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MSTR 90-08

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November 1990

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1. Introduction

Let S be a set of nontrivial permutations of $\{1, 2, \ldots, n\}$; i.e., $S \subsetneq S_n - \{id\}$, where S_n is the symmetric group on n symbols. A group G is said to be S-rewriteable if, for each n-tuple

$$(x_1, \ldots, x_n) \in G^n,$$

there exists a permutation $\sigma \in S$ such that

$$x_1x_2 \cdots x_n = x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

If $S = S_n - \{id\}$, then S-rewriteability is referred to as n-rewriteability [1]. Note that a group is 2-rewriteable if, and only if, it is abelian: $x_1x_2 = x_2x_1 = x_{\sigma(1)}x_{\sigma(2)}$, where σ is the transposition (1,2). Thus, rewriteability can be viewed as a natural generalization of commutativity; the motivation for this generalization in the context of groups and semigroups stems from automata theory [2]. Research has been done classifying groups which are 3-rewriteable and 4-rewriteable [3, 5]. In particular, Curzio, Longobardio, and Maj [3] have shown:

G is 3-rewriteable if, and only if, $|G'| \leq 2$, where G' is the commutator subgroup of G.

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*The author's work was supported by NSF grant DMS 8922674.

In this paper, we primarily consider S-rewriteablity, with $S = D_n^* = D_n - \{id\}$, where D_n is the dihedral group on n symbols. For example, a group G is D_4^*-rewriteable if, for all $(x,y,z,w) \in G^4$,

$$xyzw \in \{wxyz, zwxy, yzwx, wzyx, xwzy, yxwz, zyxw\},$$

since $D_4^* = \{(1,4,3,2), (1,3)(2,4), (1,2,3,4), (1,4)(2,3), (2,4), (1,2)(3,4), (1,3)\}$. Notice that D_3^*-rewriteability is equivalent to 3-rewriteability because $D_3 = S_3$ and, therefore, that D_3^*-rewriteable groups have been classified. We are concerned with classifying D_n^*-rewriteable groups for all n. Our main results are that, for $n \geq 3$, D_n^*-rewriteability is in fact equivalent to D_3^*-rewriteability and, further, that D_n^*-rewriteability is characterized by a proper subset of D_n^*.

2. The Equivalence of D_3^*-rewriteability and D_n^*-rewriteability

THEOREM 1. *For $n \geq 3$, a group G is D_n^*-rewriteable if, and only if, G is D_3^*-rewriteable.*

Proof: We show first that D_{n+1}^*-rewriteability implies D_n^*-rewriteability for $n \geq 3$. Choose $(x_1,e,x_2, \ldots, x_n)$ as an (n+1)-tuple in G^{n+1}, where e is the identity. The result follows because $x_1x_2\cdots x_n = x_1ex_2\cdots x_n$ is at least one of,

$$x_nx_1ex_2\cdots x_{n-1} = x_nx_1\cdots x_{n-1},$$
$$x_{n-1}x_nx_1ex_2\cdots x_{n-2} = x_{n-1}x_nx_1x_2\cdots x_{n-2},$$
$$\vdots$$
$$x_2x_3\cdots x_nx_1e = x_2x_3\cdots x_nx_1,$$
$$ex_2\cdots x_nx_1 = x_2\cdots x_nx_1,$$
$$x_nx_{n-1}\cdots x_2ex_1 = x_n\cdots x_1,$$
$$x_1x_n\cdots x_2e = x_1x_n\cdots x_2,$$

$$ex_1x_n \cdots x_2 = x_1x_n \cdots x_2,$$

$$x_2ex_1x_n \cdots x_3 = x_2x_1x_n \cdots x_3,$$

$$\vdots$$

$$x_{n-2} \cdots ex_1x_nx_{n-1} = x_{n-2} \cdots x_1x_nx_{n-1},$$

$$x_{n-1}x_{n-2} \cdots ex_1x_n = x_{n-1}x_{n-2} \cdots x_1x_n.$$

To clarify this notation, consider the instance when $n = 3$. Then, $x_1x_2x_3 = x_1ex_2x_3$ is at least one of

$$x_3x_1ex_2 = x_3x_1x_2 \qquad (\sigma = (1,3,2)),$$

$$x_2x_3x_1e = x_2x_3x_1 \qquad (\sigma = (1,2,3)),$$

$$ex_2x_3x_1 = x_2x_3x_1 \qquad (\sigma = (1,2,3)),$$

$$x_3x_2ex_1 = x_3x_2x_1 \qquad (\sigma = (1,3)),$$

$$x_1x_3x_2e = x_1x_3x_2 \qquad (\sigma = (2,3)),$$

$$ex_1x_3x_2 = x_1x_3x_2 \qquad (\sigma = (2,3)),$$

$$x_2ex_1x_3 = x_2x_1x_3 \qquad (\sigma = (1,2)).$$

It follows that D_n^*-rewriteability implies D_3^*-rewriteability.

For the converse, suppose G is D_3^*-rewriteable but not abelian (because if G is abelian, then G is clearly D_n^*-rewriteable). If G is not D_n^*-rewriteable, then there exists an n-tuple $(x_1, \ldots, x_n) \in G^n$ such that the product $p = x_1 \cdots x_n$ cannot be rewritten as the product of any of the non-trivial 2n-1 dihedral permutations of $(x_1, \ldots, x_n)$. Thus, $x_nx_1x_2 \cdots x_{n-1} \neq p$ and $x_nx_{n-1} \cdots x_2x_1 \neq p$. Note that by transposing adjacent terms in the ordered n-tuple $(x_1, x_2, \ldots, x_n)$ enough times, we can obtain the n-tuples $(x_n, x_1, x_2, \ldots, x_{n-1})$ and $(x_n, x_{n-1}, \ldots, x_1)$. How does transposing adjacent terms affect the product? We know that for all $a, b \in G$, $ab = baa^{-1}b^{-1}ab$, where $a^{-1}b^{-1}ab \in G'$. By the result of Curzio, Longobardio, and Maj, we may take $G' = \{e, z\}$ because $|G'| = 2$. Moreover, z is in the center, Z, of G because G' is a normal subgroup of G. Therefore, for all a, b $\in$ G, either ab = ba or ab = baz. Since $z \in Z$, we have $x_n x_1 x_2 \cdots x_{n-1} = pz^k$ and $x_n x_{n-1} \cdots x_2 x_1 = pz^j$ for some k and j. But, $z^2 = e$, so $x_n x_1 x_2 \cdots x_{n-1} = pz = x_n x_{n-1} \cdots x_2 x_1$, which implies $x_1 x_2 \cdots x_{n-1} x_n = x_{n-1} \cdots x_2 x_1 x_n$. In other words, the product $p = x_1 \cdots x_n$ is D_n^*-rewriteable, which gives us a contradiction. Therefore, G is D_n^*-rewriteable.

3. S-rewriteability

Let R denote the group of rotations generated by the n-cycle (1, 2, ... , n) and let $R^* = R - \{id\}$. We will refer to R^*-rewriteability as <u>n-rotateability</u>.

THEOREM 2. *G is n-rotateable if, and only if, G is abelian; i.e., 2-rewriteable.*

Proof: That abelian implies n-rotateable is trivial. So, let G be n-rotateable and consider the n-tuple $(a,e,e, \ldots, e,b) \in G^n$. The product $ab = a \cdot e \cdot e \cdots e \cdot b$ is at least one of

$$b \cdot a \cdot e \cdot e \cdots e = ba,$$

$$e \cdot b \cdot a \cdot e \cdot e \cdots e = ba,$$

$$\vdots$$

$$e \cdot e \cdots e \cdot b \cdot a = ba.$$

In fact, a similar argument shows that G is abelian if, and only if, G is U-rewriteable, where U consists of a single non-trivial permutation. It is also possible to characterize D_n^*-rewriteablility by a smaller set of permutations.

THEOREM 3. G is D_n^*-rewriteable for $n \geq 3$ if, and only if, G is T-rewriteable where T-rewriteability is determined by the permutations which require that

$$x_1 x_2 \cdots x_n \in \{x_n x_1 x_2 \cdots x_{n-2} x_{n-1}, x_n x_{n-1} \cdots x_2 x_1, x_{n-1} x_{n-2} \cdots x_2 x_1 x_n\}$$

for all $(x_1, \ldots, x_n) \in G^n$;i.e.,

$$T = \{(1, n, n-1, \ldots, 3, 2), (1, n)(2, n-1)(3, n-2) \cdots, (1, n-1)(2, n-2)(3, n-3) \cdots\}.$$

Proof: T-rewriteability implies D_n^*-rewriteability because the elements of T are symmetries (a rotation and two reflections) of the regular n-gon.

Conversely, assume G is D_n^*-rewriteable. By THEOREM 1, G is D_3^*-rewriteable and, therefore, $G' = \{e,z\}$, where $z \in Z$. It follows, just as in the proof of THEOREM 1, that $x_n x_1 x_2 \cdots x_{n-2} x_{n-1}$ and $x_n x_{n-1} \cdots x_2 x_1$ are elements of $\{p, pz\}$. If $x_n x_1 x_2 \cdots x_{n-2} x_{n-1} = p$ or $x_n x_{n-1} \cdots x_2 x_1 = p$ then, we are done. Otherwise, $x_n x_1 x_2 \cdots x_{n-2} x_{n-1} = pz = x_n x_{n-1} \cdots x_2 x_1$ which implies that $x_1 x_2 \cdots x_{n-2} x_{n-1} x_n = x_{n-1} \cdots x_2 x_1 x_n$ and we are done.

4. Conjectures and Questions

There are eight classes of groups which are 4-rewriteable [5]. Roughly speaking, for a group to be 4-rewriteable it must have an abelian subgroup of index two or a commutator subgroup of order at most eight. In contrast, we have shown that a group which is D_4^*-rewriteable has a much sharper classification: it is D_3^*-rewriteable, which is means that the order of the commutator subgroup must be at most two. Since D_4 is the group of symmetries of a square, and S_4 is the group of symmetries of a cube, the group of symmetries of a tetrahedron, A_4, is a natural "compromise" between the two. Using CAYLEY [4], a computer program which enables the user to generate groups and test them for certain properties, we have collected empirical evidence which supports a

CONJECTURE: *G is A_4^*-rewriteable if, and only if, $|G'| = 3$.*

Since one of the eight classes of 4-rewriteable groups consists of precisely those groups G for which $|G'| = 3$, the previous conjecture leads to another

CONJECTURE: *Each of the eight classes of 4-rewriteable groups is characterized by S-rewriteability for some proper subset, S, of S_4 - {id}.*

The idea of the proof of THEOREM 3 works just as well to show show that G is D_n^*-rewriteable if, and only if, G is T-rewriteable, where T is the subset of D_n^* which requires $x_1 x_2 \cdots x_n \in \{x_n x_{n-1} x_{n-2} \cdots x_1, x_1 x_n x_{n-2} \cdots x_2, x_2 x_3 \cdots x_n x_1\}$. Thus, for n = 4, the following statements are equivalent.

i) $xyzw \in \{wxyz, zwxy, yzwx, wzyx, xwzy, yxwz, zyxw\}$ for all $x,y,z,w \in G$.

ii) $xyzw \in \{wxyz, wzyx, zyxw\}$ for all x,y,z,w G.

iii) $xyzw \in \{wzyx, xwyz, yzwx\}$ for all $x,y,z,w \in G$.

QUESTION: *How many subsets of order three of* D_n^* *characterize* D_n^**-rewriteability?*

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Acknowledgement: The author wishes to acknowledge useful conversations with Jon Atkins and Jordan Ellenberg which took place in a supportive, stimulating work environment during the NSF-REU program at Rose-Hulman Institute of Technology directed by Gary Sherman.